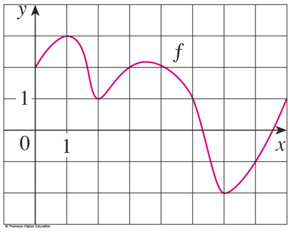
1. For each part, if the statement is always true, circle the printed capital **T**. If the statement is sometimes false, circle the printed capital **F**. For each T/F question, write a careful and clear justification or describe a counterexample.

(3) (a) If $F(x) = \int_x^1 f(t)\,dt$ (see the graph of $f$) then $F(x)$ is concave down for all $x$ between 1 and 2. **T** **F**

[Hint: If the graph of $g$ lies below all of its tangents on an interval $I$, then it is called concave downward on $I$.] **Justification:**

$F'(x) = -f(x)$ and $F''(x) = -f'(x) > 0$ in the interval $1 \le x \le 2$ from the graph(since $f(x)$ is decreasing, so $f'(x) < 0$), we get $F(x)$ is concave up for $1 \le x \le 2$. So it is False.



(3) (b) If $F(x) = \int_1^x f(t)\,dt$ and $f(a) = 0$, then $a$ is a critical point of $F$. **T** **F**

[Hint: A critical number of a function $g$ is a number $c$ in the domain of $g$ such that either $g'(c) = 0$ or $g'(c)$ does not exist.] **Justification:**

$F'(x) = f(x)$. We have $F'(a) = f(a) = 0$. So $a$ is a critical point of $F(x)$. It is True.

(3) (c) If $F(x) = \int_1^x f(t)\,dt$, $f'(a) = 0$, and $f''(a) > 0$, then $a$ is an inflection point of $F$. **T** **F**

[Hint: A point $P$ on a curve $y = g(x)$ is called an inflection point if $g$ is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at $P$.] **Justification:**

$F'(x) = f(x)$, $F''(x) = f'(x)$ and $F'''(x) = f''(x)$. By condition we get $F''(a) = 0$ and $F'''(a) = f''(a) > 0$. $F''(x)$ is an increase function for $x$ near $a$, we can get $F''(x) < 0$ for $x < a$(near $a$) and $F''(x) > 0$ for $x > a$ and near $a$. The function $F(x)$ changes from concave down to concave up when $x$ passes $a$. So $x = a$ is an inflection point. It is True.

(3) (d) If $F(x) = \int_0^x e^t\,dt$, then $\int F(x)\,dx = e^x - x + C$. **T** **F**

**Justification:**

The first integral can be easily evaluated. $F(x) = e^t|_0^x = e^x - 1$. So $\int F(x)dx = \int(e^x - 1)dx = e^x - x + C$. It is True.

(3) (e) The integral $\int_{-2}^{2} \pi(4-x^2)\,dx$ represents the volume of a sphere of radius 2. **T** **F**

**Justification:**

The sphere can be viewed as rotating the curve $y=\sqrt{4-x^2}$ with respect to $x$-axis. The cross-section is a disk which is perpendicular to $x$-axis. So $A(x)=\pi r^2=\pi y^2=\pi(\sqrt{4-x^2})^2=\pi(4-x^2)$. So the volume of the ball can be expressed as

$$\int_{-2}^{2} \pi(4-x^2)dx$$

So it is True.

(3) (f) The average value of $\sin(x)$ on $[0,\pi]$ is greater than or equal to 1/2. **T** **F**

**Justification:**

Since $\int_0^\pi \sin(x)dx = -\cos(x)|_0^\pi = 2$. By definition of average value. the average of $\sin x$ on $[0,\pi]$ is $\frac{2}{\pi} > 1/2$. So It is True.

2. Evaluate the following definite or indefinite integrals:

(4) (a) $\int_0^1 (\sqrt{x}+1)^2\,dx = \int_0^1 (x+2\sqrt{x}+1)dx = \frac{x^2}{2}+\frac{4}{3}x^{\frac{3}{2}}+x|_0^1$

So the answer is $\frac{1}{2}+\frac{4}{3}+1$.

(4) (b) $\int \sqrt{4x-3}\,dx$

Let $u=4x-3$, then $du=4dx$ with $dx=du/4$, plug in

$$\int \sqrt{4x-3}dx = \int \sqrt{u}du/4 = \frac{1}{6}u^{\frac{3}{2}}+C = \frac{1}{6}(4x-3)^{\frac{3}{2}}+C$$

(4) (c) $\int x(x^2+3)^4\,dx$

Let $u = x^2+3$, then $du = 2xdx$. or $xdx = du/2$. Plug in we can get

$$\int x(x^2+3)^4 dx = \int u^4 du/2 = \tfrac{1}{10}u^5 + C$$

$$= \tfrac{1}{10}(x^2+3)^5 + C$$

(4) (d) $\int \frac{t^2-2t+2}{1+t}\,dt$

Let $u = t+1$, $t = u-1$ and $du = dt$. We get

$$\int \tfrac{t^2-2t+2}{1+t}dt = \int \tfrac{(u-1)^2-2(u-1)+2}{u}du = \int \tfrac{u^2-4u+5}{u}du$$

$$= \int (u-4+5u^{-1})du = \tfrac{u^2}{2} - 4u + 5\ln|u| + C$$

$$= \tfrac{(t+1)^2}{2} - 4(t+1) + 5\ln|t+1| + C$$

3. Find the derivatives of the following functions:

(4) (a) $f(x) = \int_0^{4x+1} \sqrt{1+t^4}\,dt$

Assume the antiderivative of $\sqrt{1+t^4}$ is $H(t)$, i. e., $H'(t) = \sqrt{1+t^4}$. By Fundamental theorem of Calculus part 2.

$$f(x) = H(t)|_0^{4x+1} = H(4x+1) - H(0)$$

By chain rule, the derivative of $f(x)$ is

$$f'(x) = H'(4x+1)\cdot 4 = 4\sqrt{1+(4x+1)^4}$$

(8) (b) $g(x) = \int_{\sqrt{x}}^{x^2} e^{t^2}\,dt$ for $x > 0$.

Let $H'(t) = e^{t^2}$. By Fundamental theorem of Calculus part 2.

$$g(x) = H(t)|_{\sqrt{x}}^{x^2} = H(x^2) - H(\sqrt{x})$$

By chain rule

$$g'(x) = H'(x^2)\cdot 2x - H'(\sqrt{x})\cdot \tfrac{1}{2\sqrt{x}}$$

$$= 2xe^{x^4} - \tfrac{1}{2\sqrt{x}}e^x$$

4. A particle is moving along a line with velocity function $v(t) = t^2 - 4t - 5$, where $v(t)$ is measured in meters per second.

(5) (a) Find the displacement of the particle during the time interval $[2, 6]$.
By net change theorem. The displacement is

$$\int_2^6 v(t)dt = \int_2^6 (t^2 - 4t - 5)dt = \frac{t^3}{3} - 2t^2 - 5t|_2^6 = -\frac{44}{3}$$

Negative means the particle is at the left side of the initial position.

(6) (b) Find the distance traveled by the particle during the time interval $[2, 6]$.
The distance traveled is equal to

$$\int_2^6 |v(t)|dt = \int_2^6 |t^2 - 4t - 5|dt$$

We can not evaluate it directly, We need to split the integral according to the sign of $v(t)$. Set $t^2 - 4t - 5 = (t - 5)(t + 1) = 0$, we solves that $v(5) = 0$ and when $2 \leq t \leq 5$, $v(t) \leq 0$ and when $5 \leq t \leq 6$, $v(t) \geq 0$. So

$$\begin{aligned} \textstyle\int_2^6 |t^2 - 4t - 5|dt &= \textstyle\int_2^5 -(t^2 - 4t - 5)dt + \int_5^6 (t^2 - 4t - 5)dt \\ &= -(\tfrac{t^3}{3} - 2t^2 - 5t)|_2^5 + (\tfrac{t^3}{3} - 2t^2 - 5t)|_5^6 \\ &= \tfrac{124}{3} \end{aligned}$$

(3) (c) Can we fully determine the position function of particle at any time $t$ (using the given information about velocity function $v$)?
Position function $P(t)$ is an antiderivative of $v(t)$. So

$$P(t) = \frac{t^3}{3} - 2t^2 - 5t + C$$

The constant $C$ is arbitrary and can only be determined by the initial position of the particle. As no initial position is given for the particle. So we can not fully determine the position function of the particle.

(12) 5. Consider two curves $f(x) = x^3 - x$ and $g(x) = 3x$. Find the area between these two curves in the interval $-2 \leq x \leq 2$.

We first find the intersection point for the two curves, which gives $x^3 - x = 3x$ so $x = 0$ or $x = \pm 2$. By symmetric of the picture, we just need to get the area in the first quadrant where $y = 3x$ is upper curve and $y = x^3 - x$ is lower curve. The area can be expressed as an integral in $x$.

$$A = 2\int_0^2 (3x - x^3 + x)dx = 2(2x^2 - \frac{x^4}{4})|_0^2 = 2(8-4) = 8$$

(6) 6. The base of a solid $\mathcal{S}$ is the triangle bounded by the lines $y = 3x$, $3x + y = 6$, and $x = 2$. Parallel cross-sections perpendicular to the $x$-axis are squares. Write down the exact definite integrals that must be evaluated to obtain the volume of the solid $\mathcal{S}$. **Do not evaluate the integrals**.

The triangle has vertices $(1, 3)$, $(2, 6)$ and $(2, 0)$, since the cross-section is perpendicular to $x$-axis, the volume can be expressed as an integral of $x$, The cross-section perpendicular to $x$ will have a line segment on the triangle with length $a = 3x - (6 - 3x) = 6x - 6$ (since $y = 3x$ serves as upper curve and $y = 6 - 3x$ serves as lower curve). This length $a$ is the edge length of the square as the cross-section. so

$$A(x) = a^2 = (6x - 6)^2$$

We get

$$V = \int_1^2 (6x - 6)^2 dx$$

7. Consider the region $\mathcal{R}$ bounded by the graphs of $y = x$, $y = x^2$, $x = 1$ and $x = 3$.

(4) (a) Write down the exact definite integrals that must be evaluated to obtain the area of the region $\mathcal{R}$. **Do not evaluate the integrals**.

Important, you need to draw the graph and you can notice that $x = 1$ gives the intersection point of the two curves. Since $y = x^2$ is upper curve and $y = x$ is lower curve for $1 \le x \le 3$. The area can be given as

$$A = \int_1^3 (x^2 - x)dx$$

(6) (b) Form a solid by rotating the region $\mathcal{R}$ about the line $y = 0$. Write down the exact definite integrals that must be evaluated to obtain the volume of this solid. **Do not evaluate the integrals**.

When we rotate the region above with respect to $y = 0$, the cross-section of the solid is perpendicular to $x$-axis. So the volume of the solid will be expressed as an integral in $x$. As the cross-section is an annulus or (washer). We need to find the inner radius $r$ and outer radius $R$. From the graph from the first part. It is not hard to see $r$ is determined by lower curve and $R$ is determined by upper curve. For $1 \le x \le 3$

$$r = x \quad R = x^2$$

So $A(x) = \pi(R^2 - r^2) = \pi(x^4 - x^2)$

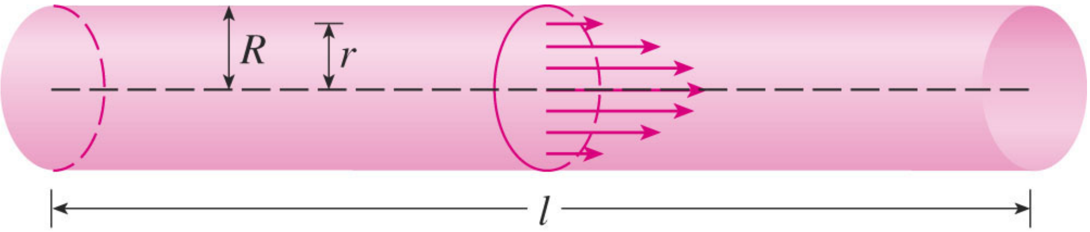
$$V = \int_1^3 A(x)dx = \int_1^3 \pi(x^4 - x^2)dx$$

(c) (BONUS) Form a solid by rotating the region $\mathcal{R}$ about the line $x = 0$. Write down the exact definite integrals that must be evaluated to obtain the volume of this solid. **Do not evaluate the integrals**. (8pts)

This time the cross-section is perpendicular to $y$-axis. So the volume of the solid is expressed by an integral in $y$ by cross-section method. The cross-section is still a washer, however the inner and outer radius is in different formula from part 2. The inner radius is given by the curve $y = x^2$, which gives $r = x = \sqrt{y}$. However the outer radius is given by different formula depending on the value of $y$. When $x = 3$, $y = 3$. We can find that when $3 \le y \le 9$, the outer radius is given by $R = 3$ and when $1 \le y \le 3$ the outer radius is determined by curve $y = x$. So $R = x = y$. We need to split the integral according to the different formula of $R$. Finally we have

$$V = \int_1^9 A(y)dy = \int_1^3 \pi(y^2 - \sqrt{y}^2)dy + \int_3^9 \pi(3^2 - \sqrt{y}^2)dy$$

8. When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius $R$ and length $l$ as illustrated in Figure 1. Because of friction at the walls of the tube, the velocity $\upsilon$ of the blood is greatest along the central axis of the tube and decreases as the distance $r$ from the axis increases until $\upsilon$ becomes 0 at the wall. The relationship between $\upsilon$ and $r$ is given by the **law of laminar flow** discovered by the French physician Jean-Louis-Marie Poiseuille in 1840. This law states that $\upsilon = (P/(4\eta l))(R^2 - r^2)$, where $\eta$ is the viscosity of the blood and $P$ is the pressure difference between the ends of the tube. If $P$ and $l$ are constant, then $\upsilon$ is a function of $r$ with domain $[0, R]$. Say, $\upsilon(r) = C(R^2 - r^2)$, where $C = P/(4\eta l)$.



© Thomson Higher Education

Figure 1: Blood flow in an artery

(6) (a) Find the average velocity (with respect to $r$) over the interval $0 \le r \le R$ ($\upsilon_{\text{ave}}$).
By definition

$$\upsilon_{\text{ave}} = \frac{1}{R-0}\int_0^2 C(R^2 - r^2)dr = \frac{1}{R}C(R^2 r - r^3/3)|_0^R = \frac{2C}{3}R^2$$

(3) (b) Compare the average velocity $\upsilon_{\text{ave}}$ with the maximum velocity $\upsilon_{\text{max}}$.
When $r = 0$, the velocity get max value $\upsilon_{\text{max}} = CR^2$. So

$$\frac{\upsilon_{\text{ave}}}{\upsilon_{\text{max}}} = \frac{2C}{3}R^2 \cdot \frac{1}{CR^2} = \frac{2}{3}$$

(3) (c) Find value $r$ in the interval $[0, R]$ which satisfies $\upsilon(r) = \upsilon_{\text{ave}}$.
Let $\upsilon(r) = \frac{2C}{3}R^2$, we get

$$R^2 - r^2 = \frac{2}{3}R^2$$

Which gives $r^2 = \frac{1}{3}R^2$, take square root on both side and notice $r \ge 0$, we get

$$r = \sqrt{\frac{1}{3}}R$$