

Instructor: J. Haga

1. For the following functions, algebraically determine the points of discontinuity of  $f$ . Justify your answer.

$$(a) f(x) = \begin{cases} x + 10 & \text{if } x \leq -2 \\ x^2 + 4 & \text{if } -2 < x < 6 \\ 3x + 2 & \text{if } 6 \leq x \end{cases}$$

**Solution:** Remember that  $f(x)$  is continuous at  $x = a$  if (and only if)

$$\lim_{x \rightarrow a} f(x) = f(a).$$

If  $\lim_{x \rightarrow a} f(x)$  doesn't exist, or if  $\lim_{x \rightarrow a} f(x)$  exists but it something other than  $f(a)$  then  $f(x)$  is not continuous at  $x = a$ .

On the intervals  $(-\infty, -2)$ ,  $(-2, 6)$  and  $(6, \infty)$   $f(x)$  is a polynomial. Since polynomials are continuous everywhere,  $f$  is continuous on those intervals. We have that

$$\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} x + 10 = -2 + 10 = 8$$

and

$$\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^+} x^2 + 4 = \lim_{x \rightarrow -2^+} (-2)^2 + 4 = 4 + 4 = 8.$$

Thus

$$\lim_{x \rightarrow -2} f(x) = \lim_{x \rightarrow -2^+} f(x) = 8$$

and so

$$\lim_{x \rightarrow -2} f(x) = 8.$$

We have that  $f(-2) = -2 + 10 = 8$  and so  $\lim_{x \rightarrow -2} f(x) = f(-2)$ , giving us that  $f$  is continuous at  $x = -2$ . We have that

$$\lim_{x \rightarrow 6^-} f(x) = \lim_{x \rightarrow 6^-} x^2 + 4 = (6)^2 + 4 = 36 + 4 = 40$$

and

$$\lim_{x \rightarrow 6^+} f(x) = \lim_{x \rightarrow 6^+} 3x + 2 = 3(6) + 2 = 18 + 2 = 20.$$

Thus  $\lim_{x \rightarrow 6^-} f(x) \neq \lim_{x \rightarrow 6^+} f(x)$  and so  $\lim_{x \rightarrow 6} f(x)$  does not exist. Thus  $f(x)$  is not continuous at  $x = 6$ . In conclusion, the only point of discontinuity of  $f(x)$  is  $x = 6$ .

$$(b) f(x) = \begin{cases} |x| & \text{if } x \leq 7 \\ x^2 + 2x + 1 & \text{if } 7 \leq x \leq 10 \\ 3x^2 - 8x & \text{if } 10 < x \end{cases}$$

**Solution:** We have that

$$|x| = \begin{cases} -x & x \leq 0 \\ x & x > 0 \end{cases}$$

and so  $f(x)$  can be rewritten:

$$f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 7 \\ x^2 + 2x + 1 & \text{if } 7 < x \leq 10 \\ 3x^2 - 8x & \text{if } 10 < x \end{cases}$$

We have the following limits:

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} -x = -0 = 0$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x = 0$$

$$\lim_{x \rightarrow 7^-} f(x) = \lim_{x \rightarrow 7^-} x = 7$$

$$\lim_{x \rightarrow 7^+} f(x) = \lim_{x \rightarrow 7^+} x^2 + 2x + 1 = (7)^2 + 2(7) + 1 = 49 + 14 + 1 = 64$$

$$\lim_{x \rightarrow 10^-} f(x) = \lim_{x \rightarrow 10^-} x^2 + 2x + 1 = (10)^2 + 2(10) + 1 = 100 + 20 + 1 = 121$$

$$\lim_{x \rightarrow 10^+} f(x) = \lim_{x \rightarrow 10^+} 3x^2 - 8x = 3(10)^2 - 8(10) = 300 - 80 = 220$$

and so  $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$ , but neither  $\lim_{x \rightarrow 7} f(x)$  nor  $\lim_{x \rightarrow 10} f(x)$  exist. Thus  $f$  is continuous everywhere except  $x = 7$  and  $x = 10$ .

2. Find the value of  $k$  so that  $f(x)$  is continuous, where

$$f(x) = \begin{cases} 3x^2 + 2x + 1 & \text{if } x \leq 1 \\ kx + 3 & \text{if } x > 1 \end{cases}$$

**Solution:** For any value of  $k$ ,  $f(x)$  is continuous on the intervals  $(-\infty, 1)$  and  $(1, \infty)$  since  $f$  behaves as a polynomial on those intervals. For  $f$  to be continuous everywhere (in particular, at  $x = 1$ ), we must have that  $\lim_{x \rightarrow 1} f(x) = f(1) = 6$ . We have that

$$\lim_{x \rightarrow 1^-} f(x) = f(1) = 6,$$

and that

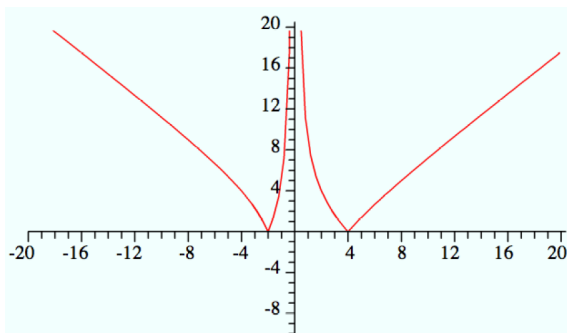
$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} kx + 3 = k + 3.$$

In order for  $\lim_{x \rightarrow 1} f(x)$  to exist, we must have that these two one-sided limits are equal. Thus,  $k + 3 = 6$  implying that  $k$  must be 3. In this case,  $f$  becomes continuous everywhere.

3. For each of the following graphs of a function  $f(x)$ , answer the following questions.

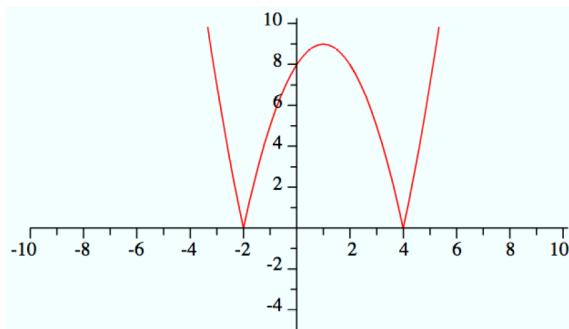
- Find the values of  $x$  where  $f$  is NOT differentiable and why.
- Find the values of  $x$  for which  $f'$  is positive.

- (c) Find the values of  $x$  for which  $f'$  is negative.  
 (d) Find the values of  $x$  for which  $f'$  is zero.



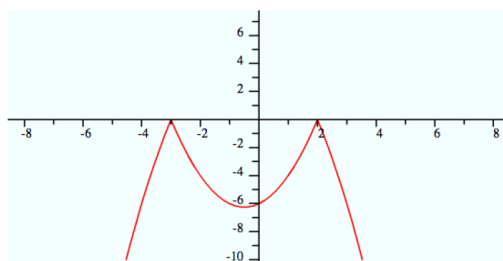
(i)

**Solution:** This function is not differentiable at  $x = -2$  or  $x = 4$  because the graph has cusps at these two points. This function is also not differentiable at  $x = 0$  because it is discontinuous there. We have that  $f(x)$  is increasing on  $(-2, 0)$  and  $(4, \infty)$  and so  $f'(x) > 0$  on these intervals. We have that  $f(x)$  is decreasing on  $(-\infty, -2)$  and  $(0, 4)$  and so  $f'(x)$  is negative on these intervals. The graph of  $f(x)$  has no horizontal tangent lines, and so  $f'(x) \neq 0$  everywhere.



(ii)

**Solution:** This function is not differentiable at  $x = -2$  or  $x = 4$  because the graph has cusps at these two points. We have that  $f(x)$  is increasing on  $(-2, 1)$  and  $(4, \infty)$  and so  $f'(x) > 0$  on these intervals. We have that  $f(x)$  is decreasing on  $(-\infty, -2)$  and  $(1, 4)$  and so  $f'(x)$  is negative on these intervals. The graph of  $f(x)$  has a horizontal tangent line at  $x = 1$ , and so  $f'(1) = 0$ .



(iii)

**Solution:** This function is not differentiable at  $x = -3$  or  $x = 2$  because the graph has cusps at these two points. We have that  $f(x)$  is increasing on  $(-\infty, -3)$  and

$(-0.5, 2)$  and so  $f'(x) > 0$  on these intervals. We have that  $f(x)$  is decreasing on  $(-3, -0.5)$  and  $(2, \infty)$  and so  $f'(x)$  is negative on these intervals. The graph of  $f(x)$  has a horizontal tangent line at  $x = -0.5$ , and so  $f'(-0.5) = 0$ .

4. Use the limit definition of the derivative to find  $f'(x)$  for the following functions.

(a)  $f(x) = \sqrt{x+5}$

**Solution:** To find the given derivative, first use the definition of the derivative and substitute in the particular function given:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h+5} - \sqrt{x+5}}{h}$$

If we try to substitute  $h = 0$  directly we get the indeterminate form  $\frac{0}{0}$ , so we must try something else. The trick here is to multiply top and bottom by the conjugate of the numerator:

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h+5} - \sqrt{x+5})(\sqrt{x+h+5} + \sqrt{x+5})}{h(\sqrt{x+h+5} + \sqrt{x+5})} \\ &= \lim_{h \rightarrow 0} \frac{(x+h+5) - (x+5)}{h(\sqrt{x+h+5} + \sqrt{x+5})} = \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h+5} + \sqrt{x+5})} \\ &= \lim_{h \rightarrow 0} \frac{-1}{(\sqrt{x+h+5} + \sqrt{x+5})} = \frac{1}{\sqrt{x+5} + \sqrt{x+5}} = \frac{1}{2\sqrt{x+5}}. \end{aligned}$$

To check that this solution is correct, you may apply differentiation rules. We can apply the chain rule to quickly get the derivative of  $f(x)$ :

$$\frac{df}{dx} = \frac{d}{dx}(x+5)^{1/2} = \frac{1}{2}(x+5)^{-1/2} \frac{d}{dx}(x+5) = \frac{1}{2}(x+5)^{-1/2} = \frac{1}{2\sqrt{x+5}}.$$

(b)  $f(x) = \frac{3x+1}{2x+4}$

**Solution:** To find the given derivative, first use the definition of the derivative and substitute in the particular function given:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3(x+h)+1}{2(x+h)+4} - \frac{3x+1}{2x+4}}{h}$$

If we try to substitute  $h = 0$  directly we get the indeterminate form  $\frac{0}{0}$ , so we must try something else. We do the only thing we really can do: simplify this complicated expression and hope that something will cancel. in the end it works. We start by combining the fractions in the numerator by obtaining a common denominator:

$$= \lim_{h \rightarrow 0} \frac{\frac{(3(x+h)+1)(2x+4)}{(2(x+h)+4)(2x+4)} - \frac{(3x+1)(2(x+h)+4)}{(2x+4)(2(x+h)+4)}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\left( \frac{(3(x+h)+1)(2x+4) - (3x+1)(2(x+h)+4)}{(2(x+h)+4)(2x+4)} \right)}{h}$$

Now we simplify the ugly numerator. First multiply everything out, then combine/cancel like terms:

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{\left( \frac{(6x^2+12x+6xh+12h+2x+4) - (6x^2+6xh+12x+2x+2h+4)}{(2(x+h)+4)(2x+4)} \right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left( \frac{10h}{(2(x+h)+4)(2x+4)} \right)}{h} = \lim_{h \rightarrow 0} \frac{10}{(2(x+h)+4)(2x+4)} \\ &= \frac{10}{(2x+4)^2} \end{aligned}$$

To check that this solution is correct, you may apply differentiation rules. We can apply the quotient rule to quickly get the derivative of  $f(x)$ :

$$\begin{aligned} \frac{df}{dx} &= \frac{d}{dx} \left( \frac{3x+1}{2x+4} \right) = \frac{(2x+4) \frac{d}{dx}(3x+1) - (3x+1) \frac{d}{dx}(2x+4)}{(2x+4)^2} \\ &= \frac{(2x+4)(3) - (3x+1)(2)}{(2x+4)^2} = \frac{10}{(2x+4)^2}. \end{aligned}$$

(c)  $f(x) = x + \frac{1}{x}$

**Solution:** To find the given derivative, first use the definition of the derivative and substitute in the particular function given:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h + \frac{1}{x+h}) - (x + \frac{1}{x})}{h}$$

We obtain the solution in a way very similar to (but somewhat easier than) the last example:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(x+h + \frac{1}{x+h}) - (x + \frac{1}{x})}{h} &= \lim_{h \rightarrow 0} \frac{h + \frac{1}{x+h} - \frac{1}{x}}{h} = 1 + \lim_{h \rightarrow 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= 1 + \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)x} - \frac{1}{x(x+h)}}{h} = 1 + \lim_{h \rightarrow 0} \frac{\left( \frac{x-(x+h)}{x(x+h)} \right)}{h} = 1 + \lim_{h \rightarrow 0} \frac{\left( \frac{-h}{x(x+h)} \right)}{h} \\ &= 1 + \lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = 1 - \frac{1}{x^2}. \end{aligned}$$

To check that this solution is correct, we can apply differentiation rules. If we apply the power rule to  $f(x)$ , we quickly obtain  $f'(x)$ :

$$\frac{df}{dx} = \frac{d}{dx} \left( x + \frac{1}{x} \right) = \frac{d}{dx} (x^1 + x^{-1}) = 1 \cdot x^{1-1} + -1 \cdot x^{-1-1} = 1 - x^{-2} = 1 - \frac{1}{x^2}.$$

(d)  $f(x) = x^2 + 5x$ .

**Solution:** To find the given derivative, first use the definition of the derivative and substitute in the particular function given:

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{((x+h)^2 + 5(x+h)) - (x^2 + 5x)}{h}$$

Multiply out the terms in the numerator and combine/cancel like terms:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{((x+h)^2 + 5(x+h)) - (x^2 + 5x)}{h} &= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2 + 5x + 5h) - (x^2 + 5x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2 + 5h}{h} = \lim_{h \rightarrow 0} 2x + h + 5 = 2x + 5. \end{aligned}$$

To check this solution, apply the power rule to  $f(x)$ .

5. For the following limits, find the  $\delta > 0$  such that if  $|x - a| < \delta$ , then  $|f(x) - L| < \epsilon$ . Please round your answers to 3 decimal places and show all work used to arrive at your answer.

(a)  $\lim_{x \rightarrow 3} (2x + 5) = 11$ , and  $\epsilon = 0.04$ .

**Solution:** If  $x$  satisfies that  $|(2x + 5) - 11| < 0.04$  then we must have that  $10.96 < 2x + 5 < 11.04$ . Now we solve this inequality:

$$10.96 < 2x + 5 < 11.04$$

$$5.96 < 2x < 6.04$$

$$2.98 < x < 3.02.$$

Therefore we should take  $\delta = 0.02$ . Indeed, we could just reverse the preceding sequence of inequalities to show that if  $|x - 3| < 0.02$  then  $|(2x + 5) - 11| < 0.04$ .

(b)  $\lim_{x \rightarrow -7} \sqrt{9 - x} = 4$ , and  $\epsilon = 0.023$ .

**Solution:** If  $x$  satisfies that  $|\sqrt{9 - x} - 4| < 0.023$  then we must have that  $3.977 < \sqrt{9 - x} < 4.023$ . Now we solve this inequality:

$$3.977 < \sqrt{9 - x} < 4.023$$

$$15.816529 < 9 - x < 16.184529$$

$$6.816529 < -x < 7.184529$$

$$-6.816529 > x > -7.184529.$$

We have that  $|-6.816529 - (-7)| = 0.183471$  and  $|-7.184529 - (-7)| = 0.184529$ . We should let  $\delta$  be the smaller of these, or 0.183471. Rounding to 3 decimal places:  $\delta = 0.183$ .

- (c)  $\lim_{x \rightarrow 4} (3x + 9) = 21$ , and  $\epsilon = 0.017$ .

**Solution:** If  $x$  satisfies that  $|(3x + 9) - 21| < 0.017$  then we must have that  $20.983 < 3x + 9 < 21.017$ . Now we solve this inequality:

$$20.983 < 3x + 9 < 21.017$$

$$11.983 < 3x < 12.017$$

$$3.994\bar{3} < x < 4.005\bar{6}.$$

We have that  $|3.994\bar{3} - (4)| = 0.005\bar{6}$  and  $|4.005\bar{6} - (4)| = 0.005\bar{6}$ . We should let  $\delta = 0.005\bar{6}$ .

6. A particle travels in a straight line according to the position function  $s(t) = 9t^2 + 5t$  meters where  $t$  is in seconds.

- (a) Find the velocity as a function of  $t$ . Include the units.

**Solution:** Apply the power rule to  $s(t)$  to find  $s'(t)$ :

$$s'(t) = 9 \cdot 2t^{2-1} + 5 \cdot 1t^{1-1} = 18t + 5 \text{ m/s.}$$

- (b) Find the acceleration as a function of  $t$ . Include the units.

**Solution:** Apply the power rule to  $s'(t)$  to find  $s''(t)$ :

$$s''(t) = 18 \cdot 1t^{1-1} = 18 \text{ m/s}^2.$$

- (c) What is the acceleration when the velocity is 0? Include the units.

**Solution:** The acceleration is *always*  $18 \text{ m/s}^2$ .

7. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after  $t$  seconds is given by  $H = 10t - 1.76t^2$ .

- (a) Find the velocity of the rock after one second.

**Solution:** The velocity of the rock is given by the first derivative of the height function, with respect to time:

$$v(t) = H'(t) = 10 \cdot 1t^{1-1} - 1.76 \cdot 2t^{2-1} = 10 - 3.52t \text{ m/s.}$$

When  $t = 1$  we have that  $v(1) = 10 - 3.52 = 6.48 \text{ m/s}$ .

- (b) Find the velocity of the rock when  $t = a$ .

**Solution:** When  $t = a$  we have that  $v(a) = 10 - 3.52a$  m/s.

(c) When will the rock hit the surface?

**Solution:** The rock will hit the surface when  $H = 0$ . We can find the roots of  $H$  just by factoring:

$$H(t) = t(10 - 1.76t).$$

The root  $t = 0$  corresponds to the fact that  $H$  is 0 when the rock is initially thrown. The rock hits the ground when  $H$  becomes 0 again, i.e. when  $10 - 1.76t = 0$ . The rock hits the ground when  $t \approx 5.68$  sec.

(d) With what velocity will the rock hit the surface?

**Solution:** Our intuition should tell us that the rock will hit the ground with velocity  $-10$  m/s because  $v(0) = 10$  gives us that the rock was initially thrown upwards with a velocity of 10 m/s. Since the height function is a parabola, the graph of  $H(t)$  has a symmetric shape. Thus, we'd expect that the slope of the tangent line at one root would be the opposite of the slope of the tangent line at the other. To verify this intuition, simply calculate  $v(5.68) = 10 - 3.52(5.68) = -9.9936 \approx -10$ . Convincing.

8. Consider a spring which is attached to a wall on the left and has a mass attached on the right. The equilibrium position when the spring is still corresponds to  $x = 0$  and the positive direction is to the right. If the mass is pulled out to the right, the mass vibrates horizontally on a smooth level surface according to the equation of motion  $x(t) = 2 \cos(t) + 7 \sin(t)$  where  $x$  is the displacement from the equilibrium position.

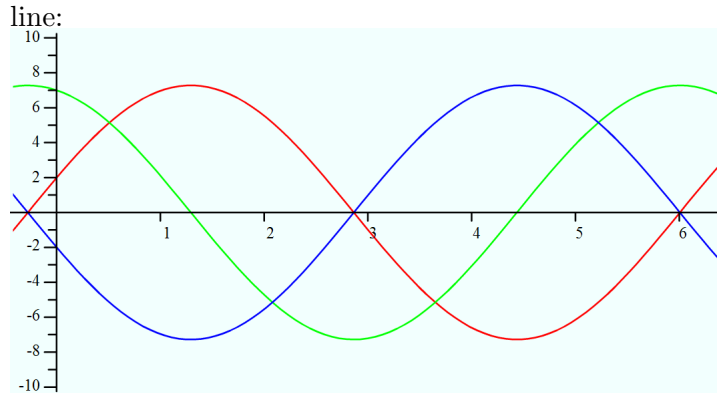
(a) Find the velocity,  $v(t)$ , and the acceleration,  $a(t)$  and graph the functions on the interval  $[0, 2\pi]$ .

**Solution:**

$$v(t) = -2 \sin(t) + 7 \cos(t)$$

$$a(t) = -2 \cos(t) - 7 \sin(t)$$

In the following picture,  $x(t)$  is the red line,  $v(t)$  is the green line and  $a(t)$  is the blue line:



(b) What is the initial position, velocity and acceleration?

**Solution:**

$$x(0) = 2$$

$$v(0) = 7$$

$$a(0) = -2$$

(c) At what times is the speed (which equals  $|v(t)|$ ) largest and what is the largest speed?

9. Compute the following limits algebraically using limit laws. If the limit is  $+\infty$  or  $-\infty$ , then say so. You must show work or justification for each of your answers.

(a)  $\lim_{x \rightarrow \infty} \frac{3e^x + 1}{4e^x - 1}$ .

**Solution:** For all parts of this question we use the following limits without justification:

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^n} = 0 \text{ for integer } n > 0.$$

$$\lim_{x \rightarrow \infty} e^x = \lim_{x \rightarrow -\infty} e^{-x} = \infty$$

$$\lim_{x \rightarrow -\infty} e^x = \lim_{x \rightarrow \infty} e^{-x} = 0$$

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1.$$

For the solution to part (a):

As  $x$  grows without bound, both  $3e^x$  and  $4e^x$  grow without bound. Thus, the numerator and denominator both tend to  $\infty$ . This limit has (at first glance) indeterminate form  $\frac{\infty}{\infty}$ . The trick here (as with MANY limits which have indeterminate form  $\frac{\infty}{\infty}$ ) is to divide the top and bottom by the “strongest” term in the denominator. In this case, the strongest term is  $e^x$ . Dividing top and bottom by  $e^x$  is the same as multiplying top and bottom by  $e^{-x}$ :

$$\lim_{x \rightarrow \infty} \frac{3e^x + 1}{4e^x - 1} = \lim_{x \rightarrow \infty} \frac{e^{-x}(3e^x + 1)}{e^{-x}(4e^x - 1)} = \lim_{x \rightarrow \infty} \frac{3 + e^{-x}}{4 - e^{-x}} = \frac{3}{4}.$$

(b)  $\lim_{x \rightarrow \infty} \frac{3e^{2x} + 1}{4e^x - 1}$ .

**Solution:** Again, here, top and bottom both tend to infinity, so we try to divide top and bottom by the “strongest” term in the denominator (i.e.  $e^x$ ). Again, this is the same as multiplying by  $e^{-x}$ :

$$\lim_{x \rightarrow \infty} \frac{3e^{2x} + 1}{4e^x - 1} = \lim_{x \rightarrow \infty} \frac{e^{-x}(3e^{2x} + 1)}{e^{-x}(4e^x - 1)} = \lim_{x \rightarrow \infty} \frac{3e^x + e^{-x}}{4 - e^{-x}} = \infty.$$

Notice that the last limit is NOT indeterminate. It is of the form  $\frac{\infty+0}{4-0}$  which is the same as  $\frac{\infty}{4}$  which is the same as  $\infty$ .

(c)  $\lim_{t \rightarrow 0} \frac{\sin 4t}{\sin 9t}$ .

**Solution:**

$$\begin{aligned}\lim_{t \rightarrow 0} \frac{\sin 4t}{\sin 9t} &= \lim_{t \rightarrow 0} \left( \frac{\sin 4t}{\sin 9t} \right) \left( \frac{4t}{4t} \right) \left( \frac{9t}{9t} \right) \\ &= \lim_{t \rightarrow 0} \left( \frac{\sin 4t}{4t} \right) \left( \frac{9t}{\sin 9t} \right) \left( \frac{4t}{9t} \right) = \frac{4}{9}.\end{aligned}$$

Why do we multiply and divide by  $4t$  and  $9t$  in the first step? Because it works! It's a trick. Learn it.

(d)  $\lim_{x \rightarrow -\infty} \frac{x^2 + 3x}{\sqrt{3x^4 + 1}}$ .

**Solution:** Again, here we have the indeterminate form  $\frac{\infty}{\infty}$  and so we divide top and bottom by the “strongest” term in the denominator. In this case, the highest power of  $x$  in the denominator is 4, but because it appears under a square root, it is effectively  $x^2$ . So we divide top and bottom by  $x^2$ :

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{x^2 + 3x}{\sqrt{3x^4 + 1}} &= \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2}(x^2 + 3x)}{\frac{1}{x^2}\sqrt{3x^4 + 1}} \\ \lim_{x \rightarrow -\infty} \frac{1 + \frac{3}{x}}{\sqrt{\frac{1}{x^4}(3x^4 + 1)}} &= \lim_{x \rightarrow -\infty} \frac{1}{\sqrt{3 + \frac{1}{x^4}}} = \frac{1}{\sqrt{3}}.\end{aligned}$$

(e)  $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x}$ .

**Solution:**

$$\lim_{x \rightarrow 0} \frac{\sin 5x}{3x} = \lim_{x \rightarrow 0} \left( \frac{\sin 5x}{3x} \right) \left( \frac{5}{5} \right) = \lim_{x \rightarrow 0} \left( \frac{\sin 5x}{5x} \right) \left( \frac{5}{3} \right) = \frac{5}{3}.$$

(f)  $\lim_{x \rightarrow \infty} \frac{2x^2 + 3x}{4x^3 + 9x + 1}$ .

**Solution:**

$$\lim_{x \rightarrow \infty} \frac{2x^2 + 3x}{4x^3 + 9x + 1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^3}(2x^2 + 3x)}{\frac{1}{x^3}(4x^3 + 9x + 1)} = \lim_{x \rightarrow \infty} \frac{\frac{2}{x} + \frac{3}{x^2}}{4 + \frac{9}{x^2} + \frac{1}{x^3}} = \frac{0}{4} = 0.$$

10. For the following functions determine the vertical and horizontal asymptotes, if they exist.

$$(a) f(x) = \frac{\sqrt{9x^4 - x}}{x^2 - 1}.$$

**Solution:** We have that

$$f(x) = \frac{\sqrt{9x^4 - x}}{x^2 - 1} = \frac{\sqrt{9x^4 - x}}{(x + 1)(x - 1)}.$$

We have that

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} \frac{\sqrt{9x^4 - x}}{(x + 1)(x - 1)} = \lim_{x \rightarrow 1^-} \frac{\sqrt{9x^4 - x}}{(x + 1)} \lim_{x \rightarrow 1^-} \frac{1}{(x - 1)} \\ &= \frac{\sqrt{8}}{2} \lim_{x \rightarrow 1^-} \frac{1}{(x - 1)} = -\infty, \end{aligned}$$

and also that

$$\begin{aligned} \lim_{x \rightarrow -1^-} f(x) &= \lim_{x \rightarrow -1^-} \frac{\sqrt{9x^4 - x}}{(x + 1)(x - 1)} = \lim_{x \rightarrow -1^-} \frac{\sqrt{9x^4 - x}}{(x - 1)} \lim_{x \rightarrow -1^-} \frac{1}{(x + 1)} \\ &= \frac{\sqrt{10}}{2} \lim_{x \rightarrow -1^-} \frac{1}{(x + 1)} = -\infty. \end{aligned}$$

Thus,  $f$  has vertical asymptotes at  $x = -1$  and  $x = 1$ . We have that

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{\sqrt{9x^4 - x}}{x^2 - 1} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} \sqrt{9x^4 - x}}{\frac{1}{x^2}(x^2 - 1)} = \lim_{x \rightarrow \infty} \frac{\sqrt{\frac{1}{x^4}(9x^4 - x)}}{1 - \frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{9 - \frac{1}{x^3}}}{1} = 3 \end{aligned}$$

and that

$$\begin{aligned} \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{\sqrt{9x^4 - x}}{x^2 - 1} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2} \sqrt{9x^4 - x}}{\frac{1}{x^2}(x^2 - 1)} = \lim_{x \rightarrow -\infty} \frac{\sqrt{\frac{1}{x^4}(9x^4 - x)}}{1 - \frac{1}{x^2}} \\ &= \lim_{x \rightarrow -\infty} \frac{\sqrt{9 - \frac{1}{x^3}}}{1} = 3. \end{aligned}$$

Thus,  $f$  has one horizontal asymptote:  $y = 3$ .

$$(b) f(x) = \frac{x^3 - 2x^2 + x}{x^2 - 3x + 2}.$$

**Solution:** We have that

$$f(x) = \frac{x^3 - 2x^2 + x}{x^2 - 3x + 2} = \frac{x(x - 1)(x - 1)}{(x - 1)(x - 2)}.$$

We have that

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} \frac{x(x-1)(x-1)}{(x-1)(x-2)} = \lim_{x \rightarrow 1} \frac{x(x-1)}{x-2} = 0.$$

Thus,  $f$  has no vertical asymptote at  $x = 1$ . We have that

$$\begin{aligned} \lim_{x \rightarrow 2^-} f(x) &= \lim_{x \rightarrow 2^-} \frac{x(x-1)(x-1)}{(x-1)(x-2)} = \lim_{x \rightarrow 2^-} \frac{x(x-1)(x-1)}{x-1} \lim_{x \rightarrow 2^-} \frac{1}{x-2} \\ &= 2 \lim_{x \rightarrow 2^-} \frac{1}{x-2} = -\infty. \end{aligned}$$

Thus  $f$  has a vertical asymptote at  $x = 2$ . We have that

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{x^3 - 2x^2 + x}{x^2 - 3x + 2} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2}(x^3 - 2x^2 + x)}{\frac{1}{x^2}(x^2 - 3x + 2)} = \lim_{x \rightarrow \infty} \frac{x - 2 + \frac{1}{x}}{1 - \frac{3}{x} + \frac{2}{x^2}} \\ &= \frac{\infty}{1} = \infty. \end{aligned}$$

We also have that

$$\begin{aligned} \lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} \frac{x^3 - 2x^2 + x}{x^2 - 3x + 2} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2}(x^3 - 2x^2 + x)}{\frac{1}{x^2}(x^2 - 3x + 2)} = \lim_{x \rightarrow -\infty} \frac{x - 2 + \frac{1}{x}}{1 - \frac{3}{x} + \frac{2}{x^2}} \\ &= \frac{-\infty}{1} = -\infty. \end{aligned}$$

Thus,  $f$  has no horizontal asymptotes.

(c)  $f(x) = \frac{1 + e^{4x}}{1 - 3e^{4x}}$ .

**Solution:** We have that  $1 + e^{4x} > 1$  and so  $f$  has a vertical asymptote wherever the denominator vanishes. This happens when  $1 - 3e^{4x} = 0$ , i.e. where  $x = \frac{1}{4} \ln(1/3)$ . We have that

$$\lim_{x \rightarrow \infty} \frac{1 + e^{4x}}{1 - 3e^{4x}} = \lim_{x \rightarrow \infty} \frac{e^{-4x}(1 + e^{4x})}{e^{-4x}(1 - 3e^{4x})} = \lim_{x \rightarrow \infty} \frac{e^{-4x} + 1}{e^{-4x} - 3} = -\frac{1}{3}.$$

We also have that

$$\lim_{x \rightarrow -\infty} \frac{1 + e^{4x}}{1 - 3e^{4x}} = \frac{1}{1} = 1.$$

Thus  $f$  has two horizontal asymptotes:  $y = 1$  and  $y = -\frac{1}{3}$ .

11. Use differentiation rules to find the derivatives of each of the following functions. Show your work.

(a)  $f(x) = \sqrt[3]{x^3 + 1}$

**Solution:**

$$f(x) = \sqrt[3]{x^3 + 1} = (x^3 + 1)^{1/3}.$$

We apply the chain and power rules:

$$f'(x) = \frac{1}{3}(x^3 + 1)^{\frac{1}{3}-1} \frac{d}{dx}(x^3 + 1) = \frac{1}{3}(x^3 + 1)^{-2/3}(3x^2).$$

(b)  $f(t) = \frac{6t^4 - 2t + 1}{\cos(t^2 + 1)}$

**Solution:** We apply the quotient rule:

$$\begin{aligned} f'(t) &= \frac{(\cos(t^2 + 1)) \frac{d}{dt}(6t^4 - 2t + 1) - (6t^4 - 2t + 1) \frac{d}{dt}(\cos(t^2 + 1))}{(\cos(t^2 + 1))^2} \\ &= \frac{(\cos(t^2 + 1))(24t^3 - 2) - (6t^4 - 2t + 1)(-\sin(t^2 + 1)(2t))}{(\cos(t^2 + 1))^2}. \end{aligned}$$

(c)  $g(x) = \sin(e^{2x})$

**Solution:** Applying the chain rule twice:

$$g'(x) = \cos(e^{2x}) \frac{d}{dx}(e^{2x}) = \cos(e^{2x})(e^{2x}) \frac{d}{dx}(2x) = 2e^{2x} \cos(e^{2x}).$$

(d)  $h(z) = z^3 + 9z + 4 + \frac{3}{z^3}$

**Solution:**

$$h'(z) = 3z^2 + 9 - \frac{9}{z^4}.$$

(e)  $f(x) = \tan^2(x^3)$

**Solution:**

$$f'(x) = 2 \tan(x^3) \sec^2(x^3)(3x^2)$$

12. If  $f(t) = te^{2t}$ , find  $f'(t)$  and  $f''(t)$ .

**Solution:** We apply the product and chain rules:

$$f'(t) = (e^{2t}) \frac{d}{dt}(t) + (t) \frac{d}{dt}(e^{2t}) = e^{2t} + (t)(e^{2t}) \frac{d}{dt}(2t) = e^{2t} + 2te^{2t}$$

We do this again to find the second derivative:

$$f''(t) = 2e^{2t} + 2e^{2t} + 4te^{2t}.$$

13. If  $s(x) = x^2 + 4x + 9$ , find  $s'(x)$  and  $s''(x)$ .

**Solution:**

$$s'(x) = 2x + 4$$

$$s''(x) = 2.$$

14. If  $f(t) = \frac{3t + 1}{4t + 1}$ , find  $f'(t)$  and  $f''(t)$ .

**Solution:** We apply the quotient rule for the first step:

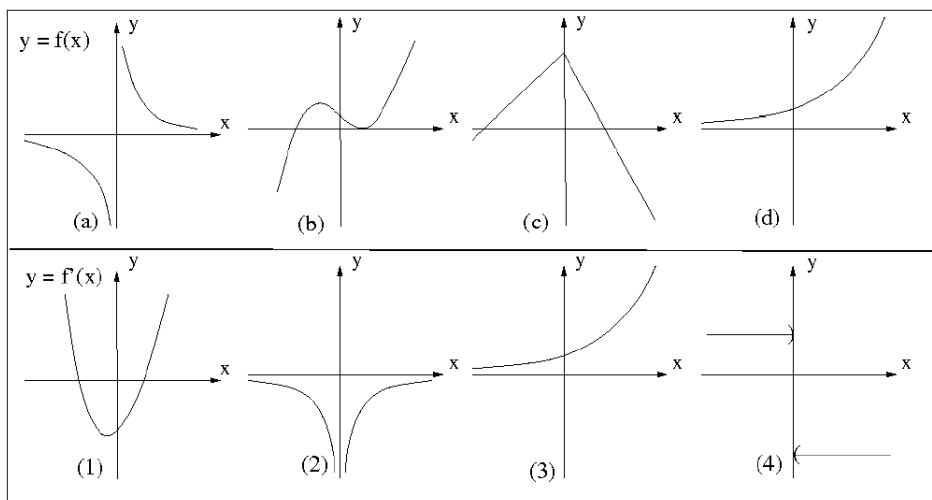
$$f'(t) = \frac{(4t + 1)\frac{d}{dt}(3t + 1) - (3t + 1)\frac{d}{dt}(4t + 1)}{(4t + 1)^2} = \frac{(4t + 1)(3) - (3t + 1)(4)}{(4t + 1)^2} = \frac{-1}{(4t + 1)^2}.$$

To find the second derivative, it's much easier to first rewrite  $f'(t) = -(4t + 1)^{-2}$  and apply the chain rule:

$$f''(t) = 2(4t + 1)^{-3}(4).$$

This is the same answer one would obtain by applying the chain rule.

15. Match the graph of each function in (a)-(d) with the graph of its derivative in (1)-(4). Give a reason for your choice.



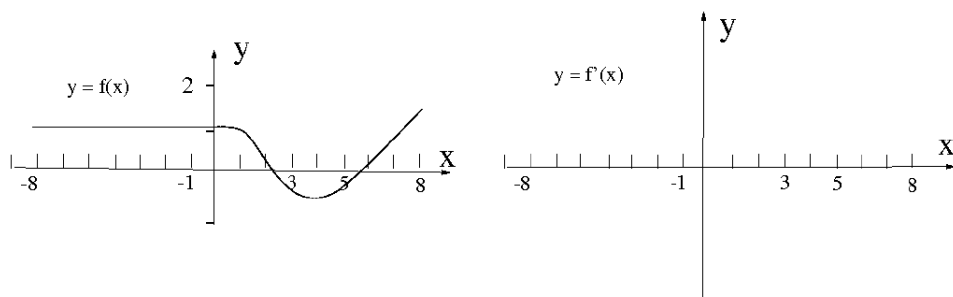
**Solution:** For (a), we see that this function is continually decreasing, and it undefined when  $x = 0$ . Thus we expect the derivative to be negative everywhere except  $x = 0$ , where it is undefined. The appropriate choice seems to match (a) with (2).

For (b), we see that the function has exactly two points with horizontal tangent lines. This means that there should be exactly two places where  $f'(x) = 0$ . The only choice for the derivative of (b), then, is (1).

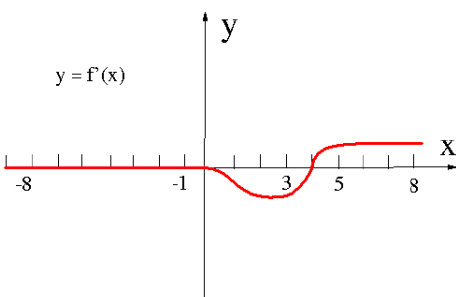
For (c), we see that the function is piecewise linear, with a cusp at  $x = 0$ , and that it is increasing for  $x < 0$  and decreasing for  $x > 0$ . Since there is a cusp at  $x = 0$ ,  $f'(0)$  is undefined. We should have that  $f'(x) = c > 0$  for  $x < 0$  and  $f'(x) = d < 0$  for  $x > 0$ . The best choice, then, for the derivative of (c) is (4).

For (d) we pick (3), by the exclusion principle. Also, this function looks like the exponential function, whose derivative is itself.

16. Given a sketch of  $y = f(x)$ , sketch a graph of  $f'(x)$ .



**Solution:**



17. Find the equation of the tangent line to the curve  $y = x^3 + 3x + 1$  at  $x = 0$ .

**Solution:** We have that

$$\frac{dy}{dx} = 3x^2 + 3.$$

When  $x = 0$ , the line tangent to the curve has slope 3. Thus, the tangent line is given by

$$y - 1 = 3(x - 0)$$

$$y = 3x + 1.$$

18. Find the equation of the tangent line to the curve  $y = \sqrt{2x^2 + 3x + 1}$  at  $x = 3$ .

**Solution:** We have that

$$\frac{dy}{dx} = \frac{1}{2}(2x^2 + 3x + 1)^{-1/2}(4x + 3)$$

and so the tangent line at  $x = 3$  has slope

$$\frac{1}{2}(2(3)^2 + 3(3) + 1)^{-1/2}(4(3) + 3) = \frac{15}{2\sqrt{28}}.$$

Thus, the equation of the tangent line is given by

$$y - \sqrt{28} = \frac{15}{2\sqrt{28}}(x - 3)$$

$$y = \frac{15x}{2\sqrt{28}} + \left(-\frac{45}{2\sqrt{28}} + \sqrt{28}\right).$$

19. Find the equation of the tangent line to the curve  $y = \frac{x^2+1}{x^2-1}$  at  $x = 0$ .

**Solution:**

$$\begin{aligned} y'(x) &= \frac{(x^2 - 1)\frac{d}{dx}(x^2 + 1) - (x^2 + 1)\frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2} \\ &= \frac{(x^2 - 1)(2x) - (x^2 + 1)(2x)}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2}. \end{aligned}$$

We have that the tangent line has slope 0 when  $x = 0$ . Thus, the desired line has equation  $y = -1$ .