Boolean Algebras

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1. What are They?
2. Where Do They Come From?
3. What are They Made Of?
4. Why Do We Care?

Boolean Algebras as Structures

Definition

A *Boolean algebra* is a structure $\mathcal{B} = (B : +, \cdot, -, 0, 1)$ satisfying associativity, commutativity, absorption, distributivity, and complementation:

$1a.$	$x + (y + z) = (x + y) + z$	$1b.$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$2a.$	$x + y = y + x$	$2b.$	$x \cdot y = y \cdot x$
$3a.$	$x + (x \cdot y) = x$	$3b.$	$x \cdot (x + y) = x$
$4a.$	$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$	$4b.$	$x + (y \cdot z) = (x + y) \cdot (x + z)$
$5a.$	$x + (-x) = 1$	$5b.$	$x \cdot (-x) = 0$

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Remark

Structures satisfying (1), (2), and (3) are called lattices. Structures also satisfying (4) are called distributive lattices.

Boolean Algebra Notation

Remark

Depending on the context, the structure is sometimes denoted $\mathcal{B} = (B : \vee, \wedge, \neg, 0, 1)$ or $\mathcal{B} = (B : \cup, \cap, \mathrm{C}, \emptyset, B)$.

Remark

If $x \cdot y = x$, then we write $x \leq y$ and say y bounds x.

Finite Boolean Algebras

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Proof.

Observe that every element of a finite Boolean algebra bounds a non-zero, but otherwise least, element. The isomorphism type of a finite Boolean algebra is determined by the number of such elements. □

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Theorem

The infinite Boolean algebras are complicated.

Alternate Axiomatizations (I)

Proposition (Huntington,1933)

The Boolean algebra axioms follow from the axioms

$$\begin{array}{ll} 1a. & x + (y + z) = (x + y) + z \\ 2a. & x + y = y + x \\ H. & n(n(x) + y) + n(n(x) + n(y)) = x \end{array}$$

Here $n(x)$ denotes the complement of x.

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Remark

Here, and in any context where join and complementation are defined but meet is not, the meet operation is assumed to be defined by $a \cdot b = n(n(a) + n(b))$.

Alternate Axiomatizations (II)

Proposition (McCune, ????)

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Proof.

Proved by *EQP*, a theorem-proving program for equational logic. The proof was found after five CPU-weeks of computer time. The successful search took 8 days and derived 49,548 equations. □

Boolean Algebras as Set Algebras

Example

Let X be any set and let $B \subseteq \mathcal{P}(X)$ be a subset of the power set of X closed under union, intersection, and complementation. Then $\mathcal{B} = (B : \cup, \cap, \complement, \emptyset, X)$ is the *set algebra of* B.

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Example

Let X be a set of cardinality n and let $B = \mathcal{P}(X)$. Then $\mathcal{B}$ is the finite Boolean algebra with 2^n elements.

Boolean Algebras as Interval Algebras

Example

Let $\mathcal{L} = (L : \prec)$ be a linear order with an associated topology Γ and let B be the set of clopen subsets of $\mathcal{L}$ under Γ. Then $\mathcal{B} = (B : \cup, \cap, \mathrm{C}, \emptyset, L)$ is the *interval (clopen) algebra of* $\mathcal{L}$.

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Example

Let $\mathcal{L}$ be the linear order consisting of n points with basic open sets $[a, b)$ where $a \in L$ and $b \in L \cup \{\infty\}$. Then $\mathrm{Int}(\mathcal{L})$ is the finite Boolean algebra with 2^n elements.

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Theorem

For every Boolean algebra $\mathcal{B}$, there is a set X and $B \subseteq \mathcal{P}(X)$ such that $\mathcal{B}$ is the set algebra of B.

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Proof.

Construct the Stone space of $\mathcal{B}$ as the set of all ultrafilters in $\mathcal{B}$ with base $\{U : U \text{ an ultrafilter containing } x\}$ for $x \in \mathcal{B}$. □

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Definition

A non-zero element x is *atomic* if x bounds no atomless element.

Examples of Atoms and Atomless Elements

Example

Let $\mathcal{B}$ be the set algebra of the power set of $\omega = \{0, 1, 2, \ldots\}$. Then the atoms of $\mathcal{B}$ are the subsets containing exactly one integer. Moreover, $\mathcal{B}$ is atomic.

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Example

Let $\mathcal{B}$ be the interval algebra of the linear order $1 + \mathbb{Q}$ with basic open sets $[a, b)$. Then $\mathcal{B}$ is atomless.

α-Atoms: The Molecules of Boolean Algebras

Definition

An element x is a 0-*atom* if it is an atom.

An element x is a 1-*atom* if it bounds infinitely many atoms but, for every element y, either xy or $x - y$ contains only finitely many atoms.

An element x is an α-*atom* for $\alpha > 0$ if it cannot be expressed as a finite join of β-atoms for $\beta < \alpha$, but for all y, either xy or $x - y$ can be expressed in this form.

A 1-Atom and a 2-Atom

Example

Let $\mathcal{L} = (\omega + 1 :<)$ with basic open sets (a, b). Then any singleton set is an atom and any cofinite set $S \subseteq L$ is a 1-atom.

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Remark

Let $B \subset \mathcal{P}(\omega)$ be the collection of all finite and cofinite subsets of the non-negative integers. Then the set algebra of B is isomorphic to the interval algebra of $\omega + 1$.

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Example

Let $\mathcal{L} = (\omega^2 + 1 :<)$ with basic open sets (a, b). Then the interval algebra of $\mathcal{L}$ is a 2-atom.

Frechet Ideals

Definition

Define the sequence of Frechet ideals by recursion with

$$\begin{aligned} F_0(\mathcal{B}) &= \{0_{\mathcal{B}}\} \\ F_{\alpha+1}(\mathcal{B}) &= \{x \in \mathcal{B} : x \bmod F_\alpha(\mathcal{B}) \text{ is in the ideal generated} \\ &\quad \text{by the atoms of } \mathcal{B} \bmod F_\alpha(\mathcal{B})\} \\ F_\lambda(\mathcal{B}) &= \cup_{\alpha<\lambda} F_\alpha(\mathcal{B}). \end{aligned}$$

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Definition

If $I \subset \mathcal{B}$ is an ideal of $\mathcal{B}$ (i.e., a subset closed downwards and under join), then $a \sim_I b$ if $(a-b)+(b-a) \in I$. The quotient algebra $\mathcal{B}/I$ is the Boolean algebra whose universe is the set of $\sim_I$ equivalence classes.

Boolean Algebras

Propositional Logic

Example

Propositional logic is an example of a Boolean algebra. The join of two propositions P and Q is their disjunction $P \vee Q$; the meet of two propositions P and Q is their conjunction $P \wedge Q$; and the complement of a proposition P is its negation $\neg P$.

Mysteriousness of $\mathcal{P}(\omega)/\textsc{Fin}$

Definition

Let $\mathcal{P}(\omega)/\textsc{Fin}$ be the set algebra of the power set of $\omega = \{0, 1, 2, \dots\}$ after identifying sets that differ by only finitely many elements.

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Is $\mathcal{P}(\omega)/\textsc{Fin} = \mathcal{P}(\omega_1)/\textsc{Fin}$?

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Remark

Note that $\mathcal{P}(\omega)/\text{FIN}$ is equal to $\mathcal{P}(\omega)/F_1(\mathcal{P}(\omega))$.

Computable Boolean Algebras

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Remark

Algebraic invariants that determine the isomorphism type of a countable Boolean algebra are known (see Ketonen), but are quite complicated.

Essentially, the invariants describe the position of α atoms with respect to all other β atoms for $\beta < \alpha$.

References

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