Research Statement

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My research interests are in the field of functional analysis. In particular, much of my work over the past few years has involved the interaction between operator theory/algebras, irreversible dynamical systems, the representation theory of groupoids and the theory of wavelets. In the three papers [30], [31], [32] that resulted from my thesis I studied the structure of different operator algebras attached to a large class of fractals. I built different such algebras and showed how they capture the dynamics of the fractals. Currently I am extending this study to a larger class of fractals as well as to a general class of Markov operators. In the recently submitted paper [37] we describe the simplicity of C^*-algebras associated to Markov operators in terms of the probabilistic properties of these operators. A closely related project I am currently undertaking relates the theory of wavelets to natural representations of groupoids attached to local homeomorphisms. Some partial results and examples have appeared in [33]. In a different direction of research, I am studying the irreducible representations and primitive ideals of C^*-algebras arising from groupoids and groupoid dynamical systems. Some of my recent achievements in this area are the description of irreducible representations and the proof of the generalized Effros-Hahn conjecture for groupoid C^*-algebras [34], [35].

Recently, I have become interested also in the study of differential and pseudo-differential operators on fractals. As I will detail below, my work is part of a larger effort to understand elliptic and hypoelliptic operators on fractals. The results I proved with my coauthors in [36] represent an important advancement for this ongoing project.

In what follows, I describe my research accomplishments in different areas, I discuss some problems that I am currently pursuing, and I provide a brief description of directions for future research.

1. Fractals and C^*-algebras

1.1. **Disertation Results.** The results from [30], [31], and [32] have evolved from my dissertation. My thesis was completed at the University of Iowa under the direction of Professor Paul S. Muhly. It was awarded the 2006 D.C. Spriestersbach Dissertation Prize for Mathematics, Physical Sciences, and Engineering at the University of Iowa.

The common subject of the papers mentioned above is the structure of certain operator algebras (self-adjoint and non-self-adjoint) associated with Mauldin-Williams graphs and the dynamical systems they determine. My primary focus was the Pimsner construction of what are known now as Cuntz-Pimsner algebras (see [65] and [55]).

Let $G = (E^0, E^1, r, s)$ be a *finite* directed graph. A *Mauldin-Williams graph* associated to G consists of a collection $\{T_v\}_{v\in E^0}$ of compact metric spaces, one for each vertex of the graph, and a collection $\{\phi_e\}_{e\in E^1}$ of contractive maps, one for each edge of the graph ([53], [20], [30]). We associate with such a system a C^*-correspondence $\mathcal{X}$ over the C^*-algebra $A = C(T)$, where $T = \bigsqcup T_v$, which reflects the dynamics of the Mauldin-Williams graph (see [30, Definition 2.2])). My interests lie in the structure of operator algebras built from this correspondence.

The first main result of [30] states that if the underlying graph G has no sources and no sinks, that is, if the maps r and s are surjective, then the Cuntz-Pimsner algebra $\mathcal{O}(\mathcal{X})$ associated to the Mauldin-Williams graph $(G, \{T_v, \rho_v\}_{v\in E^0}, \{\phi_e\}_{e\in E^1})$ is isomorphic to the Cuntz-Krieger algebra associated with the graph G [12] (see [30, Theorem 2.3]). This is a generalization of the results of [66]. My proof is different from that in [66] and yields a second theorem.

It asserts, roughly, that if one wants to build a graph-directed system where the T_v are replaced by arbitrary unital C^*-algebras A_v and where the ϕ_e are replaced by homomorphisms that are contractive in the Rieffel metric [73, 74, 75], then the resulting Cuntz-Pimsner algebra *still* is isomorphic to $C^*(G)$. In fact, I proved that in such situations, using the hypothesis that the graph G has no sinks, the C^*-algebras A_v involved are necessarily commutative.

I showed, however, in [31] that the *tensor algebra* of $\mathcal{X}$, $\mathcal{T}_+(\mathcal{X})$, is "locally" a "complete conjugacy invariant". More precisely, I proved that if $\mathcal{X}_i$, $i = 1, 2$ are the C^*-correspondences coming from two Mauldin-Williams graphs defined over the same graph G, then the associated tensor algebras $\mathcal{T}_+(\mathcal{X}_i)$ are Morita equivalent in the sense of [7] if and only if that are completely isometrically isomorphic. This, in turn, happens if and only if there is a homeomorphism between the vertex spaces which implements a conjugacy between the appropriate edge maps. This result, thus, stands in a long series of results that were inspired by Arveson's discovery [2] of the relation between conjugacy invariants for measure preserving transformations and non-self-adjoint operator algebras.

In the third paper [32], Watatani and I associate a *slightly* different C^*-correspondence to a Mauldin-Williams graph, which yields a C^*-algebra that seems to respect the dynamics more clearly. This new C^*-correspondence X over $A = C(K)$ is based on the union of the so called *cographs* of the maps ϕ_e. This approach allows us to put more emphasis on the "branch points" of the maps. These are points $(x, y) \in K \times K$ such that $\phi_e(y) = \phi_f(y) = x$ for some $e \neq f$. Assuming that the underlying graph G of a Mauldin-Williams graph is irreducible and is not a cyclic permutation, and assuming that the invariant set K satisfies a technical condition called the open set condition, we proved that the Cuntz-Pimsner algebra $\mathcal{O}(X)$ associated with the C^*-correspondence X is simple and purely infinite. Since $\mathcal{O}(X)$ is also separable, nuclear, and satisfies the Universal Coefficient Theorem, the classification theorem of Kirchberg and Phillips [47, 64] implies that the isomorphism class of $\mathcal{O}(X)$ is completely determined by its K-theory with the K-theory class of the unit. Its K-theory is closely related to the failure of the injectivity of the coding by the shift on a Cantor set. In particular, Watatani and I compute the K-theory of this C^*-algebra and show that it can be quite different from the K-theory of $C^*(G)$.

1.2. **Graph directed Markov systems and C^*-algebras.** This project is a natural extension of my thesis. A graph directed Markov system (GDMS) is a generalization of a Mauldin-Williams graph in that it allows for an infinite, but countable, number of edges, while still requiring a finite number of vertices [52]. To each vertex v one attaches a compact metric space K_v and to each edge e one attaches a contraction $\phi_e : K_{r(e)} \to K_{s(e)}$. Some examples of such dynamical systems are the so called continued fractions and the Kleinian groups of Schottky type. The main difference compared with the classical Mauldin-Williams graphs is that the invariant set and the path space of a GDMS fail, in general, to be locally compact spaces. This failure leads to the main difficulty when one tries to associate a C^*-algebra to a GDMS. The idea that I propose to overcome this difficulty is to use the groupoid model of Paterson [63] for infinite graph C^*-algebras. This leads to building a topological quiver over the closure of the invariant set. Natural questions I will like to answer include: how does the structure of the C^*-algebra associated with a graph directed Markov system depend on the underlying infinite graph? How is the dynamics of the GDMS reflected in the properties of the C^*-algebra? Is the associated C^*-algebra simple? Can one compute it's K-theory by studying particular sets of points? Based on the work I have done so far, the C^*-algebra associated to a GDMS should be, in general, different from the C^*-algebra of the underlying graph. Moreover, I have reasons to believe that these C^*-algebras are simple. This belief is substantiated by the work described in the following project While I believe that, under suitable assumptions, these C^*-algebras are purely infinite, the proof seems to require more elaborate techniques compared with the classical Mauldin-Williams graphs.

In a different direction, I plan to study in collaboration with John Quigg and Steve Kaliszewski the KMS states induced from invariant measures of the GDMS on the C^*-algebra described above. This work will extend the impressive analysis of Pinzari, Watatani, and Yonetani [66].

1.3. **Markov operators and C^*-algebras.** In a recent paper which is joint with Paul S. Muhly and Victor Vega we began the study of Markov operators and C^*-algebras [37]. We say that an operator P on $C(X)$, where X is compact, is a Markov operator in case P is unital and positive. Using a Markov operator P, we built a topological quiver and a C^*-algebra $\mathcal{O}(P)$ ([57]) on $C(X)$ using the so called support of P. This Cuntz-Pimsner algebra, $\mathcal{O}(P)$ generalizes a number of C^*-algebras associated with automorphism, endomorphisms, transfer operators, and graphs ([83], [16]-[17], [22], [23] , [32]). Our first theorem provides a characterization of the simplicity of the C^*-algebra $\mathcal{O}(P)$ in terms of the probabilistic properties of P. Namely, we proved that $\mathcal{O}(P)$ is simple if and only if there are no closed *strongly absorbing* sets for P. The second theorem of [37] states that given a compact topological quiver E with no singular vertices, there is Markov operator P such that $\mathcal{O}(E)$ is isomorphic to $\mathcal{O}(P)$. There are many other problems which I plan to investigate. All of them involve the interactions between the probabilistic features of P and properties of $\mathcal{O}(P)$. One particularly intriguing problem is deciding when two P's give rise to isomorphic Cuntz-Pimsner algebras. When the P's are finite state Markov chains, then the two algebras are isomorphic if and only if the supports are same. Whether this is true more generally seems unlikely. However, [10, Proposition 6, p. 39] suggests that in general two P's with the same support give Morita equivalent Cuntz-Pimsner algebras.

1.4. **Tensor Algebras Associated to Fractals and their Perturbations.** As I pointed out in the description of my dissertation results, a natural C^*-correspondence associated with an iterated function system or, more generally, a Mauldin-Williams graph gives a C^*-algebra that ignores the dynamics of the system or graph. However, I proved in [31] that the tensor algebra does determine the dynamics in specified ways. One question to investigate is a perturbation question: If $\mathcal{X}_i$, is the C^*-correspondence coming from a Mauldin-Williams graph, $i = 1, 2$, and if the underlying graphs are the same, so that one may identify $\mathcal{O}(\mathcal{X}_1)$ and $\mathcal{O}(\mathcal{X}_2)$, under what circumstances is $\mathcal{T}_+(\mathcal{X}_1)$ close to $\mathcal{T}_+(\mathcal{X}_2)$ in the Hausdorff metric? I expect the answer to be in terms of some sort of "closeness" for the underlying dynamics. Another question I plan to pursue is whether the non-self adjoint algebras one can associate to the graph directed Markov systems I described above will provide a topological invariant for the GDMS in a similar way with [31]. Providing an answer for a GDMS should open a new set of problems and conjectures. For example, if the Toeplitz algebra is indeed a topological invariant of the system, for which Markov operators will the result still hold? Davidson and Katsoulis proved in [14] and [15] that the result would fail if one considers, in our language, a Markov operator built from a finite number of continuous maps which are *not* contractions.

2. Wavelets and Groupoids

2.1. **Groupoids methods in wavelet theory.** One project which I am actively pursuing with Paul S. Muhly is the use of groupoid methods in wavelet theory. An outline of our work together with partial results are published in [33]. We summarize the results below. The key idea is the use of the Deaconu-Renault groupoid [16], [69] and the theory of Exel [22] concerning irreversible dynamical systems to expand on the work of Bratteli, Jorgensen, Dutkay, et. al [9], [41], [19]. Their work, in turn, relates wavelet analysis, both for classical wavelets and for wavelets on fractals, to representations of the Cuntz algebra. Our approach shows how their Cuntz representations may be tied more closely to the underlying geometry of the situations they consider. A wavelet is a function ψ in $L^2(\mathbb{R})$ such that

$$\{U^j T^k \psi \ : \ j,k \in \mathbb{N}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$, where U is the operator of dilation by 2, and T is the operator of translation by 1.

In our approach we start with a local homeomorphism T on a compact, Hausdorff space X, and define the Deaconu-Renault groupoid to be

$$G = \{(x,n,y) \in X \times \mathbb{Z} \times X \ : \ T^k(x) = T^l(y), n = k - l\}, \tag{1}$$

endowed with a suitable topology such that G becomes an étale, locally compact groupoid. Thus G carries information about the entire pseudogroup generated by T. In investigations that Muhly and I have been making (described in part in [33]) it has become clear that harmonic analysis on fractals and the analysis of wavelets can profitably exploit the representation theory of G for suitable choices of X and T. Our analysis shows that there are structures that are intrinsic to the geometric setting of a space X with a local homeomorphism T. These include the groupoid G and its C^*-algebra, the pseudogroup $\mathfrak{G}$, and the Deaconu correspondence $\mathcal{X}$ [16, 17]. These are the source of isometries and the Cuntz relations - assuming $\mathcal{X}$ has an orthonormal basis. Each choice of orthonormal basis (which we call a *filter bank*) gives Cuntz isometries S_i in $C^*(G)$. Further, we may construct the minimal unitary extension of any of the S_i *essentially within* $C^*(G)$.

The "classic" wavelet analysis arises from our groupoid perspective via the following example. Let $X = \mathbb{T}$, $Tz = z^2$, $\mu =$ Lebesgue measure on $\mathbb{T}$. Let π be the representation of $C^*(G)$ given by $(\mu, \mathcal{H}, U)$, where $\mathcal{H} = X \times \mathbb{C}$ is the trivial line bundle on X,

$$U(\gamma) : \{s(\gamma)\} \times \mathbb{C} \to \{r(\gamma)\} \times \mathbb{C},\ U(\gamma)(s(\gamma), c) = (r(\gamma), c).$$

This representation induces the classical wavelets: $\pi(f)\xi(z) = f(z)\xi(z)$, $\pi(S_i)\xi(z) = m_i(z)\xi(z^2)$, where (m_1, m_2) is a filter bank associated with T. Then one can recapture the result due to Jorgensen and, more recently, Larsen and Raeburn that the inverse Fourier transform of $m_2(e^{\pi i x})\phi(2^{-1}x)$ is the wavelet associated with the filter bank (m_1, m_2).

3. Structure of groupoid dynamical systems C^*-algebras

3.1. **Induced representations and primitive ideals.** In this ongoing project, in which I am collaborating with Dana P. Williams, we are concerned with the generalization of the famous Effros-Hahn conjecture to groupoid, Fell bundles, and groupoid dynamical systems C^*-algebras. Key to this project is understanding the theory of representations of groupoid and groupoid cross-product C^*-algebras.

In two recent papers [34] and [35], Williams and I made significant progress on this project and we proved that a generalized Effros-Hahn conjecture is true for groupoid C^*-algebras. Let me begin with a review of the "classical" Effros-Hahn conjecture and a short description of our results. I will proceed, then, with a discussion of our future plans.

A dynamical system (A, G, α), where A is a C^*-algebra, G is a locally compact group and α is a strongly continuous homomorphism of G into $\mathrm{Aut}A$, is called *EH-regular* if every primitive ideal of the crossed product $A \rtimes_\alpha G$ is induced from a stability group ([98]). In their 1967 *Memoir* [21], Effros and Hahn conjectured that if (G, X) was a second countable locally compact transformation group with G amenable, then $\big(C_0(X), G, \mathrm{lt}\big)$ should be EH-regular. This conjecture, and its generalization to dynamical systems, was proved by Gootman and Rosenberg in [26] building on results due to Sauvageot [80, 79].

In [71], Renault gives the following version of the Gootman-Rosenberg-Sauvageot Theorem for groupoid dynamical systems. Let G be a locally compact groupoid and (A, G, α) a groupoid dynamical system. If R is a representation of the crossed product $A \rtimes_\alpha G$, then Renault forms the restriction, $\hat{L}$, of R to the isotropy groups of G and forms an induced representation $\operatorname{Ind} \hat{L}$ of $A \rtimes_\alpha G$ such that $\ker R \subset \ker(\operatorname{Ind} \hat{L})$. When G is suitably amenable, then the reverse conclusion holds. This is a powerful result and allows Renault to establish some very striking results concerning the ideal structure of crossed products.

In a recent paper [35], Williams and I provide a significant sharpening of Renault's result in the case of a groupoid C^*–algebra — that is, a dynamical system where G acts on the commutative algebra $C_0(G^{(0)})$ by translation. We showed that every primitive ideal K of $C^*(G)$ is induced from a stability group. That is, we show that $K = \operatorname{Ind}_{G(u)}^{G} J$ for a primitive ideal J of $C^*\big(G(u)\big)$, where $G(u)$ is the stability group at some $u \in G^{(0)}$. This not only provides a cleaner generalization of the Gootman-Rosenberg-Sauvageot result to the groupoid setting, but gives us a much better means to study the fine ideal structure of groupoids and the primitive ideal space (together with its topology) in particular. We took the opportunity to formalize the theory of inducing representations from a general closed subgroupoid in [34]. The main result of this paper is that the induced representation of an irreducible representation of a stability group is irreducible. This result is also one of the main pillars in our proof of the Effros-Hahn conjecture for groupoids. In the case of transformation group C^*-algebras, it is well known that representations induced from irreducible representations of the stability groups are themselves irreducible [97]. The corresponding result for groupoid C^*-algebras has been proved in an *ad hoc* manner in a number of special cases (see, e.g., [69, 71, 54, 58, 59].). Thus our analysis unifies and extends these results to groupoid C^*-algebras.

The next goal is to extend our analysis to Fell bundles and groupoid dynamical systems. We believe that the proof of the Effros-Hahn conjecture for Fell bundles should follow closely the steps we preformed in [35] and [34], modulus technicalities (see [38] and [39] for some recent results on this project). We expect, though, that the study of the Effros-Hahn conjecture for groupoid dynamical systems to be substantially harder. It seems unlikely that the results of [34] will go over to this level of generality. Thus the proof will require to retool the methods of Sauvageot [80, 79] and Gootman-Rosenberg [26] to work in the context of groupoid dynamical systems. Success here should lead to an important improvement upon [71] and give more information about the structure of the primitive ideal space and simplicity of groupoid crossed product C^*-algebras.

3.2. **Groupoids and Markov operators.** I plan to use the techniques developed in the previous project to study the problem of deciding for which Markov operators P there is a groupoid such that $\mathcal{O}(P)$ is isomorphic to the C^*-algebra of the groupoid. This problem was suggested to me by Jean Renault and it should fill an important gap in the literature of topological quivers [57] and C^*-algebras. Based on preliminary work, it seems that one needs some kind of "locally finiteness" assumption for the Markov operator P. An answer to this problem might shed some light on a more general problem: given a topological quiver $E = (E^0, E^1, r, s, \lambda)$, is there a groupoid G so that $\mathcal{O}(E)$ is isomorphic to $C^*(G)$? The answer is known to be true when r is a local homeomorphism, that is, when E is a topological graph in the sense of Katsura ([45]). An answer for general topological quiver has been searched by many people working in C^*-algebras. I believe that my study of Markov operators might help provide a (negative) answer to this open question.

4. Laplace operators on fractals

I became interested in problems related to analysis on fractals during the year I spent at Cornell University in 2007-2008 as a visiting assistant professor. I had the opportunity to interact with Strichartz, Kigami, Teplyaev, Rogers, and other people working in the field. I have also been involved in the research of students participating in the R.E.U. program in fractal analysis at Cornell, which helped me understand the many possibilities of research in this new and dynamic field. I became immediately involved in some interesting projects on the subject. I will describe next a project which I began during my year at Cornell University, on which the paper [36] joint with Strichartz, Rogers, and Ruan is a first important step. Then I will discuss a few other projects that are in various stages of completion. Some of them involve joint work with Robert S. Strichartz and Luke Rogers.

A theory of analysis on certain P.C.F. self-similar fractals has been developed around the Laplace operator Δ by Kigami [46]. Recall that an iterated function system (i.f.s.) is a collection $\{F_1, \dots, F_N\}$ of contractions on $\mathbb{R}^d$. For such an i.f.s. there exist a unique invariant set X and a unique invariant measure μ on X ([29]). Kigami built on the class of P.C.F. fractals a self-similar energy $\mathcal{E}$ using the graph approximations of the fractal. Then one can define the Laplacian on these self-similar sets using the weak formulation. That is, we say that $\Delta u = f$ if

$$\mathcal{E}(u, v) = -\int_X f v d\mu,$$

for all functions v vanishing on the boundary of X.

4.1. **Resolvents and Heat Kernels on Fractals.** The heat equation, the heat kernel and heat kernel estimates have been central notions in analysis on fractals. These notions have been studied primarily with probabalistic methods [4, 46, 24, 28]. In collaboration with Pearse, Ruan, Rogers and Strichartz, we were able to give an analytic formula for the resolvent of the Laplacian on a p.c.f. fractal that by-passed probabilistic methods [36]. That is, we constructed a symmetric function $G^{(\lambda)}(x, y)$ which weakly solves $(\lambda - \Delta)^{-1} G^{(\lambda)}(x, y) = \delta(x, y)$, meaning that

$$\int_X G^{(\lambda)}(x, y) u(y) d\mu(y) = (\lambda - \Delta)^{-1} u(x).$$

For $\lambda = 0$ our construction recovers the Green function for Δ. Our main theorem provides an explicit description of the resolvent kernel. We worked out in detail our construction for a series of examples, including the unit interval and the Sierpinski gasket. Heuristically, the resolvent kernel is the sum of the weak solutions of the resolvent problem in all cells of all order.

We plan to use this formula to recapture the known estimates of heat kernels on Sierpinski gaskets and to obtain estimates for a larger class of fractals. Some initial results in this direction have been announced in [76]. The proofs given so far for the estimates of heat kernels use probability theory. A direct proof of these estimates, using analytic methods, is highly desirable in order to have a self-contained theory.

It is also our hope that using resolvent estimates we could obtain useful information about spectral operators of the form

$$\xi(\Delta) = \int_\Gamma \xi(\lambda)(\lambda I - \Delta)^{-1} d\lambda,$$

in the same manner as used by Seeley [81, 82] for the Euclidian setting. I describe an outline of some incipient work on a particular class of such operators in the following project. I believe that the resolvent estimates will be useful also for the project on pseudo-differential operators on fractals that I will discuss later.

4.2. **Complex powers of the Laplacian on PCF fractals.** This project is, to my knowledge, the first systematic attempt to study the kernels of the Riesz and Bessel potentials on P.C.F. fractals. I aim to prove the L^p boundedness of these operators using techniques analogous to [84, Chapter 5].

For $A = I - \Delta$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha < 0$ we define

$$A^\alpha u = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{-t} e^{t\Delta} u dt,$$

where $\{e^{t\Delta}\}$ is the heat semigroup with generator Δ. I defined the kernels of these operators to be

$$K_\alpha(x, y) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty h_t(x, y) t^{-\alpha-1} e^{-t} dt,$$

where $h_t(x, y)$ is the heat kernel. Using the heat kernel estimates we prove estimates for these kernels which allow us to prove the L^p-boundedness of these operators. In particular one can develop a general theory of Sobolev spaces on fractals (see also [91]).

In the second part of this project I consider the case when α is an imaginary number. I proved that in this case the kernels satisfy the following pseudo-differential type estimates:

(2) $$|K_{i\alpha}(x, y)| \lesssim R(x, y)^{-d}$$

and

(3) $$|\Delta_y K_{i\alpha}(x, y)| \lesssim R(x, y)^{-2d-1},$$

where $R(x, y)$ is the *resistance metric* on the fractal X ([46],[89]). To make the connection with the singular integral operators theory, as described, for example, in [85], is a delicate task and requires the development of new techniques, since fractal analysis lacks many tools available in real analysis. Success in this project will provide the first example, to my knowledge, of singular integral operators on fractals built out of Kigami's construction. An affirmative answer will also be key in the projects I am going to describe next.

4.3. **Pseudo-differential operators on PCF fractals.** Spectral operators on fractals have been studied in a series of recent papers ([1, 5, 60, 88, 91]) mainly using numerical methods. These papers, as well as the project I described above, lead me to take on the task on formalizing a general theory of pseudo-differential operators on P.C.F. fractals. In this joint project with Robert S. Strichartz, we define our operators not on the fractals, but on the fractafolds built on P.C.F. fractals ([88, 87, 89, 95]). The spectrum Λ of the Laplacian is well understood in these situations ([88],[95]). For $m \in \mathbb{R}$ we define the class S^m of *constant coefficient symbols* to be the set of real valued smooth functions that satisfy

$$\left| \left(\lambda \frac{d}{d\lambda}\right)^k p(\lambda) \right| \lesssim (1 + \lambda)^{\frac{m}{d+1}}$$

for all $\lambda \geq 0$. As in the classical analysis (see, for example, [84, 85, 94]), the symbol class S^m defines a class of operators, ΨDO^m, which we call the class of *constant coefficient pseudo-differential* operators. We define such an operator via

$$p(-\Delta)u = \sum_{\lambda \in \Lambda} p(\lambda) P_\lambda u,$$

where P_λ are the spectral projections. These operators extend to bounded linear operators on $L^2(\mu)$. Using the heat kernel estimates on fractals (see [4, 24, 28, 92]) and the multiplier theorem of [96] we proved that they are bounded on $L^q(\mu)$, $1 < q < \infty$, as well. Even though we made good progress in understanding the constant coefficient pseudo-differential operators, we still have a long way to go in order to provide a complete analysis of them. For example, we conjecture that these operators are given by integration with respect to kernels that are smooth away from the diagonal and satisfy estimates of the form (2) and (3). We showed that this conjecture is true for the so called Laplace transform operators. Other important questions which are still open and we plan to pursue in the near future is the pseudo-local property of such operators and the study of elliptic and hypoelliptic operators.

We plan to extend, then, the analysis to a more general class of pseudo-differential operators. The idea is to allow the symbols to depend on the variable $x \in X_\infty$ as well. That is, for $m \in \mathbb{R}$ we define the *general symbol class* S^m to consist of real valued smooth functions defined on $[0, \infty) \times K_\infty$ that satisfy

$$\left| \left(\lambda \frac{\partial}{\partial \lambda} \right)^k \Delta_x^l p(\lambda, x) \right| \lesssim (1+\lambda)^{\frac{m}{d+1}}$$

for all λ and x. Thus we define the *general class of pseudo-differential operators* on fractals via

$$p(-\Delta, x)u(x) = \sum_{\lambda \in \Lambda} \int p(\lambda, x) P_\lambda(x, y) u(y) d\mu(y).$$

We conjecture that such operators extend to bounded operators on $L^2(\mu)$. We proved this conjecture when X_∞ is a compact fractafold. Moreover, we conjecture that these operators are singular integral operators and, thus, they are bounded on $L^q(\mu)$, for $1 < q < \infty$. A positive answer to our conjecture leads naturally to the study of pseudo-differential operators on products of fractals ([93, 8, 89]).

4.4. **Fractafolds.** Of particular interest to us is the possibility for using the Deaconu-Renault groupoid G (as described in Section 2 equation (1)) to understand better the structure of fractafolds. These were defined by R. Strichartz in [88] in an effort to understand objects that look locally like known fractals. The well-known relations among atlases, pseudogroups and groupoids, realized through Morita equivalence (see, e.g., [27, 48, 67, 72]), should lead to a useful way of looking at fractafolds. This is especially evident in the case of "fractals in the large", a special class of fractafolds that Strichartz introduced in [87]. In particular, harmonic analysis on G should illuminate and help extend the theory of periodic and almost periodic functions on the Sierpinski gasket recently developed by Strichartz [90]. The point is that under favorable circumstances G is the groupoid of germs of the pseudogroup $\mathfrak{G}$ of partial homeomorphisms defined by T. I believe that the pseudogroup of partial homeomorphisms of a fractafold which is locally like X will be Morita equivalent, in the sense described by Renault in [72, Section 3], to $\mathfrak{G}$. This sense is based on work of Kumjian [48] and Haefliger [27]. At this stage, however, there still is a lot of work to be done to substantiate this belief.

In a series of papers ([61], [50]), Nistor and his collaborators defined and studied pseudo-differential operators on differential groupoids attached to manifolds. I believe that their methods can be adapted to the groupoids I plan to attach to a fractafold. This will allow us to provide a different approach than the one described in the previous project to the study of pseudo-differential operators on P.C.F. fractals. The results and techniques we developed in [34] and [35] should provide us invaluable help in pursuing this approach.

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