

Atoms: The Building Blocks of Nature (Boolean Algebras)

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Boolean Algebras

Definition

A Boolean algebra is (will be viewed as) a structure

$$\mathcal{B} = (B : \cup, \cap, ^c, 0, 1) \quad \text{or} \quad \mathcal{B} = (B : +, \cdot, -, 0, 1)$$

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We will write $x \sqcup y$ (respectively, $x \oplus y$) to denote the element $x \cup y$ (respectively, $x + y$) with the additional hypothesis that $x \cap y = 0$ (respectively, $xy = 0$).

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We will write $x \leq y$ if $x \cap y = x$.

Example

Let $\mathcal{B}_{\mathcal{P}(\mathbb{N})}$ be the structure with universe $B = \mathcal{P}(\mathbb{N})$, with operations union, intersection, and complementation as set theoretic union, intersection, and complementation, and with constants $0 = \emptyset$ and $1 = \mathbb{N}$.

Boolean Algebra Examples

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Remark

In ZFC, are the Boolean algebras $\mathcal{B}_{\mathcal{P}(\omega)/\text{FIN}}$ and $\mathcal{B}_{\mathcal{P}(\omega_1)/\text{FIN}}$ necessarily distinct?

Theorem (Circa History)

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Definition

A non-zero element x of a Boolean algebra \mathcal{B} is an atom if

$$\forall y \in \mathcal{B} [y < x \implies y = 0].$$

In other words, a non-zero element x is an atom if the only element strictly below it is 0.

Example

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Example

In $\mathcal{B}_{\mathcal{P}(\mathbb{N})/\text{FIN}}$, there are no atoms. Note that any non-zero element corresponds to an infinite set (an equivalence class of infinite sets) and that any infinite set can be written as the disjoint union of two infinite sets.

Definition

Let $\mathcal{L} = (L : \prec)$ be a linear order. The interval algebra of \mathcal{L} , denoted $\mathcal{B}_{\mathcal{L}}$, is the Boolean algebra whose universe is the set of clopen subsets of \mathcal{L} under the order topology.

The operations of join, meet, and complementation in the Boolean algebra $\mathcal{B}_{\mathcal{L}}$ are given by taking the union, intersection, and complementation of the clopen sets.

Stone Representation

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The operations of join, meet, and complementation in the Boolean algebra $\mathcal{B}_{\mathcal{L}}$ are given by taking the union, intersection, and complementation of the clopen sets.

Theorem (Stone Representation)

Trivially, every linear order \mathcal{L} has an interval algebra. Every Boolean algebra \mathcal{B} is the interval algebra of some linear order \mathcal{L} .

More Boolean Algebra Examples

Example

Let $\mathcal{B}_{\omega+1}$ be the interval algebra of the linear order $\omega + 1$ with basic open sets (a, b) .

Let \mathcal{B}_{ω^2+1} be the interval algebra of the linear order $\omega^2 + 1$ with basic open sets (a, b) .

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Remark

If $x \sqcup y = 1$ in $\mathcal{B}_{\omega+1}$, then either x or y is a finite join of atoms.

If $x \sqcup y = 1$ in $\mathcal{B}_{\mathcal{P}(\mathbb{N})}$, then both x and y may not be a finite join of atoms. Consider, for example, the elements $x = \{2n : n \in \mathbb{N}\}$ and $y = \{2n + 1 : n \in \mathbb{N}\}$.

Definition

An element x of a Boolean algebra \mathcal{B} is a 0-atom if x is an atom.

An element x is an α -atom for $\alpha > 0$ if it cannot be expressed as a finite join of β -atoms for $\beta < \alpha$, but for all y , either xy or $x(-y)$ can be expressed in this form.

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The 1 element of the Boolean algebra $\mathcal{B}_{\omega+1}$ is a 1-atom.

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Fact

The 1 element of the Boolean algebra \mathcal{B}_{ω^2+1} is not a 1-atom, but is a 2-atom.

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Example

As noted earlier, the Boolean algebra $\mathcal{B}_{\mathcal{P}(\mathbb{N})/\text{FIN}}$ is atomless.

Remark

It might seem natural that, to describe the isomorphism type of a Boolean algebra, it suffices to describe the relationship of atomless elements and α -atoms to all other atomless elements and β -atoms.

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Theorem (Ketonen)

Formalizing the above correctly, the isomorphism type of a Boolean algebra can be so characterized.

Definition

Given a linear order \mathcal{L} , the Cantor-Bendixson derivative \mathcal{L}' of \mathcal{L} is the linear order with universe

$$\mathcal{L}' = \mathcal{L} - \{x \in \mathcal{L} : x \text{ is isolated in } \mathcal{L}\}$$

and order inherited from \mathcal{L} .

Using transfinite recursion, define the α^{th} Cantor-Bendixson derivative of \mathcal{L} , denoted $\mathcal{L}^{(\alpha)}$, by

$$\mathcal{L}^{(0)} = \mathcal{L}, \quad \mathcal{L}^{(\alpha+1)} = \left(\mathcal{L}^{(\alpha)}\right)', \quad \mathcal{L}^{(\gamma)} = \bigcap_{\alpha < \gamma} \mathcal{L}^{(\alpha)}$$

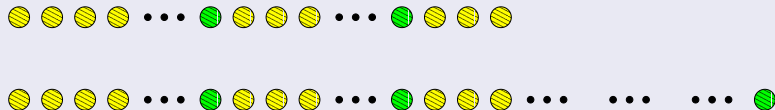
Cantor-Bendixson Example

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Theorem

If \mathcal{L} is a first-countable, zero-dimensional, compact Hausdorff space, then there is a (least) countable ordinal $\nu = \nu(\mathcal{L})$, the rank of \mathcal{L} , such that

$$\mathcal{L}^{(\nu)} = \mathcal{L}^{(\nu+1)}.$$

The linear order $\mathcal{L}^{(\nu)}$, called the perfect kernel of \mathcal{L} , will either be empty or the Cantor set \mathcal{C} .

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The linear order $\mathcal{L}^{(\nu)}$, called the perfect kernel of \mathcal{L} , will either be empty or the Cantor set \mathcal{C} .

Definition

If \mathcal{B} is a Boolean algebra, then \mathcal{B} is superatomic if it is the interval algebra of a linear order \mathcal{L} whose perfect kernel is empty.

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The class of superatomic Boolean algebras is boring.

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Corollary

The class of non-superatomic Boolean algebras is interesting.

Definition

Let \mathcal{L} be a linear order whose perfect kernel is the Cantor set. The rank function $r = r_{\mathcal{L}} : \mathcal{C} \rightarrow \omega_1$ is the function given by

$$r(x) = \min \left\{ \beta : x \notin \overline{\mathcal{L}^{(\beta)} \setminus \mathcal{L}^{(\nu)}} \right\}.$$

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Remark

The rank function computes the number of Cantor-Bendixson derivatives required for an element x of the perfect kernel to no longer be a limit of points not in the perfect kernel.

Definition

Let \mathcal{L} be a linear order with rank function $r = r_{\mathcal{L}}$. The measure $\sigma = \sigma_r : 2^{<\omega} \rightarrow \omega_1$ is the function given by

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Remark

A measure computes the maximum rank of a point within a clopen subset of the perfect kernel.

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By defining derivatives of measures and injecting them into a hierarchy, Ketonen found an algebraic invariant that characterized the isomorphism type of a uniform Boolean algebra.

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Definition

Let σ be a measure. Define maps $\Delta^\alpha \sigma$ recursively by setting $\Delta^0 \sigma(\mathbf{x}) = \sigma(\mathbf{x})$,

$$\Delta^{\alpha+1} \sigma(\mathbf{x}) = \{(\Delta^\alpha \sigma(\mathbf{x}_1), \dots, \Delta^\alpha \sigma(\mathbf{x}_n)) : \mathbf{x}_1 \sqcup \dots \sqcup \mathbf{x}_n = \mathbf{x}\},$$

and $\Delta^\gamma \sigma(\mathbf{x})$ as ... for limit ordinals γ .



Jussi Ketonen.

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