

1. Evaluate the following definite or indefinite integrals

(a) $\int_0^1 (2 + x\sqrt{x})dx = (2x + \frac{2}{5}x^{\frac{5}{2}})|_0^1 = 2 + \frac{2}{5}$

(b) $\int \frac{3x}{x^2-4}dx$

use substitution method, let $u = x^2 - 4$, then $du = 2xdx$, plug in we get

$$\int \frac{3x}{x^2-4}dx = \int \frac{3}{2u}du = \frac{3}{2} \log |u| + C = \frac{3}{2} \log |x^2 - 4| + C$$

(c) $\int_{-10}^{10} \sin^3 x dx = 0$ by the fact that $\sin^3 x$ is an odd function and the interval $[-10, 10]$ is symmetric with respect to origin.

(d) $\int \frac{t}{t^4+1}dt$

for this problem we choose $u = t^2$ (this is the only substitution that works) and $du = 2tdt$, plug in we have

$$\int \frac{t}{t^4+1}dt = \int \frac{1}{2(u^2+1)}du = \frac{1}{2} \arctan u + C = \frac{1}{2} \arctan(x^2) + C$$

(e) $\int \frac{\cos \sqrt{x}}{\sqrt{x}}dx$

choose $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$, plug in , we get

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}}dx = \int 2 \cos u du = 2 \sin u + C = 2 \sin \sqrt{x} + C$$

(f) $\int_1^9 \frac{3x-1}{\sqrt{x}}dx = \int_1^9 3\sqrt{x} - x^{-\frac{1}{2}}dx = (2x^{3/2} - 2x^{1/2})|_0^9 = 2 \times 27 - 6 = 48$

(g) $\int \frac{1+3x}{x^2+1}dx = \int \frac{1}{x^2+1}dx + \int \frac{3x}{x^2+1}dx$

you have to split the integral, for the 2nd integral , let $u = x^2 + 1$, then $du = 2xdx$,
So

$$\int \frac{3x}{x^2+1}dx = \int \frac{3}{2u}du = \frac{3}{2} \log |u| + C = \frac{3}{2} \log(x^2 + 1) + C$$

So

$$\int \frac{1+3x}{x^2+1}dx = \arctan x + \frac{3}{2} \log(x^2 + 1) + C$$

(h) $\int_0^2 x\sqrt{4-x^2}dx$

let $u = 4 - x^2$, $du = -2xdx$, when $x = 0$, $u = 4$ and when $x = 2$, $u = 0$, so

$$\int_0^2 x\sqrt{4-x^2}dx = \int_4^0 \frac{-\sqrt{u}}{2}du = \int_0^4 \frac{\sqrt{u}}{2}du = \frac{1}{3}u^{3/2}|_0^4 = 8/3$$

Note, in the above, we directly change the limit to the limits for u .

2. A particle is moving along a line with acceleration $a(t) = 4 - 2t$ with initial velocity $v(0) = 12$.

- (a) When will the particle move to left or right?
 when $v(t) > 0$, the particle moves to right and when $v(t) < 0$, the particle moves to left. since

$$v(t) = \int a(t)dt = \int (4 - 2t)dt = 4t - t^2 + C$$

by the initial $v(0) = 12$, we get $v(0) = C = 12$. So $v(t) = 4t - t^2 + 12 = -(t-6)(t+2)$.
 When $t = 6$, the particle stops and use test point for t we can get the particle moves to right when $0 \leq t < 6$ and moves to left when $t > 6$.

- (b) Find the displacement of the particle during the given time interval $0 \leq t \leq 10$.
 the displacement equals

$$\int_0^{10} v(t)dt = \int_0^{10} (4t - t^2 + 12)dt = (2t^2 + 12t - t^3/3)|_0^{10} = -\frac{40}{3}$$

negative means the particle is on the left side of the initial position and with a distance of $40/3$.

- (c) Find the distance traveled by the particle during the time interval $0 \leq t \leq 10$.
 distance traveled equals

$$\int_0^{10} |v(t)|dt = \int_0^6 v(t)dt + \int_6^{10} -v(t)dt$$

in the above we can not directly evaluate the integral due to the absolute value sign, we need to split the integral by the result of 1st part as above. Now plug in $v(t)$

$$\begin{aligned} \int_0^{10} |v(t)|dt &= \int_0^6 (4t - t^2 + 12)dt - \int_6^{10} (4t - t^2 + 12)dt \\ &= (2t^2 + 12t - t^3/3)|_0^6 - (2t^2 + 12t - t^3/3)|_6^{10} \\ &= 72 + 40/3 \end{aligned}$$

- (d) At what time will the particle return to the initial place?
 the particle returns to the initial place means we need to find the time when the particle has zero displacement. Assume the time value is b , we need to solve

$$\int_0^b v(t)dt = 0 = (2t^2 - t^3/3 + 12t)|_0^b = b(2b - b^2/3 + 12) = 0$$

Which gives $b = 0$ (this is certainly true as at the initial time, the particle is at initial position, this time is not what we need) or $b^2/3 - 2b - 12 = 0$. use the quadratic formula, we can solve for b from the 2nd equation. $b = 3 \pm 3\sqrt{5}$. Since $b > 0$, throw away the negative solution, we finally get $b = 3 + 3\sqrt{5}$. This means when $t = 3 + 3\sqrt{5}$, the particle returns to the initial position.

3. Find the derivatives of the following functions (Do not expect that you can find explicit formula of the function)

(a) $f(x) = \int_1^{e^x} \sqrt{t+t^3} dt$

For all those problems, we need to set up the antiderivative function, then use fundamental theorem of calculus part 2 and then differentiate using chain rule.

Let $F'(t) = \sqrt{t+t^3}$ (this is as same as to say $F(t)$ is the antiderivative of $\sqrt{t+t^3}$, use fundamental theorem of calculus part 2.

$$f(x) = F(t)|_1^{e^x} = F(e^x) - F(1)$$

Now differentiate in x , by chain rule we get

$$f'(x) = F'(e^x) \times e^x = e^x \sqrt{e^x + e^{3x}}$$

(b) $g(x) = \int_{2x}^{3x+2} \sin(t^4) dt$

Let $G'(t) = \sin(t^4)$, then

$$g(x) = \int_{2x}^{3x+2} \sin(t^4) dt = G(t)|_{2x}^{3x+2} = G(3x+2) - G(2x)$$

Differentiate $g(x)$ by chain rule, we get

$$g'(x) = G'(3x+2) \cdot 3 - G'(2x) \cdot 2 = 3 \sin((3x+2)^4) - 2 \sin((2x)^4)$$

(c) $h(x) = \int_0^{x^4} \sqrt{t^2+1} dt$

Let $H'(t) = \sqrt{t^2+1}$, then

$$h(x) = \int_0^{x^4} \sqrt{t^2+1} dt = H(t)|_0^{x^4} = H(x^4) - H(1)$$

So

$$h'(x) = H'(x^4) \cdot 4x^3 = 4x^3 \sqrt{x^8+1}$$

4. Consider the region enclosed by $x = 0$, $y = x$ and $y = \sqrt{16-x^2}$. Find the volume of the solid by rotating the region with respect to x -axis

Due to the reason that i need to do a lot of latex work if I want to put a picture here. So all the problems related to area or volumes, there will be no pictures. However, you need to draw the graph when you do those kind of problems. Since the region is rotated with respect to x -axis, so the cross-section is perpendicular to x -axis. So the volume should be set up as

$$V = \int_a^b A(x) dx$$

where $a \leq x \leq b$ is the interval for x where you can draw those kind of cross-sections perpendicular to x -axis. To find a and b , you need to find the intersection point of $y = x$ and $y = \sqrt{16-x^2}$, from which we get

$$x = \sqrt{16-x^2}$$

we get $x^2 = 16 - x^2$, this gives $x = 2\sqrt{2}$. So $b = 2\sqrt{2}$, since the region is in the 1st quadrant, $a = 0$. Now at x , the cross-section is an annulus with the larger radius determined by upper curve $y = \sqrt{16 - x^2}$ so $R = y - 0 = \sqrt{16 - x^2}$ (since the rotation is with respect to x -axis), the smaller radius is determined by $y = x$ which gives $r = y - 0 = x$. So at x

$$A(x) = \pi R^2 - \pi r^2 = \pi(16 - x^2) - \pi x^2 = \pi(16 - 2x^2)$$

So

$$V = \int_0^{2\sqrt{2}} \pi(16 - 2x^2) dx = \pi(16x - \frac{2}{3}x^3) \Big|_0^{2\sqrt{2}} = \frac{64\sqrt{2}}{3}\pi$$

5. Consider the solid with base (on xy -plane) as a triangle with vertices $(0, 0)$, $(5, 0)$ and $(2, 3)$. Assume that the parallel cross-section, which is perpendicular to the base and is parallel to x -axis is of the following shape, find the volume of the solid in each cases. since the cross-sections are perpendicular to y -axis. So the volume should be in the form of

$$V = \int_a^b A(y) dy = \int_0^3 A(y) dy$$

where a and b can be determined by the triangle with $a = 0$ and $b = 3$. From the given vertices, the side line equations are $y = 0$, $y = \frac{3}{2}x$ this one is the left curve; and $y = -(x - 5)$ this gives right curve. Solve for x from the left curve and right curve, we get $x = 5 - y$ (right curve) and $x = \frac{2}{3}y$ (left curve). Now for fixed y , if we draw the cross-section parallel to x -axis, it will give a horizontal line segment on the triangle. This line segment has length $a = (5 - y) - \frac{2}{3}y = 5 - \frac{5}{3}y$

- (a) the cross-sections are squares.

at y the value a serves as the side of the square cross-section. So in this case $A(y) = a^2 = (5 - \frac{5}{3}y)^2$, so

$$V = \int_0^3 (5 - \frac{5}{3}y)^2 dy = \int_0^3 25(1 - \frac{2}{3}y + \frac{1}{9}y^2) dy$$

I didn't finish the definite integral, you should know how to evaluate.

- (b) the cross-sections are equilateral triangles.

at y , a serves as the side of equilateral triangle. So $A(y) = \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}(5 - \frac{5}{3}y)^2$, then

$$V = \int_0^3 \frac{\sqrt{3}}{4}(5 - \frac{5}{3}y)^2 dy = \dots$$

- (c) the cross-sections are isosceles right triangles with hypotenuse in the base.

in this case a serves as hypotenuse. Since the other angles of the triangle cross-section is $\frac{\pi}{4}$. It is not hard to get $A(y) = \frac{1}{2}(\frac{\sqrt{2}}{2}a)^2 = \frac{1}{4}(5 - \frac{5}{3}y)^2$, hence

$$V = \int_0^3 \frac{1}{4}(5 - \frac{5}{3}y)^2 dy = \dots$$

6. Find the area of the region enclosed by

(a) $y = 2 - x^2$ and $y = |x|$.

Draw the picture, you will see it is type 1 region. So the integral should be in x . $y = 2 - x^2$ serves as upper curve and $y = |x|$ serves as lower curve. Find the intersection point of those two curves, we have by cancel x ,

$$y = 2 - y^2$$

so $y^2 + y - 2 = (y - 1)(y + 2) = 0$, we have $y = 1$ or $y = -2$. then find x value of the intersection point, when $y = 1$, $x = \pm 1$, when $y = -2$, impossible as $y = |x| \geq 0$, we get $-1 \leq x \leq 1$. using the difference of upper curve and lower curve. The area

$$A = \int_{-1}^1 (2 - x^2 - |x|)dx = 2 \int_0^1 (2 - x^2 - x)dx = 2(2x - \frac{1}{3}x^3 - \frac{1}{2}x^2)|_0^1 = 7/3$$

in the above, we use $2 - x^2 - |x|$ is an even function and $[-1, 1]$ is symmetric about origin. We also use $|x| = x$ when $x \geq 0$.

(b) $x = 1 + y^2$ and $y = x - 3$

This is a type 2 region. The integral should be in y . $x = 1 + y^2$ is the left curve and $x = y + 3$ is the right curve. The intersect of those two curves are given by

$$x = 1 + y^2 = y + 3$$

Solve the quadratic equation, we can get $y = 2$ or $y = -1$, which gives the upper/lower limit of y for the integral. The area for the bounded region is

$$\int_{-1}^2 (y + 3) - (1 + y^2)dy = (2y + \frac{1}{2}y^2 - \frac{1}{3}y^3)|_{-1}^2 = 9/2$$

7. Find the average value of the function $f(x) = (x - 3)^2$ in the interval $[2, 5]$. Then find value c in the interval $[2, 5]$ which satisfies $f(c) = f_{ave}$.

By definition of the average value of a function in a given interval,

$$f_{ave} = \frac{1}{5 - 2} \int_2^5 (x - 3)^2 dx = \frac{1}{3} \int_2^5 (x - 3)^2 dx$$

There are two ways to evaluate the definite integral above. One is by expansion $(x - 3)^2 = x^2 - 6x + 9$, then you can use power rule to find antiderivative; The other way is by substitution method, let $u = x - 3$ then $du = dx$. No matter in which way, you can get

$$\int_2^5 (x - 3)^2 dx = \frac{1}{3}(x - 3)^3|_2^5 = 3$$

So $f_{ave} = \frac{1}{3} \cdot 3 = 1$. Now let $f(x) = 1$ we solve for x , which gives

$$(x - 3)^2 = 1$$

We have two solutions $x = 4$ or $x = 2$ and all these solutions are in the given interval $[2, 5]$. So $c = 2$ or $c = 4$.