Δ^0_2-CATEGORICITY OF EQUIVALENCE STRUCTURES

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Abstract. We exhibit computable equivalence structures, one Δ^0_2-categorical and one not Δ^0_2-categorical, having unbounded character, infinitely many infinite equivalence classes, and no s_1-function. This offers a natural example where Δ^0_2-categoricity and relative Δ^0_2-categoricity differ.

1. Introduction and Results

In [2], Calvert, Cenzer, Harizanov, and Morozov investigate effective categoricity of computable equivalence structures. We quickly recall a computable structure $\mathcal{A}$ is Δ^0_α-*categorical* if, given any computable presentations $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$, there is a Δ^0_α-computable isomorphism $\pi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$; and a computable structure $\mathcal{A}$ is *relatively* Δ^0_α-*categorical* if, given any computable presentation $\mathcal{A}_1$ and an arbitrary presentation $\mathcal{A}_2$, there is a $\Delta^0_\alpha(\mathcal{A}_2)$-computable isomorphism $\pi : \mathcal{A}_1 \rightarrow \mathcal{A}_2$.

For $\alpha = 1$ and $\alpha = 3$, the paper characterizes which computable equivalence structures are Δ^0_α-categorical and relatively Δ^0_α-categorical.

Theorem 1.1 ([2])**.** *A computable equivalence structure $\mathcal{E}$ is computably categorical (also relatively computably categorical) if and only if there is a cardinality κ such that $\mathcal{E}$ has only finitely many classes not of size κ. Every computable equivalence structure $\mathcal{E}$ is Δ^0_3-categorical (also relatively Δ^0_3-categorical).*

For $\alpha = 2$, the paper characterizes which computable equivalence structures are relatively Δ^0_2-categorical.

Theorem 1.2 ([2])**.** *A computable equivalence structure $\mathcal{E}$ is relatively Δ^0_2-categorical if and only if it has bounded character or finitely many infinite equivalence classes.*

However, the paper fails to provide a complete characterization of which computable equivalence structures are Δ^0_2-categorical.

Theorem 1.3 ([2])**.** *A computable equivalence structure $\mathcal{E}$ is Δ^0_2-categorical if it has finitely many equivalence classes or bounded character.*

A computable equivalence structure $\mathcal{E}$ is not Δ^0_2-categorical if it has infinite many infinite equivalence classes and an s_1-function.

The reason computable equivalence structures with infinitely many infinite classes and no s_1-function are not characterized is because the set

$$(\dagger) \qquad \left\{\mathrm{FIN}^{\mathcal{E}_1} : \mathcal{E}_1 \text{ is a computable presentation of } \mathcal{E}\right\}$$

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was not sufficiently well understood.

Before continuing, we introduce the relevant notions.

Definition 1.4. If $\mathcal{E}$ is an equivalence structure, its *character* $\chi_{\mathcal{E}}$ is the set of all pairs (n, k) such that $\mathcal{E}$ has at least k many classes of size n.

If there are arbitrarily large integers n such that $(n, 1) \in \chi_{\mathcal{E}}$, the equivalence structure $\mathcal{E}$ is said to have *unbounded character*.

Definition 1.5. If $\mathcal{E}_1$ is a computable presentation of a computable equivalence structure $\mathcal{E}$, the set of elements of $\mathcal{E}_1$ in finite equivalence classes is denoted $\mathrm{FIN}^{\mathcal{E}_1}$.

Definition 1.6. A (strictly increasing) function $F : \omega \to \omega$ is *(strictly increasing) limitwise monotonic* if there is a total computable function $f : \omega \times \omega \to \omega$ satisfying $f(x, s) \leq f(x, s+1)$ and $F(x) = \lim_s f(x, s)$.

A function f witnessing that F is (strictly increasing) limitwise monotonic is called a *(strictly increasing) limitwise monotonic approximation.*

A set $S \subseteq \omega$ is *(strictly increasing) limitwise monotonic* if it is the range of a (strictly increasing) limitwise monotonic function.

We also use the following historically motivated terminology.

Definition 1.7 ([6])**.** An equivalence structure $\mathcal{E}$ is said to have an *s_1-function* if the set $\{n : (n, 1) \in \chi_{\mathcal{E}}\}$ contains a strictly increasing limitwise monotonic subset.

In this paper, we demonstrate the following theorems by partially controlling the set in (†).

Theorem 1.8. *There is a computable equivalence structure $\mathcal{E}$ that is Δ^0_2-categorical having unbounded character, infinitely many infinite equivalence classes, and no s_1-function.*

Theorem 1.9. *There is a computable equivalence structure $\mathcal{E}$ that is not Δ^0_2-categorical having unbounded character, infinitely many infinite equivalence classes, and no s_1-function.*

There are two noteworthy consequences of these theorems. First, it is noteworthy that the equivalence structure of Theorem 1.8 is Δ^0_2-categorical but not relatively Δ^0_2-categorical, and thus the class of equivalence structures offers examples where Δ^0_2-categoricity and relative Δ^0_2-categoricity diverge. Though other examples are well-known (see [4], [5], and [3], for example), previous examples have utilized nonclassical classes of algebraic structures.

Second, these theorems suggest Σ^0_2 sets having no s_1-function do not all share the same algebraic properties. This suggests that our understanding of these sets is far too coarse.

We refer the reader to [1] for background on computable structures and to [2] for background on computable equivalence structures, a history of the study of effective categoricity (note [3] is too recent to appear in it), and the proofs of Theorem 1.1 and Theorem 1.2.

2. Proof of Theorem 1.8

We exhibit an appropriate computable equivalence structure that is Δ^0_2-categorical by constructing an isomorphism type $\mathcal{E}$ for which $\mathrm{FIN}^{\mathcal{E}_1}$ is Π^0_1 in every computable presentation $\mathcal{E}_1$ of $\mathcal{E}$. This suffices (as observed in [2]) as the size of an equivalence class can be determined by $\mathbf{0}'$ if it is known to be finite.

Fact 2.1. If $\mathcal{E}$ is a computable equivalence structure, it is possible to uniformly associate with $\mathcal{E}$ a computable function $f = f_{\mathcal{E}}$ with domain $\omega \times \omega$ and range ω, where $f(x, s)$ is an approximation from below of the size of the equivalence class of x.

Proof of Theorem 1.8. Fix an effective enumeration $\{\mathcal{E}_i\}_{i\in\omega}$ of all computable presentations of computable equivalence structures and the corresponding enumeration of computable functions $\{f_i\}_{i\in\omega}$. The structure $\mathcal{E}$ is defined so that if $\mathcal{E} \cong \mathcal{E}_i$, then $\mathrm{FIN}^{\mathcal{E}_i}$ is Π^0_1. This is done by setting a computable threshhold for each element and guaranteeing if the size of its equivalence class rises beyond that threshhold, then if $\mathcal{E} \cong \mathcal{E}_i$ is to be possible, the equivalence class must become infinite.

Construction: At stage zero, the structure $\mathcal{E}$ starts empty. At stage $s > 0$, the construction operates in three steps. First, it ensures that $\mathcal{E}$ has no equivalence class of size $f_i(n, s)$ for all $i, n < s$ for which $f_i(n, s) > 2^{i+n}$. It does so by turning any class in $\mathcal{E}$ of such size into an infinite class. Second, it ensures that $\mathcal{E}$ does have an equivalence class of size k for each $k < s$ that is not within the set

$$\{f_i(n, s) : i, n < s \text{ and } f_i(n, s) > 2^{i+n}\}.$$

It does so by simply building such a class in $\mathcal{E}$ with fresh elements if such a class does not already exist. Third, it starts a new infinite equivalence class.

Verification: As $\mathcal{E}$ was built with infinitely many infinite equivalence classes, it remains only to verify that $\mathcal{E}$ has unbounded character and that $\mathcal{E} \cong \mathcal{E}_i$ implies $\mathrm{FIN}^{\mathcal{E}_i}$ is Π^0_1. Note that Theorem 1.3 implies that $\mathcal{E}$ has no s_1-function.

The reason $\mathcal{E}$ has unbounded character is combinatorial. Fixing a positive integer k, there are at most $(1 + \log k)^2$ many pairs (i, n) such that $2^{i+n} \leq k$. Thus at any stage s, the set

$$\left\{m \leq k : (\exists i, n < s)\left[m = f_i(n, s) > 2^{i+n}\right]\right\}$$

has size less than $(1+\log k)^2$. At some stage s_0, this set will cease changing, as the value of $f_i(n, s)$ is monotonically increasing in s. Consequently, by stage s_0, the second substage will have built at least $k - (1 + \log k)^2$ many equivalence classes of distinct sizes k or less which will never change in size. Thus the structure $\mathcal{E}$ has unbounded character as $\lim_{k\to\infty}\left[k - (1 + \log k)^2\right] = \infty$.

The reason $\mathcal{E} \cong \mathcal{E}_i$ implies $\mathrm{FIN}^{\mathcal{E}_i}$ is Π^0_1 is by nature of the construction. For $x \in \mathcal{E}_i$, and denoting $\lim_s f_i(x, s)$ by $F_i(x)$ (possibly infinite), it suffices to show either

$$x \in \mathrm{FIN}^{\mathcal{E}_i} \text{ if and only if } (\forall s)\left[f_i(x, s) \leq 2^{i+x}\right] \qquad \text{or} \qquad \mathcal{E} \ncong \mathcal{E}_i.$$

Of course, it is immediate that $x \in \mathrm{FIN}^{\mathcal{E}_i}$ if $(\forall s)\left[f_i(x, s) \leq 2^{i+x}\right]$. Conversely, if $x \in \mathrm{FIN}^{\mathcal{E}_i}$, either $F_i(x) \leq 2^{i+x}$ or $F_i(x) > 2^{i+x}$. In the former case, we have $(\forall s)\left[f_i(x, s) \leq 2^{i+x}\right]$; in the latter case, we have $\mathcal{E} \ncong \mathcal{E}_i$ as $\mathcal{E}$ will have no equivalence class of size $F_i(x)$ as a consequence of the first substage. It follows that if $\mathcal{E} \cong \mathcal{E}_i$, then $x \in \mathrm{FIN}^{\mathcal{E}_i}$ if and only if $(\forall s)\left[f_i(x, s) \leq 2^{i+x}\right]$. □

3. Proof of Theorem 1.9

We exhibit an appropriate computable equivalence structure that is not Δ_2^0-categorical by constructing an isomorphism type having computable presentations $\mathcal{E}_1$ with $\mathrm{FIN}^{\mathcal{E}_1} \leq_T \mathbf{0}'$ and $\mathcal{E}_2$ with $\mathrm{FIN}^{\mathcal{E}_2} \not\leq_T \mathbf{0}'$. Achieving the former is automatic; the latter is more difficult. Of course, this suffices as a $\mathbf{0}'$-isomorphism $\pi : \mathcal{E}_1 \to \mathcal{E}_2$ would be a bijection between $\mathrm{FIN}^{\mathcal{E}_1}$ and $\mathrm{FIN}^{\mathcal{E}_2}$.

Lemma 3.1 ([2])**.** *Every computable equivalence structure $\mathcal{E}$ has a computable presentation $\mathcal{E}_1$ for which $FIN^{\mathcal{E}_1}$ is Π_1^0.*

Fact 3.2. There is an effective enumeration $\{f_i\}_{i\in\omega}$ of total computable functions $f_i : \omega \times \omega \to \omega$ with $f_i(x, s) \leq f_i(x, s + 1)$ and $f_i(x, s) < f_i(y, s)$ whenever $x < y$ whose limit functions $\{F_i\}_{i\in\omega}$ contain all the strictly increasing limitwise monotonic functions.

Proof of Theorem 1.9. By Lemma 3.1, it suffices to build a computable equivalence structure $\mathcal{E}$ and a computable presentation $\mathcal{E}_2$ of it with $\mathrm{FIN}^{\mathcal{E}_2} \not\leq_T \mathbf{0}'$. Towards this, fix an effective enumeration $\{f_i\}_{i\in\omega}$ of candidate strictly increasing limitwise monotonic approximation functions $f_i : \omega \times \omega \to \omega$ (as in Fact 3.2) and an effective enumeration $\{g_i\}_{i\in\omega}$ of limit approximation functions $g_i : \omega \times \omega \to \{0, 1\}$ to all Δ_2^0 sets (we choose these functions to be total).

The idea is to build a computable presentation $\mathcal{E}_2$ of a computable equivalence structure $\mathcal{E}$ meeting a *monotonic diagonalization requirement* $\mathcal{M}_i$ for each $i \in \omega$ and a *complexity diagonalization requirement* $\mathcal{C}_j$ for each $j \in \omega$.

- $\mathcal{M}_i$: There is an integer x for which either $F_i(x)$ fails to exist or $\mathcal{E}$ has no equivalence class of size $F_i(x)$.
- $\mathcal{C}_j$: The function $G_j(n)$ is not the characteristic function of $\mathrm{FIN}^{\mathcal{E}}$.

The requirements will have priority order given by $\mathcal{M}_0 \prec \mathcal{C}_0 \prec \mathcal{M}_1 \prec \mathcal{C}_1 \prec \ldots$.

The strategy to meet $\mathcal{M}_i$ will be to choose an appropriate column x, increase the size of all (lower priority) classes currently of size the current approximation to $F_i(x)$, and prevent any (lower priority) classes of size the current approximation to $F_i(x)$ from being built. The strategy to meet $\mathcal{C}_j$ will be to choose a set of elements $\{n_\ell\}_{\ell<2^{j+1}}$ and ensure $G_j(n_\ell)$ is incorrect for at least one of them. Of course, conflict occurs when a $\mathcal{C}_j$ strategy wishes to prevent a class from growing that is the current approximation to a chosen $F_i(x)$.

Strategy for Requirement $\mathcal{M}_i$: When started at stage s_0, the strategy searches for the least column $x = x_i$ such that $f_i(x, s_0)$ is not the size of class built by a higher priority $\mathcal{C}_i$ requirement. At each stage $s \geq s_0$, it computes the *exclusion size* $f_i(x, s)$. If there is an equivalence class in $\mathcal{E}_2$ of the exclusion size which has been built by a higher priority $\mathcal{C}_i$ requirement, the strategy resets. Otherwise, the strategy adds an element to each equivalence class in $\mathcal{E}_2$ of the exclusion size. Finally, it prohibits any lower priority $\mathcal{C}_i$ requirement from building an equivalence class in $\mathcal{E}_2$ of the exclusion size.

Strategy for Requirement $\mathcal{C}_j$: When started at stage s_0, the strategy associates, for each $S \subseteq \{0, \ldots, j\}$, a substrategy $\mathcal{C}_{j,S}$ which works with the hypothesis that $F_i(x_i)$ is finite if $i \in S$ and infinite if $i \notin S$. A substrategy believes its hypothesis only when

$$\max\{f_i(x_i, s) : i \in S\} + 1 < \min\{f_i(x_i, s) : i \notin S\}.$$

Note that if $\{0,\dots,j\}$ is partitioned correctly, then the substrategy will believe its hypothesis cofinitely often; if $\{0,\dots,j\}$ is partitioned incorrectly, then the substrategy may or may not believe its hypothesis.

At each stage $s > s_0$, each substrategy determines whether or not it believes its hypothesis and acts accordingly.

- If it does not, any equivalence class in $\mathcal{E}_2$ built on behalf of this substrategy is made infinite and no longer associated with this substrategy.
- If it does but did not at the previous stage, an equivalence class of size $\max\{f_i(x_i,t) : i \in S\}+1$ is created in $\mathcal{E}_2$. Denote by n_S the least element in this equivalence class.
- If it does and did at the previous stage, its behavior depends on $g_j(n_S, s)$:
  - If $g_j(n_S,s) = 0$, the equivalence class of n_S is increased to size $\max\{f_i(x_i,s) : i \in S\}+1$ if it was of smaller size, but is otherwise unchanged.
  - If $g_j(n_S,s) = 1$, the equivalence class of n_S is increased to size $\min\{f_i(x_i,s) : i \notin S\}-1$ if it was of smaller size.

Construction: At stage zero, the structure $\mathcal{E}_2$ begins empty. At each stage $s > 0$, the requirements $\{\mathcal{M}_i\}_{i<s}$ and $\{\mathcal{C}_j\}_{j<s}$ act in priority order as described. A new infinite equivalence class is also started.

Verification: As the construction yields a computable presentation $\mathcal{E}_2$ with infinitely many infinite classes, it remains only to verify that $\mathcal{E}$ (the isomorphism type of $\mathcal{E}_2$) has unbounded character, that $\mathcal{E}$ has no s_1-function, and that $\mathrm{FIN}^{\mathcal{E}_2} \not\leq_T \mathbf{0}'$.

The following two claims are proved by simultaneous induction.

Claim 3.2.1. For a given substrategy $\mathcal{C}_{j,S}$ of a given strategy $\mathcal{C}_j$, let $h(s)$ be the size of the class associated with this substrategy at stage s, or the most recent finite value if no class is associated at stage s. If no class has ever been associated, let $h(s)$ be zero. Then $h(s)$ is either eventually constant or $\liminf_s h(s) = \infty$.

Claim 3.2.2. For a given strategy $\mathcal{M}_i$, let $e(s)$ be the exclusion size at stage s. Then $e(s)$ is either eventually constant or $\liminf_s e(s) = \infty$.

Proof of Claim 3.2.1. By Claim 3.2.2, either $\max\{f_i(x_i,s) : i \in S\}$ is eventually constant or $\liminf_s \max\{f_i(x_i,s) : i \in S\} = \infty$. In the latter case, if classes are associated with $\mathcal{C}_{j,S}$ infinitely often, then $\liminf_s h(s) = \infty$; otherwise $h(s)$ is eventually constant. In the former, consider $\min\{f_i(x_i,s) : i \notin S\}$. If this is eventually constant, then $h(s)$ will be eventually constant. If $\liminf_s \min\{f_i(x_i,s) : i \notin S\} = \infty$, then $h(s)$ will either be eventually constant or increase without bound, depending on the behavior of g_j. □

Proof of Claim 3.2.2. Note that by a pigeon-hole argument, every time strategy $\mathcal{M}_i$ resets, it will choose its next column x with $x \leq i$. Choose a sufficiently large stage s' such that:

- For each $y \leq i$, if $F_i(y)$ exists then $F_i(y) = f_i(y, s')$.
- For each h associated with a substrategy of some $\mathcal{C}_j$ with $j < i$, if $h(s)$ is eventually constant, then $h(s) = h(s')$ for any $s > s'$.
- For each $y \leq i$ and h associated with a substrategy of $\mathcal{C}_j$ with $j < i$, if $F_i(y)$ exists and $h(s)$ is not eventually constant, then $h(s) > F_i(y)$ for any $s > s'$.

If, after stage s', the strategy $\mathcal{M}_i$ is ever reset so that $F_i(x)$ exists for its witness column x, then $e(s)$ will henceforth be constant with $e(s) = F_i(x)$. Otherwise, its witness column will only be reset to x for which $f_i(x, s)$ increases without bound, and thus $\liminf_s e(s) = \infty$. □

To see that $\mathcal{E}$ will have unbounded character, fix an integer N. Fix an integer $i = i_N$ such that $F_i(x) = x + N$ and an integer $j > i$ such that g_j is identically zero. Then the correct substrategy of $\mathcal{C}_j$ will create a finite equivalence class of size greater than N.

To see that $\mathcal{E}$ has no s_1-function, it suffices to show that for each i, strategy $\mathcal{M}_i$ meets its requirement. If the exclusion size $e(s)$ is eventually constant, then by construction the requirement is met. Otherwise, there is some x such that $F_i(x)$ does not exist, and thus the requirement is met automatically.

Also, the presentation $\mathcal{E}_2$ constructed satisfies $\mathrm{FIN}^{\mathcal{E}_2} \not\leq_T \mathbf{0}'$. For if $\mathrm{FIN}^{\mathcal{E}_2} \leq_T \mathbf{0}'$, by the Limit Lemma there would be a computable approximation $g(n, s)$ to $\mathrm{FIN}^{\mathcal{E}_2}$. However, this cannot be the case as the construction explicitly diagonalized against every such function $g(n, s)$. In particular, for the correct partition S, we have

$$\lim_s \left[\max\{f_{i,s}(x_i, t) : i \in S\}\right] < \infty \text{ and } \lim_s \left[\min\{f_{i,s}(x_i, t) : i \notin S\}\right] = \infty.$$

As a result, the equivalence class of n_S will disagree with $G(n_S)$ as it will have finite size if $G(n_S) = 0$ and infinite size if $G(n_S) = 1$. □

Remark 3.3. Closer inspection of the construction reveals that $\mathrm{FIN}^{\mathcal{E}_2} >_T \mathbf{0}'$.

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