4.2 Null Spaces, Column Spaces, and Linear Transformations

**Definition** The **null space** of an \(m \times n\) matrix \(A\), denoted \(\text{Nul}A\), is the set of all solutions of the homogeneous equation \(A\vec{x} = \vec{0}\). In set notation, we write

\[
\text{Nul}A = \{\vec{x} : \vec{x} \text{ is in } \mathbb{R}^n \text{ and } A\vec{x} = \vec{0}\}.
\]

**Example** We let \(A\) be the matrix

\[
\begin{bmatrix}
2 & 2 & 1 \\
4 & 1 & 0
\end{bmatrix}
\]

and determine if the vectors

\[
\vec{v}_1 = \begin{bmatrix}
1 \\
1 \\
0
\end{bmatrix}, \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix}
1 \\
-4 \\
6
\end{bmatrix}
\]

are in \(\text{Nul}A\). First note that

\[
A\vec{v}_1 = \begin{bmatrix}
2 + 2 + 0 \\
4 + 1 + 0
\end{bmatrix} = \begin{bmatrix}
4 \\
5
\end{bmatrix},
\]

so \(\vec{v}_1 \notin \text{Nul}A\). However,

\[
A\vec{v}_2 = \begin{bmatrix}
2 - 8 + 6 \\
4 - 4 + 0
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

so that \(\vec{v}_2 \in \text{Nul}A\).

**Theorem** The null space of an \(m \times n\) matrix \(A\) is a subspace of \(\mathbb{R}^n\). Equivalently, the set of all solutions to a system \(A\vec{x} = \vec{0}\) of \(m\) homogeneous equations in \(n\) variables is a subspace of \(\mathbb{R}^n\).

**Proof** The proof of this fact requires verifying the three properties of subspaces, namely: (i) \(\vec{0} \in \text{Nul}A\); (ii) \(\text{Nul}A\) is closed under addition; (iii) \(\text{Nul}A\) is closed under scalar multiplication.

For (i), note that \(A\vec{0} = \vec{0}\), so \(\vec{0} \in \text{Nul}A\).

For (ii), if \(\vec{u}\) and \(\vec{v}\) are in \(\text{Nul}A\), then \(A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}\), so \(\vec{u} + \vec{v} \in \text{Nul}A\).

For (iii), if \(c\) is a scalar and \(\vec{u} \in \text{Nul}A\), then \(A(c\vec{u}) = cA\vec{u} = c\vec{0} = \vec{0}\), so \(c\vec{u} \in \text{Nul}A\).

**Example** Let \(H\) be the set of vectors \((a, b, c, d)\) in \(\mathbb{R}^4\) that satisfy \(a + b = c\) and \(2a - b = c + d\). To show that \(H\) is a vector subspace of \(\mathbb{R}^4\), simply rearrange the equations to get a homogeneous system:

\[
\begin{align*}
a + b - c &= 0 \\
2a - b - c - d &= 0.
\end{align*}
\]

It follows by the theorem that \(H\) is a vector subspace of \(\mathbb{R}^4\).

**Note** The null space of a matrix \(A\) is defined *implicitly*. We know the null space always exists (since it contains at least the zero vector), but we have no idea from the definition what it looks like. To get an *explicit* description of \(\text{Nul}A\), we typically write the set of vectors \(\vec{x}\) satisfying \(A\vec{x} = \vec{0}\) in parametric vector form.

**Definition** The **column space** of an \(m \times n\) matrix \(A\), denoted \(\text{Col}A\), is the set of all linear combinations of the columns of \(A\). If \(A = \begin{bmatrix} \vec{a}_1 & \ldots & \vec{a}_n \end{bmatrix}\), then

\[
\text{Col}A = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}.
\]

Since the span of a set of vectors is a subspace, we have the following theorem.

**Theorem** The column space of an \(m \times n\) matrix \(A\) is a subspace of \(\mathbb{R}^m\).
Another way to think of the column space is as the set of all vectors $\vec{b}$ in $\mathbb{R}^m$ such that $\vec{b} = A\vec{x}$ for some $\vec{x}$ in $\mathbb{R}^n$. In terms of linear transformations, the column space of the matrix $A$ is the range of the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\vec{x}) = A\vec{x}$. From this vantage point we see that the column space of an $m\times n$ matrix $A$ is all of $\mathbb{R}^m$ if and only if the equation $A\vec{x} = \vec{b}$ is consistent for each $\vec{b}$ in $\mathbb{R}^m$. This happens if and only if $T(\vec{x})$ is onto. What is the difference between the null space and column space of a matrix $A$? We illustrate with an example.

**Example** Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}.$$  

We answer the following questions:

(a) For what value $k$ is $\text{Col} A$ a subspace of $\mathbb{R}^k$?

(b) For what value $k$ is $\text{Nul} A$ a subspace of $\mathbb{R}^k$?

(c) Find a nonzero vector in $\text{Col} A$. Is this always possible?

(d) Find a nonzero vector in $\text{Nul} A$. Is this always possible?

(e) Let

$$\vec{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}.$$  

(i) Determine if $\vec{u}$ is in $\text{Nul} A$. Is it also in $\text{Col} A$?

(ii) Determine if $\vec{v}$ is in $\text{Col} A$. Is it also in $\text{Nul} A$?

For (a) and (b), we consider that $A$ is a $3\times 4$ matrix. Hence $\text{Col} A$ is a subspace of $\mathbb{R}^3$, and $\text{Nul} A$ is a subspace of $\mathbb{R}^4$.

For (c), we can simply take any nonzero column of $A$, say the first column

$$\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}.$$  

This is *not* possible in the case that $A$ is the zero matrix.

For (d), first note that there must be a nonzero vector in $\text{Nul} A$, since there are more columns than rows (i.e., the number of column vectors is greater than the number of entries in each vector, so there must be some linear dependence). In order to actually find a nonzero vector, we row reduce $A$ to

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  

Considering this as a homogeneous system ($A\vec{x} = \vec{0}$) in the variables $x_1, x_2, x_3, x_4$, we see that $x_3$ must be a free variable. Hence

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}.$$  

Picking $x_3 = 1$ (or any nonzero number) gives us a nonzero vector in $\text{Nul} A$.

For (e)(i), we simply compute $A\vec{u}$ and check if it is zero:

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \vec{0}.$$
Hence \( \bar{u} \) is not in \( \text{Nul}\ A \). It also could not be in \( \text{Col}\ A \), since \( \bar{u} \) contains 4 entries, while those in \( \text{Col}\ A \) contain 3.

For (e)(ii), we want to see if there’s an \( \bar{x} \) such that \( A\bar{x} = \bar{v} \). From above, we see that \( A \) has a pivot in each row, so there must be a solution to this matrix equation. It follows that \( \bar{v} \in \text{Col}\ A \). However, since \( \bar{v} \) has 3 entries whereas vectors in \( \text{Nul}\ A \) have 4, we see that \( \bar{v} \) is not in \( \text{Nul}\ A \).

**Note** The table on page 204 of your textbook gives a nice list of contrasting properties of \( \text{Nul}\ A \) and \( \text{Col}\ A \).

**Definition** A linear transformation \( T \) from a vector space \( V \) into a vector space \( W \) is a rule that assigns to each vector \( \bar{x} \) in \( V \) a unique vector \( T(\bar{x}) \) in \( W \) such that

(i) \( T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v}) \) for all \( \bar{u}, \bar{v} \in V \), and

(ii) \( T(c\bar{u}) = cT(\bar{u}) \) for all \( \bar{u} \in V \).

Notice that this definition is exactly the same as what we saw before, except that we replace \( \mathbb{R}^n \to \mathbb{R}^m \) with \( V \to W \).

**Definition** The **kernel** of \( T \) is the set of all \( \bar{u} \in V \) such that \( T(\bar{u}) = 0 \). The **range** of \( T \) is the set of vectors \( \bar{y} \in W \) such that \( T(\bar{x}) = \bar{y} \) for some \( \bar{x} \in V \). Notice that if \( T \) can be written as a matrix transformation \( T(\bar{x}) = A\bar{x} \) (which is always the case if \( T: \mathbb{R}^n \to \mathbb{R}^m \), but not necessarily otherwise), then

\[
\text{kernel of } T = \text{null space of } A, \quad \text{and} \quad \text{range of } T = \text{column space of } A.
\]

**Note** The kernel of \( T \) is a subspace of \( V \), whereas the range of \( T \) is a subspace of \( W \).

**Example** Let \( V \) be the vector space of real-valued functions \( f \) defined on an interval \( [a, b] \) with derivatives \( f' \) that are also continuous on \( [a, b] \). Let \( W \) be the vector space of all real-valued continuous functions on \( [a, b] \), and let \( D: V \to W \) be the transformation that takes a function \( f \) to its derivative \( f' \), i.e. \( D(f) = f' \). By properties of derivatives, we have that

\[
D(f + g) = f' + g' = D(f) + D(g), \quad \text{and} \quad D(cf) = cf' = cD(f),
\]

so \( D \) is a linear map of vector spaces. The kernel of \( D \) consists of the constant functions (functions whose derivative is zero), and the range of \( D \) is all of \( W \). To see why the range of \( D \) is all of \( W \), take an arbitrary function \( h \in W \). Then \( h \) is a continuous function on \( [a, b] \), so we can define \( H(x) = \int_a^x h(t)dt \), which is real-valued and defined on \( [a, b] \). By the Fundamental Theorem of Calculus, \( D(H) = H'(x) = h(x) \), so \( D \) is onto.