COMPUTABLE SHUFFLE SUMS OF ORDINALS

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Abstract. The main result is that for sets $S \subseteq \omega + 1$, the following are equivalent:
(1) The shuffle sum $\sigma(S)$ is computable.
(2) The set S is a liminf set, i.e. there is a total computable function $g(x,t)$ such that $f(x) = \liminf_t g(x,t)$ enumerates S.
(3) The set S is a limitwise monotonic set relative to $\mathbf{0}'$, i.e. there is a total $\mathbf{0}'$-computable function $\tilde{g}(x,t)$ satisfying $\tilde{g}(x,t) \leq \tilde{g}(x,t+1)$ such that $\tilde{f}(x) = \lim_t \tilde{g}(x,t)$ enumerates S.

Other results discuss the relationship between these sets and the Σ^0_3 sets.

1. Introduction

A countable linear order is said to be computable if its universe can be identified with ω in such a way that the order is a computable relation on $\omega \times \omega$. The class of computable linear orders has been studied extensively; see [4] for an overview. In this paper we discuss the class of linear orders that are the shuffle sums of ordinals.

Definition 1.1. The *shuffle sum* of a countable set of linear orders $S = \{L_i\}_{i\in\omega}$, denoted $\sigma(S)$, is the (unique) linear order obtained by partitioning the rationals into dense sets $\{Q_i\}_{i\in\omega}$ and replacing each rational of Q_i by the linear order L_i.

Equivalently, the shuffle sum of $S = \{L_i\}_{i\in\omega}$ is the linear order obtained by interleaving copies of each L_i densely and unboundedly between each other.

The class of shuffle sums of ordinals has yielded various results in computable model theory. In [1], the authors use shuffle sums to produce, for each computable ordinal $\alpha \geq 2$, a linear order A_α such that A_α has αth jump degree but not βth jump degree for any $\beta < \alpha$. In [5], shuffle sums of ordinals are used to exhibit a linear order with both a computable model and a prime model, but no computable prime model.

We characterize which shuffle sums of the finite order types and the order type ω are computable. In order to do so, we need the following notions.

Definition 1.2. A set $S \subseteq \omega + 1$ is a limit infimum set, written LimInf set, if there is a total computable function $g(x,t)$ such that S is the range of the function $f(x)$ given by

$$f(x) = \liminf_t g(x,t).$$

We use the convention that $f(x) = \omega$ if $\liminf_t g(x,t) = \infty$.

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Definition 1.3. A set $S \subseteq \omega+1$ is a limitwise monotonic set relative to a degree $\mathbf{a}$, written LimMon($\mathbf{a}$) set, if there is a total $\mathbf{a}$-computable function $\tilde{g}(x,t)$ satisfying $\tilde{g}(x,t) \leq \tilde{g}(x,t+1)$ for all x and t such that S is the range of the function $\tilde{f}(x)$ given by

$$\tilde{f}(x) = \lim_t \tilde{g}(x,t).$$

Again we use the convention that $\tilde{f}(x) = \omega$ if $\lim_t \tilde{g}(x,t) = \infty$.

Although the notion of LimInf sets is new, LimMon$(\mathbf{0}')$ sets have been previously studied. Limitwise monotonic functions were first introduced in [6], relativized in [2], and further studied in [3], [5] and [7], for example.

Blurring the distinction between an ordinal α and the linear order of order type α (which we will do throughout the paper), we note that $\sigma(S) = \sigma(S \cup \{0\})$ for any set S of linear orders. In order to avoid complications in the proofs, we assume the following conventions.

Convention 1.4. *Any set S of ordinals is assumed to not contain 0. Any set S of linear orders is assumed to not contain the empty linear order. Any* LimInf *witnessing function $g(x,t)$ is assumed to satisfy $g(x,t) \neq 0$ for all x and t. Any* LimMon$(\mathbf{0}')$ *witnessing function $\tilde{g}(x,t)$ is assumed to satisfy $\tilde{g}(x,t) \neq 0$ for all x and t.*

We briefly mention that the following facts justify that all the results mentioned in this paper are correct as stated, without needing to invoke Convention 1.4.

Fact 1.5. *If S is a Σ^0_3 set, then $S \backslash \{0\}$ is a Σ^0_3 set.*

Fact 1.6. *If S is a* LimInf *set, then $S \backslash \{0\}$ is a* LimInf *set. If S is a* LimMon$(\mathbf{0}')$ *set, then $S \backslash \{0\}$ is a* LimMon$(\mathbf{0}')$ *set.*

Fact 1.7. *If S is a* LimInf *set not containing 0, then there is a total computable fuction $g(x,t)$ witnessing this satisfying $g(x,t) \neq 0$ for all x and t. If S is a* LimMon$(\mathbf{0}')$ *set not containing 0, then there is a total $\mathbf{0}'$-computable function $\tilde{g}(x,t)$ witnessing this satisfying $\tilde{g}(x,t) \neq 0$ for all x and t.*

As the proofs of these facts are all straightforward, we leave them to the reader. Having introduced all the relevant notions, we are now in a position to state the main results of the paper. The first result is in computable model theory. In particular, it provides a necessary and sufficient condition for the shuffle sum $\sigma(S)$ to be computable in terms of the new computability-theoretic notion of LimInf sets.

Theorem 1.8. *For sets $S \subseteq \omega + 1$, the shuffle sum $\sigma(S)$ is computable if and only if S is a* LimInf *set.*

The second result is in classical computability theory. It provides an alternate characterization of LimInf sets, showing their equivalence with the pre-existing notion of LimMon$(\mathbf{0}')$ sets.

Theorem 1.9. *A set $S \subseteq \omega+1$ is a* LimInf *set if and only if S is a* LimMon$(\mathbf{0}')$ *set.*

In Section 2 we prove Theorem 1.8, and in Section 3 we prove Theorem 1.9. In Section 4 we discuss the relationship between the LimInf and LimMon$(\mathbf{0}')$ sets and the Σ^0_3 sets, making use of previous work in [2] and [7]. We mention that in [2] the authors show that for sets $S \subseteq \omega$, if the shuffle sum $\sigma(S)$ is computable, then S is a LimMon$(\mathbf{0}')$ set.

2. Proof of Theorem 1.8

We prove Theorem 1.8 by proving the forwards and backwards directions separately.

Proposition 2.1. *If $S \subseteq \omega + 1$ is a* LimInf *set, then the shuffle sum $\sigma(S)$ is computable.*

Proof. Let $g(x,t)$ be a total computable function witnessing that S is a LimInf set. Fix a uniformly computable partition of the rationals into dense sets $\{Q_x\}_{x\in\omega}$ with $Q_x = \{q_{x,y}\}_{y\in\omega}$. We build a computable copy of $\sigma(S)$ in stages t using $g(x,t)$.

The basic idea is to build the finite linear order $g(x,t+1)$ at rationals $q_{x,y}$ at stage t. If $g(x,t+1)$ is larger than $g(x,t)$, then the appropriate number of points are added to the linear order already built for $q_{x,y}$. If $g(x,t+1)$ is smaller than $g(x,t)$, then the extra points already built for $q_{x,y}$ are no longer associated with $q_{x,y}$; instead they eventually become associated with some other rational at a later stage. In order to track whether a point is currently associated with some rational $q_{x,y}$, the states *associated* and *unassociated* will be used.

More specifically, at each stage t we build a computable linear order L_t such that $L_t \subseteq L_{t+1}$ for all t. With $L = \bigcup_t L_t$, we aim for $L = \sigma(S)$. At stage 0 we begin with the empty linear order, i.e., L_0 is the empty linear order. At stage $t+1$ we work on behalf of all rationals $q_{x,y}$ with $x, y < t$. This work is done in t^2 substages, with a substage devoted to each such rational $q_{x,y}$ (in lexicographic order). Fixing a rational $q_{x,y}$ with $x, y < t$, we compare the value of $g(x,t+1)$ and $g(x,t)$; our action is determined by which is greater and whether or not work has already been done for the rational $q_{x,y}$.

If $g(x,t+1) \geq g(x,t)$ and work has already been done for $q_{x,y}$, then we insert the appropriate number of new points (namely $g(x,t+1) - g(x,t)$) at the right end of the linear order built at $q_{x,y}$ and give these inserted points the state *associated*.

If $g(x,t+1) < g(x,t)$ and work has already been done for $q_{x,y}$, then we split off the appropriate number of points (namely $g(x,t) - g(x,t+1)$) from the right end of the linear order built at $q_{x,y}$. The points split off have their state switched to *unassociated* and receive a priority amongst all points *unassociated* based first on the stage (lower stage, higher priority) and then their position in the linear order (further left, higher priority).

If no work has been done for $q_{x,y}$, then we insert the linear order $g(x,t+1)$ at $q_{x,y}$. In particular, we note whether or not there are any *unassociated* points greater than the greatest *associated* point to the left of $q_{x,y}$ and less than the least *associated* point to the right of $q_{x,y}$. If there are such *unassociated* points, we use the one with highest priority for the first point of the linear order built at $q_{x,y}$ and insert the appropriate number of new points (namely $g(x,t+1) - 1$) immediately to the right of the first point for the rest of the linear order built at $q_{x,y}$. If there are no such *unassociated* points, we insert new points for the entire linear order built at $q_{x,y}$. All points inserted at $q_{x,y}$ are given the state *associated*, including the previously *unassociated* point if one was used. This completes the construction.

Since the construction is computable, it suffices to show that $L = \sigma(S)$. In order to demonstrate this equality, we verify the following two claims. The first implies that no extra points are built, and the second states that the rational $q_{x,y}$ has the order $f(x) = \liminf_t g(x,t)$ built at it in L.

Claim. *Every point has state* unassociated *for at most finitely many stages.*

Proof. When a point changes its state to *unassociated*, there are at most finitely many *unassociated* points with higher priority. Moreover, as priority is determined first by stage, no later point will receive a higher priority.

As a consequence of the density of the rationals and that only those rationals $q_{x,y}$ with $x, y < t$ have had work done for them by stage t, at some later stage the point will meet the criterion for becoming the first point built for some rational which is having work done for it for the first time. When the point does meet this criterion, it will never again become *unassociated* as by convention $g(x, t) > 0$ for all t, and thus will never be split off. Thus each point becomes *unassociated* at most once and eventually becomes *associated* permanently at some later stage. □

Claim. *In L, the linear order $f(x) = \liminf_t g(x, t)$ is built at the rational $q_{x,y}$.*

Proof. Since $f(x) = \liminf_t g(x, t)$, there is a stage $\hat{t}$ such that $g(x, t) \geq f(x)$ for all $t > \hat{t}$. As a result, the rational $q_{x,y}$ will have at least $f(x)$ points built at it at every stage $t > \hat{t}$. On the other hand, no other points will remain permanently *associated* with $q_{x,y}$ as infinitely often the value of $g(x, t)$ will drop to $f(x)$, causing all other points to no longer be *associated* with $q_{x,y}$.

As the rationals are dense, eventually these points split off will be separated from the $f(x)$ points permanently *associated* with $q_{x,y}$. Thus the linear order $f(x)$ is built at the rational $q_{x,y}$ in L. □

It follows from the first claim that every point of the linear order eventually becomes *associated* permanently with some rational $q_{x,y}$. As each $q_{x,y}$ has the correct linear order built at it by the second claim, we conclude that $L = \sigma(S)$. □

Before demonstrating the converse, we introduce some vocabulary which will simplify the language in the proof.

Definition 2.2. In a linear order, a *maximal block* is a maximal collection of points such that for every pair of points a and b in the collection, the interval $[a, b]$ is discrete.

The *block size* of an element x, denoted $\mathrm{BlockSize}(x)$, is the number of points in the (unique) maximal block containing x.

Proposition 2.3. *If the shuffle sum $\sigma(S)$ is computable with $S \subseteq \omega + 1$, then S is a* LimInf *set.*

Proof. Assume $\sigma(S)$ is computable and let $\mathcal{A}$ be a computable presentation of $\sigma(S)$ with universe $\{a_x\}_{x\in\omega}$. In order to show that S is a LimInf set, we define a total computable function $g(x, t)$ witnessing this. Before defining $g(x, t)$ we first define, for each element a_x, auxiliary functions $\ell_{a_x}(t)$ and $r_{a_x}(t)$ by

$$\ell_{a_x}(t) = \begin{cases} |\{j : 0 \leq j \leq t \;\;\&\;\; a_j < a_x\}| & \text{if } a_x \leq a_t, \\ |\{j : 0 \leq j \leq t \;\;\&\;\; a_t < a_j < a_x\}| & \text{if } a_t < a_x. \end{cases}$$

and

$$r_{a_x}(t) = \begin{cases} |\{j : 0 \leq j \leq t \;\;\&\;\; a_x < a_j\}| & \text{if } a_t \leq a_x, \\ |\{j : 0 \leq j \leq t \;\;\&\;\; a_x < a_j < a_t\}| & \text{if } a_x < a_t. \end{cases}$$

The idea is that ℓ_{a_x} and r_{a_x} attempt to approximate the number of points to the left and right of a_x in its maximal block. The difficulty is that all linear orders of a fixed finite cardinality are isomorphic. This obstacle is resolved by usually having ℓ_{a_x}

(respectively r_{a_x}) be too large by counting all points to the left of a_x (respectively to the right of a_x) seen so far, but for infinitely many values of t dropping the value of ℓ_{a_x} (respectively r_{a_x}) by counting points only between a_x and a_t.

From the functions $\ell_{a_x}(t)$ and $r_{a_x}(t)$ we define the function $g(x,t)$. The idea is to add $\ell_{a_x}(t)$ and $r_{a_x}(t)$ to obtain an approximation to the maximal block size of a_x, but we cannot do so directly as $\ell_{a_x}(t)$ and $r_{a_x}(t)$ will never be at their correct values simultaneously. Fixing a_x and t, we let v be the most recent time before t such that r_{a_x} took the value $r_{a_x}(t)$. More formally, we define $v = v_{x,t}$ to be the greatest integer u less than t such that $\ell_{a_x}(u) = \ell_{a_x}(t)$ if one exists; otherwise we define $v = v_{x,t}$ to be t. We then define $g(x,t)$ by

$$g(x,t) = \ell_{a_x}(t) + 1 + \min_{s\in[v,t]} r_{a_x}(s).$$

Since $\mathcal{A}$ is a computable presentation of $\sigma(S)$, it is clear that $\ell_{a_x}(t)$ and $r_{a_x}(t)$ are computable, from which it follows that $g(x,t)$ is computable. We claim that the range of $f(x) = \liminf_t g(x,t)$ is exactly S, which we will show by demonstrating that $\liminf_t g(x,t) = \text{BlockSize}(a_x)$. We consider the cases when $\text{BlockSize}(a_x)$ is finite and infinite separately.

Claim. *If BlockSize(a_x) is finite, then* $\liminf_t g(x,t) = \textit{BlockSize}(a_x)$.

Proof. If $\text{BlockSize}(a_x) = n$, then there is a $\hat{t}$ such that $\{a_0,\dots,a_{\hat{t}}\}$ includes the elements of the maximal block of a_x. Denote the elements in a_x's maximal block by $\{a_{x_1} < \cdots < a_x = a_{x_i} < \cdots < a_{x_n}\}$. Then $\{a_{x_1},\dots,a_{x_n}\} \subseteq \{a_0,\dots,a_{\hat{t}}\}$. Note that for any $t > \hat{t}$, the functions ℓ_{a_x} and r_{a_x} satisfy $\ell_{a_x}(t) \geq i-1$ and $r_{a_x}(t) \geq n-i$.

When a new element is enumerated directly to the left of a_{x_1}, we have $\ell_{a_x}(t) = i-1$; similarly, when a new element is enumerated directly to the right of a_{x_n}, we have $r_{a_x}(t) = n-i$. Because of the dense nature of shuffle sums, such points will be enumerated infinitely often. Thus $\liminf_t \ell_{a_x}(t) = i-1$ and $\liminf_t r_{a_x}(t) = n-i$, from which it follows that $\liminf_t g(x,t) = (i-1)+1+(n-i) = n$. □

Claim. *If BlockSize(a_x) is infinite, then* $\liminf_t g(x,t) = \textit{BlockSize}(a_x)$.

Proof. If $\text{BlockSize}(a_x) = \infty$, then a_x belongs to a maximal block of order type ω. For every k, there is a $\hat{t} = \hat{t}_k$ such that $\{a_0,\dots,a_{\hat{t}}\}$ includes the k points immediately to the right of a_x in $\sigma(S)$. Then $r_{a_x}(t) \geq k$ for all $t > \hat{t}$. Since there is a stage $\hat{t} = \hat{t}_k$ for every k, it follows that $\lim_t r_{a_x}(t) = \infty$. We conclude that $\liminf_t g(x,t) = \infty$ as $g(x,t) > r_{a_x}(t)$. □

Thus the range of $f(x)$ is S, so that S is a LimInf set.

□

3. Proof of Theorem 1.9

We prove Theorem 1.9, again by proving the forwards and backwards directions separately.

Proposition 3.1. *If $S \subseteq \omega+1$ is a* LimInf *set, then S is a* LimMon $(\mathbf{0}')$ *set.*

Proof. Let $g(x,t)$ be a total computable function witnessing that $S \subseteq \omega+1$ is a LimInf set. Define a function $\tilde{g}(x,t)$ by setting $\tilde{g}(x,t)$ to be the infimum of the set $\{g(x,\hat{t}) : \hat{t} \geq t\}$. Note that $\tilde{g}(x,t)$ is total, increasing in t, and computable in $\mathbf{0}'$. Moreover, $\lim_t \tilde{g}(x,t) = \liminf_t g(x,t)$ so that the range of $\tilde{f}(x) = \lim_t \tilde{g}(x,t)$ is the same as the range of $f(x) = \liminf_t g(x,t)$. It follows that S is a LimMon $(\mathbf{0}')$ set. $\square$

Proposition 3.2. *If $S \subseteq \omega + 1$ is a* LimMon $(\mathbf{0}')$ *set, then S is a* LimInf *set.*

Proof. Let $\tilde{g}(x,t)$ be a total $\mathbf{0}'$-computable function witnessing that $S \subseteq \omega + 1$ is a LimMon $(\mathbf{0}')$ set. By the Limit Lemma, there is a computable function $\tilde{h}(x,t,k)$ such that $\lim_k \tilde{h}(x,t,k) = \tilde{g}(x,t)$. Define the function $g(x,s)$ by

$$g(x,s) = \max\left\{\tilde{h}(x,i,s) : 0 \leq i \leq t_s\right\}$$

where $t_0 = 0$ and for $s > 0$, t_s is the least t not greater than t_{s-1} such that $\tilde{h}(x,t,s) \neq \tilde{h}(x,t,s-1)$ if such a t exists, and otherwise t_s is $t_{s-1}+1$.

As $g(x,s)$ is clearly total and computable, it suffices to show that $\liminf_s g(x,s) = \lim_t \tilde{g}(x,t)$. We begin with a claim.

Claim. *For any t, there are at most finitely many s with t_s less than t.*

Proof. We use induction. For $t = 0$, we have $t_s = 0$ when $s = 0$ and when $\tilde{h}(x,0,s) \neq \tilde{h}(x,0,s-1)$. Since $\lim_k \tilde{h}(x,0,k)$ exists, the latter condition occurs at most finitely often. Thus $t_s = 0$ for only finitely many s.

For $t+1$, we have $t_s = t+1$ possibly when $\tilde{h}(x,t+1,s) \neq \tilde{h}(x,t+1,s-1)$ and possibly when $t_{s-1} = t$. Since $\lim_k \tilde{h}(x,t+1,k)$ exists, the former condition happens at most finitely often. The inductive hypothesis assures that the latter condition happens at most finitely often. Thus $t_s = t+1$ for only finitely many s. $\square$

We continue by arguing that for every t there is an $s = s_t$ such that $g(x,j) \geq \tilde{g}(x,t)$ for all $j \geq s$, establishing that $\liminf_s g(x,s) \geq \lim_t \tilde{g}(x,t)$. Fixing t, let s be such that $t_j \geq t$ for all $j \geq s$, which is possible by the claim. As $\lim_k \tilde{h}(x,t,k)$ exists, we may further assume that s satisfies $\tilde{h}(x,t,k) = \tilde{g}(x,t)$ for all $k \geq s$. Then for $j \geq s$ we have

$$\begin{aligned} g(x,j) &= \max\left\{\tilde{h}(x,i,j) : 0 \leq i \leq t_j\right\} \\ &\geq \max\left\{\tilde{h}(x,i,j) : 0 \leq i \leq t\right\} \\ &\geq \tilde{h}(x,t,j) = \tilde{g}(x,t). \end{aligned}$$

Thus for every t there is an $s = s_t$ such that $g(x,j) \geq \tilde{g}(x,t)$ for all $j \geq s$.

We next argue that for every t there is an $s = s_t$ such that $g(x,s) = \tilde{g}(x,t)$ (with $s_t \neq s_{t'}$ if $t \neq t'$), establishing that $\liminf_s g(x,s) \leq \lim_t \tilde{g}(x,t)$. Again fixing t, let $\hat{s}$ be minimal such that $\tilde{h}(x,i,j) = \tilde{g}(x,i)$ for all $i \leq t$ and $j \geq \hat{s}$. Then there is an $i = i_t$ not greater than t such that $\tilde{h}(x,i,\hat{s}) = \tilde{g}(x,i) \neq \tilde{h}(x,i,\hat{s}-1)$. Let s be the least number greater than or equal to $\hat{s}$ such that $t_s = t$. Then

$$\begin{aligned} g(x,s) &= \max\left\{\tilde{h}(x,i,s) : 0 \leq i \leq t_s\right\} \\ &= \max\left\{\tilde{h}(x,i,s) : 0 \leq i \leq t\right\} \\ &= \max\left\{\tilde{g}(x,i) : 0 \leq i \leq t\right\} \\ &= \tilde{g}(x,t). \end{aligned}$$

Moreover, the values of s_t will be distinct for distinct values of t as the value of t_s is well defined. Thus for every t there is an $s = s_t$ such that $g(x,s) = \tilde{g}(x,t)$, with $s_t \neq s_{t'}$ if $t \neq t'$.

We conclude that $\liminf_s g(x,s) = \lim_t g(x,t)$, so that S is a LimInf set. □

4. LimInf and LimMon (0′) Sets

With a characterization of the computable shuffle sums of subsets $S \subseteq \omega + 1$ in terms of LimInf and LimMon $(\mathbf{0}')$ sets completed, it is natural to ask which subsets are LimInf and LimMon $(\mathbf{0}')$ sets. We note that a LimInf set (and thus a LimMon $(\mathbf{0}')$ set) can be no more complicated than a Σ^0_3 set. For if $g(x,t)$ witnesses that S is a LimInf set, then

$$\begin{aligned} n \in S \quad & \text{iff} \quad \exists x \left[\liminf_t g(x,t) = n\right] \\ & \text{iff} \quad \exists x \big[\exists s_0 \forall s > s_0 \left[g(x,s) \geq n\right] \ \& \ \forall t \exists t_0 > t \left[g(x,t_0) = n\right]\big]. \end{aligned}$$

As the last predicate is clearly Σ^0_3, membership in S cannot be more complicated than Σ^0_3. We next show that if $\omega \in S$, then the LimInf sets (and thus the LimMon $(\mathbf{0}')$ sets) coincide exactly with the Σ^0_3 sets.

Proposition 4.1. *If $S \subseteq \omega$ is a Σ^0_3 set, then $S \cup \{\omega\}$ is a* LimInf *set.*

Proof. Let S be a Σ^0_3 set witnessed by the predicate $\exists s \exists^\infty t\, R(n,s,t)$ where R is a computable relation.[1] Define a function $g(x,t)$ by

$$g(x,t) = g(\langle n,s \rangle, t) = \begin{cases} n & \text{if } R(n,s,t), \\ t & \text{otherwise.} \end{cases}$$

Note that $g(x,t)$ is computable as R is computable.

If $n \in S$, we have $\exists s \exists^\infty t\, R(n,s,t)$. Letting s_0 witness this, we have $g(\langle n, s_0 \rangle, t) = n$ for infinitely many t. As t will be less than n only a finite number of times, it follows that $\liminf_t g(\langle n, s_0\rangle, t) = n$. Thus n is in the range of $f(x) = \liminf_t g(x,t)$.

If instead $n \notin S$, we have $\forall s \exists^{<\infty} t\, R(n,s,t)$. For any $x = \langle n,s \rangle$, it follows that $g(x,t) = n$ for only finitely many t. Thus $\liminf_t g(x,t) = \infty$, and so ω is in the range of $f(x)$ and n is not in the range of $f(x)$.

In the extreme case when $S = \omega$, we can (non-uniformly) arrange to have the range of $f(x) = \liminf_t g(x,t)$ be $\omega \cup \{\omega\}$. □

It follows immediately from Theorem 1.8 and Proposition 4.1 that $\sigma(S \cup \{\omega\})$ is computable for every Σ^0_3 set S, a result shown in [1]. However $\sigma(S)$ is not computable for every Σ^0_3 set S, a corollary of our results and the following result found in [2] (which is a relativization of a result in [7]).

Proposition 4.2 (Coles, Downey, and Khoussainov)**.** *There is a Σ^0_3 set S that is not a* LimMon $(\mathbf{0}')$ *set. Moreover, the set S can be made to be a d.c.e. set.*

We conclude by leaving open two questions.

Question 4.3. *For which subsets $S \subseteq \omega_1^{\mathrm{CK}} + 1$ is $\sigma(S)$ computable?*

Question 4.4. *Which subsets $S \subseteq \omega + 1$ are Σ^0_3 sets but not* LimInf *sets, or equivalently* LimMon $(\mathbf{0}')$ *sets?*

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[1] See [8] for a justification that every Σ^0_3 set is of this form.

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