Embeddings of Computable Structures

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Outline

Background

Theorem

If τ is an infinite order type, then τ has a subset of order type ω or ω^*.

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There is a computable presentation of the order type $\omega + \omega^*$ having no computable subset of order type ω or ω^*.

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If τ is an infinite order type, then τ has a subset of order type ω or ω^*.

Theorem (Denisov [3]; Tennenbaum [8])

There is a computable presentation of the order type $\omega + \omega^*$ having no computable subset of order type ω or ω^*.

Theorem (Lerman [7]; Rosenstein [8])

If $\mathcal{L}$ is a computable presentation of an infinite order type τ, then $\mathcal{L}$ has a computable subset of order type ω, ω^*, $\omega + \omega^*$, or $\omega + \eta \cdot \zeta + \omega^*$.

Moreover, all of these order types are necessary.

Constructing $\omega + \omega^*$

Theorem (Denisov [3]; Tennenbaum [8])

There is a computable presentation of the order type $\omega + \omega^*$ having no computable subset of order type ω or ω^*.

Proof.

Construct a computable presentation of the order type $\omega + \omega^*$ meeting, for each e, the following requirement $\mathcal{R}_e$.

$\mathcal{R}_e$: *If W_e is infinite, then $W_e \not\subseteq \omega$ and $W_e \not\subseteq \omega^*$.*

Meet $\mathcal{R}_e$ by putting one element of W_e into ω and one element into ω^*. To facilitate this, maintain a virtual *fence* indicating the current boundary between ω and ω^*. □

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Start 0 2 4 . . . . . . 5 3 1

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See $2, 4 \in W_0$ 0 2 4 . . . . . . 5 3 1

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After $2, 4 \in W_0$ 0 2 ... ... 4 5 3 1

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Work towards $\omega + \omega^*$ 0 2 7 9 . . . . . . 8 6 4 5 3 1

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See $3, 5 \in W_2$ 0 2 7 9 . . . . . . 8 6 4 5 3 1

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See $6, 8 \in W_2$ 0 2 7 9 ... ... 8 6 4 5 3 1

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See $7, 9 \in W_1$ 0 2 7 9 8 . . . . . . 6 4 5 3 1

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The Questions

Question (#1)

Are there computable order types τ_1 and τ_2 having computable presentations $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1$ does not computably embed into $\mathcal{L}_2$?

Question (#2)

Are there computable order types τ_1 and τ_2 such that $\mathcal{L}_1$ does not computably embed into $\mathcal{L}_2$ for any computable presentations $\mathcal{L}_1$ and $\mathcal{L}_2$?

Generalizing Question #2

Remark

Of course, there is no reason attention should be restricted to the context of linear orders.

Question

If $\mathcal{C}$ is a class of computable algebraic structures, are there $\overline{\mathcal{S}}_1, \overline{\mathcal{S}}_2 \in \mathcal{C}$ such that $\mathcal{S}_1$ does not computably embed into $\mathcal{S}_2$ for any computable presentations $\mathcal{S}_1$ and $\mathcal{S}_2$?

Outline

Emmbeddings of Directed Graphs

Theorem (Kach and Miller [6])

If $\mathcal{C}$ is the class of computable directed graphs, then there are structures $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$ in $\mathcal{C}$ such that for no hyperarithmetic presentations of $\mathcal{S}_1$ and $\mathcal{S}_2$ does $\mathcal{S}_1$ hyperarithmetically embed into $\mathcal{S}_2$.

Definition

If $T \subseteq \omega^{<\omega}$ is any tree, define $\mathcal{G}_T$ to be the directed graph whose universe is

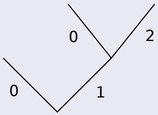
$$\{z_\sigma : \sigma \in T\} \cup \{x_{\sigma^\frown i,0}, \ldots, x_{\sigma^\frown i,i}, y_{\sigma^\frown i,0}, \ldots, y_{\sigma^\frown i,i} : \sigma^\frown i \in T\}$$

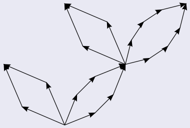
and whose edge relations include $E(z_\sigma, x_{\sigma^\frown i,0})$, $E(z_\sigma, y_{\sigma^\frown i,0})$, $E(x_{\sigma^\frown i,j}, x_{\sigma^\frown i,j+1})$, $E(y_{\sigma^\frown i,j}, y_{\sigma^\frown i,j+1})$, $E(x_{\sigma^\frown i,i}, z_{\sigma^\frown i})$, and $E(y_{\sigma^\frown i,i}, z_{\sigma^\frown i})$ for $\sigma \in T$, $\sigma^\frown i \in T$, and $0 \leq j < i$.

Embeddings of Directed Graphs (Continued...)

Example

If T is as on the left, then $\mathcal{G}_T$ is as on the right.





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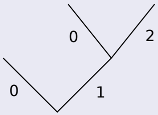

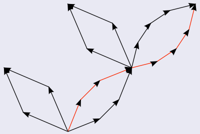

Proof.

Let $\overline{\mathcal{S}}_1$ be the graph of exactly one (directed) infinite path and let $\overline{\mathcal{S}}_2$ be the graph $\mathcal{G}_T$ where $T \subseteq \omega^{<\omega}$ is a computable tree with infinite paths but no hyperarithmetic paths.

Then $\overline{\mathcal{S}}_1$ classically embeds into $\overline{\mathcal{S}}_2$, but there cannot be hyperarithmetic presentations with a hyperarithmetic embedding. For if there was a hyperarithmetic embedding π between hyperarithmetic presentations, there would be a hyperarithmetic path in T. □

Embeddings of Directed Graphs (Continued...)

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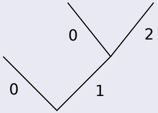

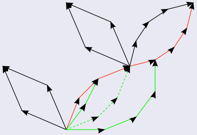

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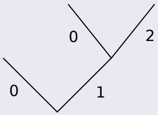

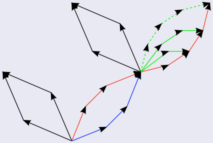

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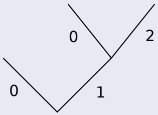

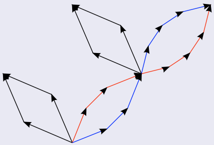

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Embeddings of Universal Structures

Corollary (With Hirschfeldt, Khoussainov, Shore, and Slinko [4])

If $\mathcal{C}$ is the class of computable

- *Directed graphs,*
- *Undirected graphs,*
- *Commutative rings,*
- *Two-step nilpotent groups,*
- *Integral domains, or*
- *Commutative semigroups,*

then there are structures $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$ in $\mathcal{C}$ such that for no hyperarithmetic presentations $\mathcal{S}_1$ and $\mathcal{S}_2$ does $\mathcal{S}_1$ hyperarithmetically embed into $\mathcal{S}_2$.

Outline

Remark

Of particular (and natural) interest are the special cases when $\mathcal{L}_1 = \eta$ and $\mathcal{L}_1 = \omega^*$.

Embeddings of Linear Orders

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Proposition

The order type η is computably categorical.

The standard presentation of the order type ω^* computably embeds into any computable presentation of the order type ω^*.

Embeddings of Linear Orders (Non-Scattered)

Theorem (Kach and Miller [6])

There is a computable non-scattered order type τ_η that is intrinsically hyperarithmetically scattered, i.e., there is a computable order type τ_η into which the rationals embed, but for which the rationals do not hyperarithmetically embed into any hyperarithmetic presentation of τ_η.

Definition

Fix a computable bijection $f : \omega^{<\omega} \to \omega$. If $T \subseteq \omega^{<\omega}$ is any tree, define order types τ_σ for $\sigma \in \omega^{<\omega}$ via corecursion by

$$\tau_\sigma = \omega + f(\sigma) + \zeta + \left(\sum_{\substack{i \in \omega \\ \sigma^\frown i \in T}} \tau_{\sigma^\frown i} \right)^* + \left(\sum_{\substack{i \in \omega \\ \sigma^\frown i \in T}} \tau_{\sigma^\frown i} \right) + \zeta + f(\sigma) + \omega^*.$$

Let τ_T be the order type τ_ϵ, where ϵ denotes the empty string.

Embeddings of Linear Orders (Non-Scattered)

Definition

$$\tau_\sigma = \omega + f(\sigma) + \zeta + \left(\sum_{\substack{i \in \omega \\ \sigma ^\frown i \in T}} \tau_{\sigma ^\frown i} \right)^* + \left(\sum_{\substack{i \in \omega \\ \sigma ^\frown i \in T}} \tau_{\sigma ^\frown i} \right) + \zeta + f(\sigma) + \omega^*.$$

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$$\tau_T = \tau_\epsilon = \omega + f(\varepsilon) + \zeta + \left(\tau_0 + \tau_1\right)^* + \left(\tau_0 + \tau_1\right) + \zeta + f(\varepsilon) + \omega^*$$

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$$\tau_0 = \omega + f(0) + \zeta + \left(\right)^* + \left(\right) + \zeta + f(0) + \omega^*$$

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$$\tau_\sigma = \omega + f(\sigma) + \zeta + \left(\sum_{\substack{i \in \omega \\ \sigma^\frown i \in T}} \tau_{\sigma^\frown i} \right)^* + \left(\sum_{\substack{i \in \omega \\ \sigma^\frown i \in T}} \tau_{\sigma^\frown i} \right) + \zeta + f(\sigma) + \omega^*.$$

$$\tau_T = \tau_\epsilon = \omega + f(\varepsilon) + \zeta + \left(\tau_0 + \tau_1 \right)^* + \left(\tau_0 + \tau_1 \right) + \zeta + f(\varepsilon) + \omega^*$$

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$$\tau_1 = \omega + f(1) + \zeta + \left(\tau_{10} + \tau_{12} \right)^* + \left(\tau_{10} + \tau_{12} \right) + \zeta + f(1) + \omega^*$$

Embeddings of Linear Orders (Non-Scattered) (Continued...)

Theorem (Kach and Miller [6])

There is a computable non-scattered order type τ_η that is intrinsically hyperarithmetically scattered.

Proof.

Let $T \subseteq \omega^{<\omega}$ be a computable tree with infinite paths but no hyperarithmetic paths. Then τ_T is computable as the definition of τ_σ depended only on knowing whether $\sigma ^\frown i \in T$. Also τ_T is non-scattered as T had an infinite path. Finally τ_T is intrinsically hyperarithmetically scattered as a $(\Delta^0_4(\mathcal{L}_T) \oplus \pi)$-computable path in T can be recovered from a hyperarithmetic embedding $\pi : \eta \to \mathcal{L}_T$. □

Embeddings of Linear Orders (Non-Well-Ordered)

Theorem (Kach and Miller [6])

There is a computable, non-well-ordered order type τ_{ω^*} that is intrinsically computably well-ordered, i.e., there is a computable order type τ_{ω^*} into which the negative integers embed, but for no computable presentation of τ_{ω^*} do the negative integers computably embed.

Definition (Montalbán)

If $F : \omega \to \omega$ is any function, define τ_F to be the order type

$$\cdots + \omega^n \cdot F(n) + \cdots + \omega^2 \cdot F(2) + \omega \cdot F(1) + F(0)$$

Proof (Sketch).

Show that, for carefully chosen $\emptyset^{(\omega)}$-computable functions F, the order type τ_F is not computable but $\tau_{\omega^*} := \omega^\omega + \tau_F$ is computable. □

Embeddings of Linear Orders (Non-Well-Ordered) (Continued...)

Proof (Sketch).

Demonstrate that a function $F : \omega \to \omega$ is $\Delta^0_{(2n+1)}$-limit infimum (equivalently $\Delta^0_{(2n+2)}$-limitwise monotonic) if and only if the linear order $\omega^\omega + \mathcal{L}_F$ is computable. Note that the forwards direction is difficult; the backwards direction is relatively straightforward.

Also demonstrate the existence of such a function F for which $\mathcal{L}_F$ is not computable by diagonalizing against all linear orders that look "like" $\mathcal{L}_F$ for some function F. Note that higher priority strategies have access to more powerful oracles, and can thus determine the success or failure of lower priority strategies. □

Embeddings of Linear Orders (Non-Well-Ordered) (Continued...)

Corollary

For each computable ordinal α, there is a computable, non-well-ordered order type that is intrinsically $\emptyset^{(\alpha)}$-computably well-ordered.

Proof.

Relativizing the construction of τ_{ω^*}, build a $\emptyset^{(\alpha)}$-computable presentation that is intrinsically $\emptyset^{(\alpha)}$-computably well-ordered. Then $\omega^{\alpha} \cdot \tau_{\omega^*}$ is computable and still intrinsically $\emptyset^{(\alpha)}$-computably well-ordered. □

Embeddings of Linear Orders (Non-Well-Ordered) (Continued...)

Theorem (Harrison)

If a presentation of a computable, non-well-ordered linear order has no hyperarithmetic descending sequence, then it has the form $\omega_1^{\mathrm{CK}}(1+\eta)+\beta$ for some computable ordinal β.

Corollary

There is no computable, non-well-ordered linear order that is intrinsically hyperarithmetically well-ordered.

Embeddings of Ordered Fields

Theorem (Calvert, Kach, Levin, and Solomon [2])

If $\mathcal{C}$ is the class of computable ordered fields, then there are structures $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$ in $\mathcal{C}$ such that for no hyperarithmetic presentations $\mathcal{S}_1$ and $\mathcal{S}_2$ does $\mathcal{S}_1$ hyperarithmetically embed into $\mathcal{S}_2$.

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Definition

An ordered field $\mathcal{F} = (F : +, \cdot, 0, 1, <)$ is a field $(F : +, \cdot, 0, 1)$ with an order $<$ that behaves as it should.

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Definition

An ordered field $\mathcal{F} = (F : +, \cdot, 0, 1, <)$ is a field $(F : +, \cdot, 0, 1)$ with an order $<$ that behaves as it should.

Definition

If τ is any order type, define $\overline{\mathcal{F}}_\tau$ to be the ordered field whose universe is generated by $\mathbb{Q} \cup \{x_i : i \in \tau\}$ and whose order is generated by $x_i^m <_{\overline{\mathcal{F}}_\tau} x_j^n$ if $i <_\tau j$ as well as $q < x_i$.

Embeddings of Ordered Fields (Continued...)

Theorem (Calvert, Kach, Levin, and Solomon [2])

If $\mathcal{C}$ is the class of computable ordered fields, then there are structures $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$ in $\mathcal{C}$ such that for no hyperarithmetic presentations $\mathcal{S}_1$ and $\mathcal{S}_2$ does $\mathcal{S}_1$ hyperarithmetically embed into $\mathcal{S}_2$.

Proof.

The ordered fields $\overline{\mathcal{F}}_\eta$ and $\overline{\mathcal{F}}_{\tau_\eta}$ suffice. In order to (somewhat) effectively recover an embedding of η into τ_η from a hyperarithmetic embedding of $\mathcal{F}_\eta$ into $\mathcal{F}_{\tau_\eta}$, use *Archimedean power classes*. □

Embeddings of Trees (Viewed as Posets)

Definition

A *tree* is a partial order $(T :\prec)$ with least element such that for all $x \in T$, the set $\{y \in T : y \prec x\}$ is finite and linearly ordered.

Theorem (Binns, Kjos-Hanssen, Lerman, Schmerl, Solomon [1])

There is an infinite computable binary branching tree S with no isolated paths such that any nontrivial self-embedding computes $\mathbf{0}''$.

Corollary

If $\mathcal{C}$ is the class of computable trees, then there are structures $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$ in $\mathcal{C}$ such that for no computable presentations $\mathcal{S}_1$ and $\mathcal{S}_2$ does $\mathcal{S}_1$ computably embed into $\mathcal{S}_2$.

Embeddings of Equivalence Relations

Theorem (Calvert, Kach, Levin, and Solomon [2])

If $\mathcal{C}$ is the class of computable equivalence structures, for all structures $\overline{\mathcal{E}}_1$ and $\overline{\mathcal{E}}_2$ in $\mathcal{C}$, there are computable presentations $\mathcal{E}_1$ and $\mathcal{E}_2$ such that $\mathcal{E}_1$ computably embeds into $\mathcal{E}_2$.

Proof.

If $\overline{\mathcal{E}}_2$ has bounded character, then $\overline{\mathcal{E}}_1$ has bounded character and the result is immediate.

If $\overline{\mathcal{E}}_2$ has unbounded character but only finitely many infinite equivalence classes, a finite injury argument suffices to build computable presentations $\mathcal{E}_1$ and $\mathcal{E}_2$ and a computable embedding between them.

Finally, if $\overline{\mathcal{E}}_2$ has infinitely many infinite equivalence classes, then the addition of countably more infinite equivalence classes provides a place for the image of $\overline{\mathcal{E}}_1$. $\square$

Embeddings of Boolean Algebras

Theorem (Calvert, Kach, Levin, and Solomon [2])

If $\mathcal{C}$ is the class of computable Boolean algebras, for all structures $\overline{\mathcal{B}}_1$ and $\overline{\mathcal{B}}_2$ in $\mathcal{C}$, there are computable presentations $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1$ computably embeds into $\mathcal{B}_2$.

Proof.

If $\overline{\mathcal{B}}_2$ is superatomic, then $\overline{\mathcal{B}}_1$ is superatomic and the result is immediate.

If $\overline{\mathcal{B}}_2$ is not superatomic, then there is a computable ordinal α such that

$$\overline{\mathcal{B}}_2 \cong \overline{\mathcal{B}}_2 \oplus \mathrm{IntAlg}(\omega^\alpha(1 + \eta))$$

as a consequence of work by Ketonen. There is a nice presentation of the latter into which the countable atomless Boolean algebra (and thus $\mathcal{B}_1$) computably embeds. □

Outline

More Questions

Remark

Thus far, we have been considering pairs of structures $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$ such that $\overline{\mathcal{S}}_1$ classically embeds into $\overline{\mathcal{S}}_2$. Until now, we have concerned ourselves with attempting to make sure *no* embedding is computable for *any* computable presentations $\mathcal{S}_1$ and $\mathcal{S}_2$.

If this is possible, it is natural to ask what further restrictions are necessary before such a phenomena is no longer possible.

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By restricting the computable presentations to *fixed* computable presentations but allowing the embedding to vary, we arrive at Question #1.

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By restricting the computable presentations to *fixed* computable presentations but allowing the embedding to vary, we arrive at Question #1.

By restricting the embedding to a *fixed* embedding but allowing the presentations to vary, we arrive at new questions.

More Questions (Cont...)

Question (#3)

If $\mathcal{C}$ is a class of computable algebraic structures, are there structures $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$ in $\mathcal{C}$ and presentations $\mathcal{S}_1 \subseteq \mathcal{S}_2$ such that for no automorphism $\pi : \mathcal{S}_2 \to \mathcal{S}_2$ is $\pi(\mathcal{S}_1)$ computably enumerable?

Question (#4)

If $\mathcal{C}$ is a class of computable algebraic structures, are there structures $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$ in $\mathcal{C}$ and presentations $\mathcal{S}_1 \subseteq \mathcal{S}_2$ such that for no automorphism $\pi : \mathcal{S}_2 \to \mathcal{S}_2$ is $\pi(\mathcal{S}_1)$ computable?

Remarks on the Third and Fourth Question

Remark

Note that a positive answer to Question $\#2$ trivially implies a positive answer to Question $\#3$ and Question $\#4$.

Remark

If a class of computable algebraic structures has a positive answer to Question $\#3$ or Question $\#4$ but a negative answer to Question $\#2$, then it “is not” possible to code into isomorphism types but it “is” possible to code within how an isomorphism type fits inside another isomorphism type in a fixed manner.

Separating Question #2 and Question #3

Proposition (Calvert, Kach, Levin, and Solomon [2])

If $\mathcal{C}$ is the class of computable equivalence structures, Boolean algebras, or abelian p-groups (of length below ω^2), then $\mathcal{C}$ has a positive answer to Question #3 and a negative answer to Question #2.

Proof.

For equivalence structures, let $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$ be the equivalence structure with exactly one class of every finite size and no infinite equivalence class. Have $\mathcal{S}_1$ be the substructure of $\mathcal{S}_2$ where the class of size i in $\mathcal{S}_1$ is a subset of the class of size $2i$ in $\mathcal{S}_2$ if $i \in S$ and a subset of the class of size $2i + 1$ otherwise.

For Boolean algebras, let $\overline{\mathcal{S}}_1$ be IntAlg(ω) and $\overline{\mathcal{S}}_2$ be IntAlg(ω^2). Have $\mathcal{S}_1$ be a substructure of $\mathcal{S}_2$ such that there is an atom of $\mathcal{S}_1$ bounding exactly i atoms in $\mathcal{S}_2$ if and only if $i \in S$. □

Outline

Summary

Remark

The (computable) embedding phenomena happens when $\mathcal{C}$ is the class of computable

- Directed graphs (or any universal class).
- Linear orders.
- Ordered fields.
- Trees.

The embedding phenomena fails to happen when $\mathcal{C}$ is the class of computable

- Equivalence structures.
- Boolean algebras.

Questions

Question

Does Question #2 have a positive answer when $\mathcal{C}$ is the class of computable fields? When $\mathcal{C}$ is the class of computable abelian *p*-groups?

Does Question #2 have a positive answer at the hyperarithmetic level when $\mathcal{C}$ is the class of computable trees?

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Question

(Conjecture) Is there a computable, non-scattered linear order that is intrinsically computably well-ordered?

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Question

(Conjecture) Is there a computable, non-scattered linear order that is intrinsically computably well-ordered?

Question

Is there, for every computable order type τ_1, a computable order type τ_2 such that for no computable presentations $\mathcal{L}_1$ and $\mathcal{L}_2$ does $\mathcal{L}_1$ computably embed into $\mathcal{L}_2$?

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