SYMMETRIC QUADRUPLE PHASE TRANSITIONS

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1. Introduction and preliminaries

1.1. **Motivation.** The grain boundary of crystalline material has been an object of intensive study by material scientists as well as by mathematicians. The two dimensional models of grain boundary dynamics for multiple phases focus in particular on the pattern of triple junctions. In [**?**], a vector-valued partial differential equation model is introduced to model the formation and dynamics of such triple junctions. The finer structure of triple junctions for this model is shown to exist in [**?**] as a solution to the Euler-Lagrange equation on the entire plane

$$-\Delta u + (DF(u))^T = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}^2 \tag{1.1}$$

where $F : \mathbb{R}^2 \to \mathbb{R}^+$ is a nonnegative triple well potential with symmetry of an equilateral triangle, and the subscript ^{T} means the transpose of a row vector to column vector. In the study of grain boundary structures, three dimensional geometries are more realistic, in particular when there are more than one layers of grains existing in the polycrystal. Indeed, polycrystalline materials contain complicated three dimensional joints. Among them are quadruple junctions which are the points of contact between four grains and six grain boundaries, or, equivalently, the junction of four triple junctions. Figure **??** below shows such a quadruple junction with one of the four grains highlighted. (See [**?**], [**?**] and [**?**] for the discussion of quadruple junctions in crystalline materials.)

In this paper, we try to model the quadruple junction via generalized Allen-Cahn equation as in [**?**]. We introduce a quadruple well potential W with each well (global minimum point) representing a phase of grain. Assuming that all phases of the grain interior are of equal status, the potential W can be chosen so that it has the symmetry of a regular tetrahedron. The physical state of a crystalline material may be represented by an order parameter V which is a $\mathbb{R}^3$ -vector valued function. The order parameter V has a constant value $\mathbf{u}_0$ in each grain, where $\mathbf{u}_0$ corresponds to one of the wells of the potential. We may use

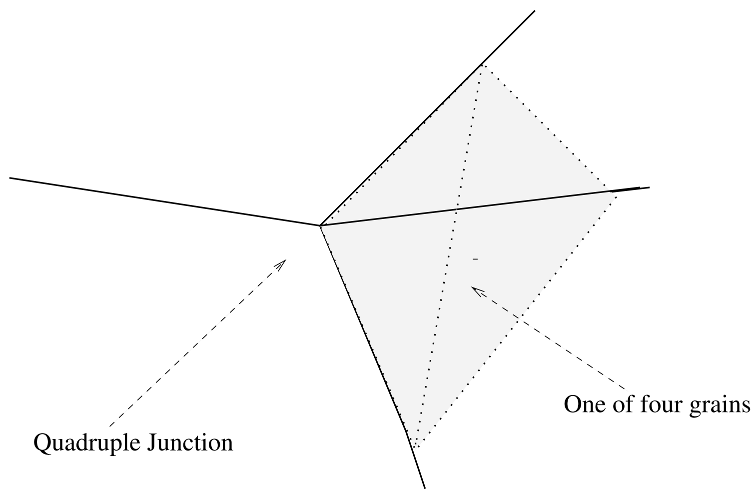

FIGURE 1. A quadruple junction with four grains.

the following energy functional to gauge the physical state

$$E_{\Omega,\epsilon}(V) = \int_\Omega \epsilon|\nabla V|^2 + \frac{1}{\epsilon}W(V)dx \tag{1.2}$$

where $\epsilon > 0$ is a small physical constant which is related to the difussion rate and therefore the thickness of the grain boundary. The dynamics of the physical state can be modeled by the gradient flow of the energy functional, i.e.

$$V_t = \epsilon\Delta V - \frac{1}{\epsilon}(DW(V))^T, \quad x \in \Omega,\, t > 0. \tag{1.3}$$

The finer structure of quadruple junction can then be expressed as a scaling of a solution of the Euler -Lagrange equation

$$-\Delta U + (DW(U))^T = 0, \quad U : \mathbb{R}^3 \to \mathbb{R}^3 \tag{1.4}$$

to $\mathbb{R}^3$ with suitable asymptotic behavior at infinity which shows a quadruple structure. In this paper, we shall construct rigorously such a quadruple junction solution with proper symmetry, in the same spirit as done in [?] for triple junction solutions. However, the quadruple junction solution turns out to be much more complicated to analyze, due to both the complexity of the three dimensional geometries and the structure of the solution. More interesting phenomena arise for the three dimensional problem of quadruple junction compared to the two dimensional problem of triple junctions. We have to construct first one dimensional transition profile between two phases (heteroclinic solution) with target space $\mathbb{R}^3$ and study its special properties; then we reproduce a triple junction solution with target space $\mathbb{R}^3$, which is not just the trivial generalization of [**?**] due to the extra dimension of the target space and the fourth well of the potential. Finally, the structure of the quadruple junction solution has a subtle two dimensional transition layer connecting different triple junctions in different faces, which requires delicate analysis. Below we shall describe the details of each step in the construction.

1.2. **Hypotheses and notations.** The letters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ denote the vertices of a regular tetrahedron in $\mathbb{R}^3$. Without loss of generality, we will assume that the side of this tetrahedron has length 2 and that its center of gravity is at the origin of the coordinates. We will denote henceforth $\mathcal{T}_1$ for this unit tetrahedron and $\mathcal{T}_L = L\mathcal{T}_1$ for the tetrahedron with size L for any positive real number L. The tetrahedron shall be used for both the domain space and target space.

For $\mathbf{x} \neq \mathbf{y}$ belonging to $\mathbf{X} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ we denote by $\gamma_{\mathbf{xy}}$ the reflection which exchanges $\mathbf{x}$ and $\mathbf{y}$ and leaves invariant the two other vertices. The group of symmetries Γ is the group generated by all the reflections $\gamma_{\mathbf{xy}}$; for three distinct letters $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$, we denote by $\Gamma(\mathbf{x}, \mathbf{y})$ the group generated by $\gamma_{\mathbf{xy}}$ and by $\Gamma(\mathbf{x}, \mathbf{y}, \mathbf{z})$ the group generated by $\gamma_{\mathbf{xy}}$, $\gamma_{\mathbf{yz}}$ and by $\gamma_{\mathbf{zx}}$.

We now assume that

(H1): *The potential W is a nonnegative function of class C^3 from $\mathbb{R}^3$ to $\mathbb{R}$ which is invariant under Γ, i.e.*

$$\forall \gamma \in \Gamma : \quad W \circ \gamma = W.$$

(H2): *W vanishes only at the points in $\mathbf{X}$ in $\mathbb{R}^3$, and the Hessian $D^2W(\mathbf{x})$ is nondegenerate for all $\mathbf{x} \in \mathbf{X}$. The eigenvalues of $D^2W(\mathbf{x})$ are the strictly positive numbers $\lambda_1 \leq \lambda_2 \leq \lambda_3$ for $\mathbf{x} \in \mathbf{X}$.*

(H3): *There is a constant $R_0 > 0$ such that*

$$DW(u) \cdot u := (u, DW(u)^T) \geq 0, \quad \forall u \in \mathbb{R}^3 \text{ with } |u| \geq R_0 \tag{1.5}$$

Hypothesis **(H3)** may be made more general as in [**?**]. For simplicity, we do not include the general form here.

For u - a function from $\mathbb{R}^d$ to $\mathbb{R}^3$- we define a Landau density of energy in a d-dimensional space by

$$\mathcal{L}_d(u) = \frac{1}{2}\sum_{j=1}^{d}\left|\frac{\partial u}{\partial x_j}\right|^2 + W(u).$$

It is well known that there exists a heteroclinic connection from $\mathbf{a}$ to $\mathbf{b}$: it is obtained by minimizing the energy functional

$$E_1(u) = \int_{\mathbb{R}} \mathcal{L}_1(u)\,dx$$

among all the functions whose derivatives are square integrable and which tend to $\mathbf{a}$ at $-\infty$ and to $\mathbf{b}$ at $+\infty$; see for instance [**?**] for the existence and [**?**], [**?**] for further analysis of the minimizer. Let z be a heteroclinic connection from $\mathbf{a}$ to $\mathbf{b}$; it satisfies the Euler-Lagrange equation

$$-z'' + DW(z)^T = 0. \tag{1.6}$$

The energy of a minimizer is denoted e_1:

$$e_1 = E_1(z).$$

The function $\gamma_{\mathbf{cd}} z$ is also a heteroclinic connection from $\mathbf{a}$ to $\mathbf{b}$. In this article, we will assume that, up to translation, there are at most two heteroclinic connections from $\mathbf{a}$ to $\mathbf{b}$; then Lemma **??** ensures that

- either there are two distinct heteroclinic connections and $\gamma_{\mathbf{cd}}$ exchanges them; in that case, one of the heteroclinic connections takes its values in the open half-space $\{\mathbf{x} \in \mathbb{R}^3 : (\mathbf{x}, \mathbf{d}-\mathbf{c}) > 0\}$; it will be denoted by $z_{\mathbf{ab},\mathbf{d}}$ and the other one takes its values in the symmetric half-space and is denoted $z_{\mathbf{ab},\mathbf{c}}$.
- or there is only one heteroclinic connection and it is $\gamma_{\mathbf{cd}}$-invariant. It will be denoted by $z_{\mathbf{ab}}$.

The operator A defined by

$$D(A) = H^2(\mathbb{R})^3, \quad Av = -v'' + D^2W(z_{\mathbf{ab},\mathbf{c}})v \tag{1.7}$$

is a self-adjoint nonnegative operator; the lower bound of its essential spectrum is governed by its behavior at infinity: it is equal to the lower bound of the spectrum of $D^2W(\mathbf{a})$, i.e. λ_1.

Differentiating (**??**) with $z = z_{\mathbf{ab},\mathbf{c}}$, we see that $z'_{\mathbf{ab},\mathbf{c}}$ satisfies the linearized equation $Az'_{\mathbf{ab},\mathbf{c}} = 0$. Since $z'_{\mathbf{ab},\mathbf{c}}$ and its derivatives decay exponentially fast at infinity, it belongs to the kernel of A.

The main assumption is the nondegeneracy of the heteroclinic connection $z_{\mathbf{ab},\mathbf{c}}$, namely

(H4): the kernel of A is spanned by $z'_{\mathbf{ab},\mathbf{c}}$.

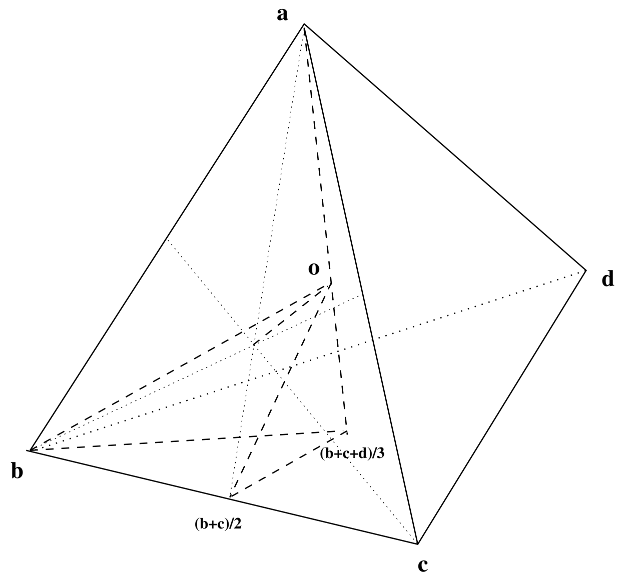

FIGURE 2. The tetrahedron **abcd** with the marking of the middle of one edge, the center of gravity of the bottom face and the center of gravity of the tetrahedron.

As the lower bound of the essential spectrum is strictly positive, it is plain that the lower bound of the strictly positive elements of the spectrum of A is a strictly positive number: it will be convenient to call this number ν^2, with $\nu > 0$. Of course, we have the inequalities

$$0 < \nu \leq \sqrt{\lambda_1}. \tag{1.8}$$

The convex hull of a finite set of points $\mathbf{x}_1, \ldots, \mathbf{x}_k$ will be denoted by $[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k]$ by analogy with the interval notation in dimension 1.

1.3. **Geometric constants.** We need a number of geometric constants:

- the distance from $\mathbf{a}$ to any of the edges $\mathbf{bc}$, $\mathbf{cd}$, $\mathbf{db}$ is $\sqrt{3}$;
- the distance from $\mathbf{a}$ to the face $\mathbf{bcd}$ is $2\sqrt{2/3}$;
- the distance from the center of gravity of a face to the edges of the face is $1/\sqrt{3}$;
- the distance from the center of gravity of the tetrahedron to each of the faces is $1/\sqrt{6}$.

The tetrahedron is pictured at Figure **??**, together with a number of distinguished points. The subtetrahedron $[0, \mathbf{b}, (\mathbf{b}+\mathbf{c})/2, (\mathbf{b}+\mathbf{c}+\mathbf{d})/3]$ has a volume which is a twenty-fourth of the total volume of the tetrahedron.

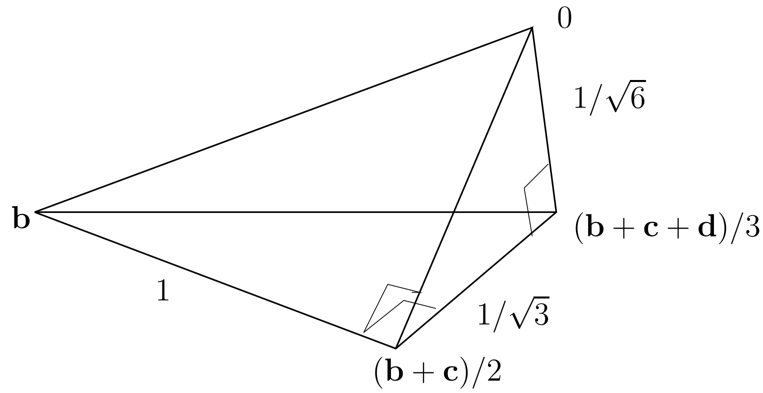

FIGURE 3. The small tetrahedron appearing in Figure **??** with the length of its sides.

On Figure **??**, we can see that the area of the triangle $[0, \mathbf{b}, (\mathbf{b}+\mathbf{c})/2]$ is equal to $\sqrt{2}/4$.

We define the following constants: the segment $[0, (\mathbf{b}+\mathbf{c}+\mathbf{d})/3]$ from the center of gravity of the tetrahedron to the center of gravity of one face has length $1/\sqrt{6}$; the total length of the four analogous segments is

$$\delta_2 = \frac{2\sqrt{6}}{3};$$

the segment $[0, (\mathbf{b}+\mathbf{c})/2]$ from the center of gravity of the tetrahedron to the center of gravity of the segment $[\mathbf{b}, \mathbf{c}]$ has length $1/\sqrt{2}$; the total length of the analogous segments is

$$\delta_2' = 3\sqrt{2};$$

the triangle $[0, (\mathbf{b}+\mathbf{c})/2, (\mathbf{b}+\mathbf{c}+\mathbf{d})/3]$ has area

$$\frac{1}{2} \times \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{6}} = \frac{\sqrt{2}}{12},$$

there are 12 analogous triangles and their total area is

$$\delta_1 = \sqrt{2}.$$

1.4. **The exponential dichotomy.** An exponential dichotomy is a situation where the principal part of a solution of an ordinary differential equation is a combination of an exponentially small term and of an exponentially large term; when a condition tells us that the solution is bounded, we may be able to conclude that it is in fact exponentially decaying. The following lemma helps to make this argument quantitative, even when the interval of interest is finite.

Lemma 1.1. *Let g be a bounded nonnegative function on $[0, L]$ and let f be a nonnegative function belonging to $L^p(0, L)$ for some $p \in [1, \infty]$. If g satisfies the ordinary differential equation*

$$g'' - \mu^2 g = f, \text{ over } [0, L] \tag{1.9}$$

with a Neumann boundary condition at L:

$$g'(L) = 0, \tag{1.10}$$

then, there exists a constant ρ depending only on p and μ such that

$$\forall t \in [0, L], \quad g(t) \le 2e^{-\mu t}(g(0) + \rho |f|_{L^p}). \tag{1.11}$$

*If $L = \infty$, the same conclusion holds without assuming (**??**), and without the constant 2 in (**??**).*

Proof. The general solution of (**??**) is

$$g(t) = Ae^{\mu t} + Be^{-\mu t} - \frac{1}{2\mu}\int_0^t e^{\mu(s-t)} f(s)\, ds - \frac{1}{2\mu}\int_t^L e^{\mu(t-s)} f(s)\, ds. \tag{1.12}$$

The value of A and B are obtained from the boundary conditions:

$$B = (1 + e^{-2\mu L})^{-1}\left(g(0) + \frac{1}{2\mu}\int_0^L \left[e^{-\mu s} + e^{\mu s - 2\mu L}\right] f(s)\, ds,\right) \tag{1.13}$$

$$A = Be^{-2\mu L} - \frac{e^{-\mu L}}{2\mu}\int_0^L e^{\mu(s-L) f(s)\, ds}. \tag{1.14}$$

Substituting (**??**) and (**??**) into (**??**), and using the positivity assumptions, we find the following identity

$$\begin{aligned} g(t) = {} & \frac{e^{-\mu t} + e^{\mu(t-2L)}}{1 + e^{-2\mu L}}\left(g(0) + \frac{1}{2\mu}\int_0^L \left[e^{-\mu s} + e^{\mu s - 2\mu L}\right] f(s)\, ds\right) \\ & - \frac{1}{2\mu}\left(\int_0^L e^{\mu(t+s-2L)} f(s)\, ds + \int_0^t e^{\mu(s-t)} f(s)\, ds + \int_t^L e^{\mu(t-s)} f(s)\, ds\right), \end{aligned}$$

from which we infer the inequality

$$g(t) \le e^{-\mu t}\left(2g(0) + \frac{1}{\mu}\int_0^L \left[e^{-\mu s} + e^{\mu(s-2L)}\right] f(s)\, ds\right).$$

By Hölder inequality, assuming $1 < p < \infty$,

$$\int_0^L e^{-\mu s} f(s)\, ds \le \frac{1}{(p'\mu)^{1/p'}}\|f\|_{L^p},$$

and

$$\int_0^L e^{\mu s} f(s)\, ds \le \frac{e^{\mu L}}{(p'\mu)^{1/p'}}\|f\|_{L^p},$$

with p' the conjugate exponent to p. The conclusion for the case when L is bounded is then clear, and we may take

$$\rho = \frac{1}{\mu(\mu p')^{1/p'}}.$$

Obvious modification are to be performed when $p = 1$ or $p = \infty$. When L is unbounded, A necessarily vanishes, and we obtain

$$B = g(0) + \frac{1}{2\mu} \int_0^\infty e^{-\mu s} f(s)\, ds,$$

and the conclusion holds. □

Lemma **??** will be repeatedly used in the course of this article.

2. One-dimensional results

This section is devoted to the careful study of heteroclinic connections, starting with general geometric properties as in [**?**], and then specializing to the linearized problem and to problem on bounded intervals with Neumann and other boundary conditions. These results will be used in the construction of triple junctions and transition layer solutions with three dimensional target space $\mathbb{R}^3$ and two-dimensional domain space $\mathbb{R}^2$.

2.1. **Properties of the heteroclinic connections.** Let us start with a very general result:

Lemma 2.1. *Let W be a nonnegative function of class C^3 from $\mathbb{R}^d$ to $\mathbb{R}$ which satisfies the nondegeneracy condition* **(H2)** *and the coercivity assumption* **(H3)** . *Let z be a minimizer of $E_1(u)$ such that $z(-\infty)$ is equal to $\mathbf{a}$ and $z(\infty)$ is equal to $\mathbf{b}$. Then z cannot have a self-intersection. Moreover, if z_1 and z_2 are minimizers and there exist x_1 and x_2 such that $z_1(x_1)$ is equal to $z_2(x_2)$, then z_1 is equal to a translation of z_2.*

Proof. Assume that z is a minimizer which has a self-intersection, i.e. there exist x_1 and $x_2 > x_1$ such that $z(x_1) = z(x_2)$. Then we define a new minimizer by

$$z_1(x) = \begin{cases} z(x) & \text{if } x < x_1, \\ z(x - x_2 + x_1) & \text{if } x > x_2. \end{cases}$$

Figure **??** makes it clear that the energy of z_1 is strictly smaller than that of z, while the boundary conditions at infinity still hold for z_1. This contradicts the assumption that z was a minimizer.

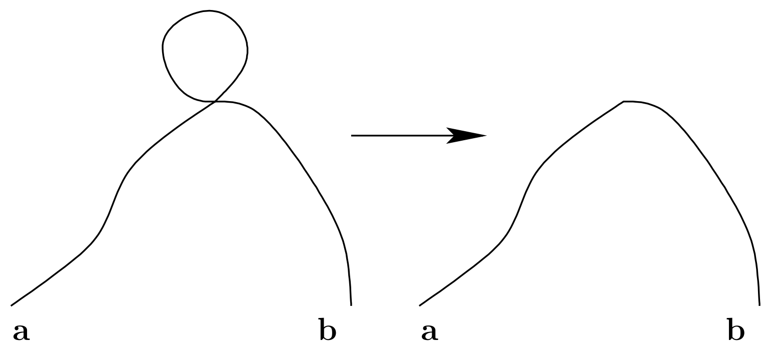

FIGURE 4. How to decrease the energy of a heteroclinic connection with a self-intersection.

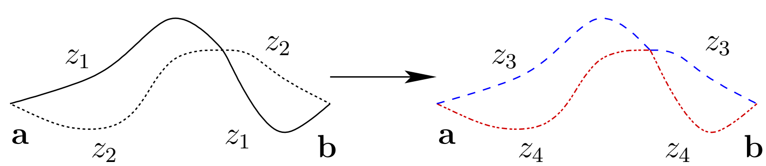

FIGURE 5. Two pairs of connections: z_1 as a solid line, z_2 as a dotted line, z_3 as a dashed line and z_4 as a dash-dot-dot line.

Assume now that z_1 and z_2 are two heteroclinic connections which intersect; and define

$$z_3 = \begin{cases} z_1(x) & \text{if } x \le x_1, \\ z_2(x - x_1 + x_2) & \text{if } x \ge x_1; \end{cases} \quad z_4(x) = \begin{cases} z_2(x) & \text{if } x \le x_2, \\ z_1(x - x_2 + x_1) & \text{if } x \ge x_2. \end{cases}$$

Figure **??** displays the geometrical situation described here. The sum of the energy of z_3 and z_4 is equal to $2e_1$. If the energy of one of them is strictly less than e_1 we get a contradiction. Therefore, z_3 and z_4 must be minimizers; thus they satisfy Euler-Lagrange equation, and the derivative of z_3 must be continuous at x_1. This implies immediately that z_3 is equal to both z_1 and z_2 by uniqueness of the solution of ordinary differential equations. □

This lemma has several corollaries:

Corollary 2.2. *Assume that W is Γ-invariant. Then any minimizer z of E_1 which tends to* **a** *at $-\infty$ and to* **b** *at $+\infty$ must be symmetric with respect to $\gamma_{\mathbf{ab}}$ after a proper translation, i.e., z satisfies*

$$\forall x \in \mathbb{R}, \quad \gamma_{\mathbf{ab}} z(x - x_0) = z(-x + x_0). \tag{2.1}$$

for some $x_0 \in \mathbb{R}$*.*

Proof. Observe that $z_1(x) = \gamma_{\mathbf{ab}} z(-x)$ is also a minimizer of E_1, with the same boundary data. By a connectivity argument, the orbit of z must cross the plane through 0, $\mathbf{c}$ and $\mathbf{d}$, for some value x_0 of x; this means that $z(x_0)$ and $z_1(-x_0)$ coincide, hence Lemma **??** gives the conclusion. □

Remark 2.3. The group $\Gamma(\mathbf{a}, \mathbf{b})$ operates on $\mathbb{R}$ according to the rule

$$\gamma_{\mathbf{ab}}(x) = -x.$$

With this convention, which we will use systematically, (**??**) can be rewritten as, modulo a translation,

$$\gamma_{\mathbf{ab}} z = z \circ \gamma_{\mathbf{ab}}, \tag{2.2}$$

i.e. z is $\Gamma(\mathbf{a}, \mathbf{b})$-equivariant.

Next corollary proves the alternative stated in the introduction:

Corollary 2.4. *Under the assumptions of Corollary* **??***, any minimizer of* E_1 *taking the value* $\mathbf{a}$ *at* $-\infty$ *and the value* $\mathbf{b}$ *at* $+\infty$ *satisfies the following alternative:*

- *either* $\gamma_{\mathbf{cd}} z = z$*,*
- *or* $z(x) - \gamma_{\mathbf{cd}} z(x)$ *never vanishes.*

Proof. Assume that there is a point such that

$$\gamma_{\mathbf{cd}} z(x_0) = z(x_0),$$

and define a new minimizer by

$$z_1(x) = \begin{cases} z(x) & \text{if } x \le x_0, \\ \gamma_{\mathbf{cd}} z(x) & \text{if } x \ge x_0. \end{cases}$$

As z_1 satisfies the Euler-Lagrange equation, this shows that the derivative of z_1 must be continuous at x_0, i.e.

$$\gamma_{\mathbf{cd}} z'(x_0) = z'(x_0),$$

which is equivalent to

$$z'(x_0) \perp \mathbf{d} - \mathbf{c}; \tag{2.3}$$

by construction, we have also

$$z(x_0) = \gamma_{\mathbf{cd}} z(x_0),$$

i.e.

$$z(x_0) \perp \mathbf{d} - \mathbf{c}. \tag{2.4}$$

Since W is invariant under $\gamma_{\mathbf{cd}}$, DW satisfies the relation

$$DW(\gamma_{\mathbf{cd}}u)\gamma_{\mathbf{cd}}v = DW(u)v, \quad \forall u, v \in \mathbb{R}^3$$

or equivalently

$$\gamma_{\mathbf{cd}} DW(\gamma_{\mathbf{cd}}u)^T = DW(u)^T, \quad \forall u \in \mathbb{R}^3. \tag{2.5}$$

Relation (**??**) implies that for all v orthogonal to $\mathbf{d}-\mathbf{c}$, $DW(v)^T$ is also orthogonal to $\mathbf{d}-\mathbf{c}$. Therefore, with initial conditions satisfying (**??**) and (**??**), the Euler-Lagrange equation (**??**) possesses a solution which is orthogonal to $\mathbf{d}-\mathbf{c}$, for all values of the argument. By uniqueness of the solution of Cauchy's problem, z must be orthogonal to $\mathbf{d}-\mathbf{c}$ for all values of its argument. This implies that $\gamma_{\mathbf{cd}}z = z$. This proves the corollary. □

The final result of this subsection describes the position of the minimizers relative to the planes $\mathcal{P}$**oad**, $\mathcal{P}$**obd**, $\mathcal{P}$**oac** and $\mathcal{P}$**obc**:

Corollary 2.5. *Under the assumptions of Corollary* **??**, *no minimizer of E_1 which tends to* **a** *at $-\infty$ and to* **b** *at $+\infty$ can take values in any of the planes $\mathcal{P}$**oad**, $\mathcal{P}$**obd**, $\mathcal{P}$**oac** and $\mathcal{P}$**obc**. In particular, the minimizer must belong entirely in the cone* **Coabc** *or* **Coabd**.

Proof. We shall prove that z does not intersect the plane $\mathcal{P}$**oac**. Assume the contrary, that $z(x_0)$ belongs to the plane $\mathcal{P}$**oac**, i.e.

$$z(x_0) \cdot (\mathbf{a} \times \mathbf{c}) = 0. \tag{2.6}$$

We define

$$z_2(x) = \begin{cases} z(x) & \text{if } x \geq x_0, \\ \gamma_{\mathbf{bd}} z(x) & \text{if } x \leq x_0. \end{cases}$$

It is easy too see that z_2 has same value at infinity as z since **a** is invariant under $\gamma_{\mathbf{bd}}$ and z_2 coincide with z for $x \geq x_0$. By the invariance of W under $\gamma_{\mathbf{bd}}$, we know $z_2(x)$ is also a minimizer of E_1 and hence satisfies (**??**). By the uniqueness of solutions to the initial value problem of ordinary equations, we have $z(x) = z_2(x)$ In particular, we have

$$z'(x_0) \cdot (\mathbf{a} \times \mathbf{c}) = 0. \tag{2.7}$$

As in the proof of Corollary **??**, the solution of the ordinary differential equation (**??**) with initial data satisfying (**??**) and (**??**) takes its values in the plane $\mathcal{P}$**oac**, which implies that z cannot tend to **b** as x tends to $+\infty$. This is a contradiction, which proves the corollary. □

The last useful property of z is its decay at infinity: by a classical stable manifold argument or arguments in [**?**], we see that there exist C_0 and κ_0 such that

$$|z_{\mathbf{ab}}(x) - \mathbf{a}| + |z'(x)| + |z''(x)| \le C_0 e^{-\kappa_0 |x|}, \quad \forall x < 0. \tag{2.8}$$

The square of κ_0 is an eigenvalue of $D^2W(\mathbf{a})$; therefore, in particular, we have

$$\kappa_0 \ge \nu. \tag{2.9}$$

where ν^2 is the infimum of the positive spectrum of the linearized operator A at z defined in (**??**).

2.2. **The operator A_L and its spectrum.** In this subsection and the next, we write for simplicity z instead of $z_{\mathbf{ab},\mathbf{c}}$ or $z_{\mathbf{ab}}$.

Lemma 2.6. *Define an operator A_L in $L^2(-L,L)^3$ by*

$$D(A_L) = \{v \in H^2(-L,L)^3 : \quad v'(\pm L) = 0, \quad v \circ \gamma_{\mathbf{ab}} = -\gamma_{\mathbf{ab}} v\},$$
$$A_L v = -v'' + D^2W(z)v.$$

As L tends to infinity, the infimum of the spectrum of A_L tends to ν^2.

Proof. Since A_L is self-adjoint, it suffices to prove that for any $\alpha \in (0,\nu^2)$ and all L large enough, A_L has no eigenvalues in $(-\infty,\alpha]$. Assume thus that for some $\beta \le \alpha$ there exists $v \not\equiv 0 \in L^2(-L,L)^3$ such that

$$\gamma_{\mathbf{ab}} v = -v \circ \gamma_{\mathbf{ab}}, \tag{2.10a}$$

$$-v'' + D^2W(z)v = \beta v, \tag{2.10b}$$

$$v'(\pm L) = 0. \tag{2.10c}$$

Without loss of generality, we may assume that the norm of v in $L^2(-L,L)^3$ is equal to 1; if the L^∞ norm of $D^2W(z)$ is defined as

$$\|D^2W(z)\|_{L^\infty} = \sup\{|D^2W(z(x))\mathbf{x}_1 \otimes \mathbf{x}_2| : x \in \mathbb{R}, |\mathbf{x}_1| = |\mathbf{x}_2| = 1\},$$

we infer from equation (**??**) that

$$\|v''\|_{L^2} \le \alpha + \|D^2W(z)\|_{L^\infty}.$$

We have the identity

$$\int_{-L}^{L} |v'|^2\,dx = -\int_{-L}^{L} v \cdot v''\,dx \le \alpha + \|D^2W(z)\|_{L^\infty}. \tag{2.11}$$

By the Sobolev Imbedding Theorem for one dimensional space, we have

$$\|v\|_{L^\infty} \le C_1. \tag{2.12}$$

Indeed, we can argue directly as follows. If $2L \geq 1/\sqrt{\alpha + \|D^2W(z)\|_{L^\infty}}$, there exists a point x_0 such that

$$|v(x_0)|^2 \leq \frac{\|v\|_{L^2}^2}{2L} \leq \sqrt{\alpha + \|D^2W(z)\|_{L^\infty}}.$$

Assume that the maximum of $|v(x)|$ in $[-L, L]$ is attained at x_1. Then

$$|v(x_1)|^2 - |v(x_0)|^2 = 2|\int_{x_0}^{x_1} v \cdot v' dx| \leq 2\|v\|_{L^2} \cdot \|v'\|_{L^2} \leq 2\sqrt{\alpha + \|D^2W(z)\|_{L^\infty}}.$$

Therefore we obtain the maximum norm estimate (**??**) with C_1 given below, which is independent of L:

$$C_1 = \sqrt{3}\big(\alpha + \|D^2W(z)\|_{L^\infty}\big)^{1/4}$$

We now choose $\eta > 0$ small enough so that $\lambda - \alpha - \eta > 0$ and define

$$\mu = \sqrt{2(\lambda_1 - \alpha - \eta)}.$$

We also choose l large enough so that

$$D^2W(z(x))v \otimes v \geq (\lambda_1 - \eta)v^2, \quad \forall x \geq l, \quad \forall v \in \mathbb{R}^2.$$

Define

$$g(x) = |v(x)|^2.$$

Thus, it is clear that g satisfies the ordinary differential equation

$$g''(x) = \mu^2 g(x) + f(x),$$

where, on the interval $[l, L]$, f is satisfies

$$f(x) = 2\,|v'(x)|^2 + 2D^2W(z(x))v(x) \otimes v(x) + 2(\alpha - \beta - \lambda_1 + \eta)|v(x)|^2 \geq 0.$$

Therefore, thanks to our choice of l and the fact that f is nonnegative, and we may estimate the L^1 norm of f over $[l, L]$ by

$$\|f\|_{L^1} \leq (\alpha + \|D^2W(z)\|_{L^\infty}) + \|D^2W(z)\|_{L^\infty}.$$

Note that the integral is only over $[l, L]$ and $\alpha - \beta - \lambda_1 + \eta < 0$. By the symmetry of v the estimates in all terms above are just half that of the whole interval.

We apply now Lemma **??** to the function $g(x) = |v(x)|^2$ and obtain the following estimate on $v(L)$ by the above L^1 estimate of f:

$$|v(L)| \leq C_2 e^{-\mu L}$$

where C_2 is defined by

$$C_2 = 2e^{\mu l}\big(C_1^2 + \rho(\alpha + 2\|D^2W(z)\|_{L^\infty})\big).$$

We extend now v to $\mathbb{R}$ by setting

$$v(x) = v(L)e^{-x+L} \text{ for } x \geq L,$$

and by assuming that $\gamma_{\mathbf{ab}} \circ v = v \circ \gamma_{\mathbf{ab}}$. Then we can calculate the linearized energy of v:

$$\int_{\mathbb{R}} \left[|v'|^2 + D^2W(z)v^{\otimes 2} \right] dx = \beta + 2\int_L^\infty \left[|v(L)|^2 + D^2W(z)v(L)^{\otimes 2} \right] e^{-2(x-L)}\, dx.$$

Thanks to the regularity assumption on W, there exists a constant C_3 such that for $|x| \geq L$:

$$|D^2W(z(x))v^{\otimes 2}| \leq \lambda_3(1 + C_3 e^{-\kappa_0 L})|v|^2.$$

By definition of ν, we must have

$$\nu^2 \left(1 + |v(L)|^2\right) \leq \alpha + |v(L)|^2 \left(1 + \lambda_3(1 + C_3 e^{-\kappa_0 L})\right). \tag{2.13}$$

For L large enough, relation (**??**) contradicts the assumption $\alpha < \nu^2$, proving thus the desired conclusion. □

2.3. **The minimum of the one-dimensional energy on a finite interval.** We will study now minimizers of the one-dimensional energy on $[-L, L]$, with Neumann boundary conditions. The idea is very similar to Theorem 3.2 in [**?**]. The discussion here, however, emphasizes the difference in details. First, it is plain that there exists a constant C_4 such that

$$\begin{aligned} E_1(z; (-L, L)) &\stackrel{\text{def}}{=} \int_{-L}^{L} \mathcal{L}_1(z)\, dx \\ &= e_1 - 2\int_L^\infty \left[\frac{|z'|^2}{2} + W(z) \right] dx \leq e_1 - C_4 e^{-2\kappa_0 L}, \end{aligned}$$

thanks to estimate (**??**).

We first prove that there exists a nontrivial minimizer of the one dimensional energy over $[-L, L]$ under appropriate boundary conditions. We denote below $1_{(-L,L)}$ the characteristic function on $(-L, L)$.

Lemma 2.7. *For all L large enough, there exists a nontrivial minimizer z_L of $E_1(\cdot; (-L, L))$; moreover, the maximum norm of $z_L - z_{\mathbf{ab},\mathbf{c}} 1_{(-L,L)}$ or the maximum norm of $z_L - z_{\mathbf{ab},\mathbf{d}} 1_{(-L,L)}$ tends to 0 as L tends to infinity.*

Proof. We take a minimizing sequence $\{z_n\}_n$ for the functional $E_1(\cdot; (-L, L))$ under the conditions

$$\gamma_{\mathbf{ab}} z_n = z_n \circ \gamma_{\mathbf{ab}} \text{ and } \min\bigl(|z_n(x) - \mathbf{c}|, |z_n(x) - \mathbf{d}|\bigr) \geq \delta, \quad \forall x \in [-L, L],$$

where $\delta > 0$ is a fixed sufficiently small constant. For all large enough n, we see that

$$E(z_n; (-L, L)) \leq e_1.$$

We extract from the sequence $\{z_n\}_n$ a subsequence that converges in the weak topology of $H^1(-L, L)^3$; let z_L be the limit of this sequence; then

$$\begin{aligned} E_1(z_L; (-L, L)) = \inf\Big\{ & E_1(u; (-L, L)) : \gamma_{\mathbf{ab}} u = u \circ \gamma_{\mathbf{ab}}, \text{and} \\ & \min\big(|u(x) - \mathbf{c}|, |u(x) - \mathbf{d}|\big) \geq \delta, \quad \forall x \in [-L, L]\Big\} \\ \leq & \; e_1 - C_4 e^{-2\kappa_0 L}. \end{aligned}$$

We have now to describe precisely the behavior of z_L; it is immediate that

$$\text{meas}\{x \in (-L, L) : W(z_L(x)) \geq \beta\} \leq e_1/\beta;$$

therefore, the number x_+ defined by

$$x_+ = \inf\{x > 0 : W(x) \leq \beta\}$$

where β will be chosen later, satisfies the inequality

$$x_+ \leq e_1/2\beta.$$

Then $z_L(x_+)$ must be close to one of the points $\mathbf{a}$ or $\mathbf{b}$ if β is chosen sufficiently small. More precisely, let

$$\varepsilon(\beta) = \max\{|v - \mathbf{b}| : \quad W(v) \leq \beta, \quad |v - \mathbf{b}| \leq \delta\};$$

it is plain that $\varepsilon(\beta) = O(\sqrt{\beta})$ and that the distance of $z_L(x_+)$ to $\mathbf{a}$ or $\mathbf{b}$ is at most equal to $\varepsilon(\beta)$.

Assume first that $z_L(x_+)$ is close to $\mathbf{b}$. Then, the energy $E_1(z_L, (x_+, L))$ can be estimated from above by using the test function

$$y(x) = \mathbf{b} + (z_L(x_+) - \mathbf{b}) \exp(-\sqrt{\lambda_1}(x - x_+)).$$

Let us define

$$\eta = |z_L(x_+) - \mathbf{b}|.$$

Then

$$\begin{aligned} & E_1(y; (x_+, L)) \\ & \quad \leq \int_{x_+}^{L} \Big(\frac{\lambda_1}{2} + \frac{\lambda_3(1 + C_3\eta)}{2}\Big) \eta^2 e^{-2\sqrt{\lambda_1}(x - x_+)}\, dx \\ & \quad \leq \frac{\lambda_1 + \lambda_3(1 + C_3\eta)}{4\sqrt{\lambda_1}} \eta^2. \end{aligned}$$

The comparison is justified since $\min(|y(x) - \mathbf{c}|, |y(x) - \mathbf{d}|) \geq \delta$ for $x \in [x_+, L]$. Thus, we shall have

$$E_1(z_L, (x_+, L)) \leq \frac{\lambda_1 + \lambda_3(1 + C_3\eta)}{4\sqrt{\lambda_1}} \eta^2. \tag{2.14}$$

On the other hand, (**??**) implies a bound on $|z_L - \mathbf{b}|$, uniformly on $[x_+, L]$, provided that $z(x_+) - \mathbf{b}$ is small enough. Indeed, let m be a positive number such that

$$(m^2 - 1)\sqrt{\lambda_1} \geq \frac{\lambda_1 + \lambda_3}{2\sqrt{\lambda_1}}. \tag{2.15}$$

Assume that there is a value $x > x_+$ such that $|z_L(x) - \mathbf{b}| \geq m|z_L(x_+) - \mathbf{b}|$; then, we can find a maximal interval $[x', x'']$ such that

$$|z_L(x') - \mathbf{b}| = \eta, \quad |z_L(x'') - \mathbf{b}| = m\eta,$$
$$\eta \leq |z_L(x) - \mathbf{b}| \leq m\eta, \quad \forall x \in [x', x''].$$

It suffices to choose

$x'' = \inf\{x \geq x_+ : |z_L(x) - \mathbf{b}| \geq m\eta\}$ and $x' = \sup\{x \leq x'' : |z_L(x) - \mathbf{b}| \leq \eta\}$.

Then, we may estimate from below the energy on (x', x''): there exists a constant C_5 such that

$$|v - \mathbf{b}| \leq m\eta \Longrightarrow W(v) \geq \frac{\lambda_1 |v - \mathbf{b}|^2 (1 - C_5 m\eta)}{2};$$

then, we may write

$$\begin{aligned} E(z_L; (x', x'')) &\geq \int_{x'}^{x''} \frac{|z_L'|^2 + \lambda_1(1 - C_5 m\eta)|z_L - \mathbf{b}|^2}{2}\, dx \\ &\geq \int_{x'}^{x''} |z_L'||z_L - \mathbf{b}|\sqrt{\lambda_1(1 - C_5 m\eta)}\, dx \\ &\geq \frac{(m^2 - 1)\eta^2 \sqrt{\lambda_1(1 - C_5 m\eta)}}{2}. \end{aligned}$$

However, under condition (**??**), for η small enough, we get a contradiction, and we have proved that if we choose β small enough, then for all $x \geq x_+$,

$$|z_L(x) - \mathbf{b}| \leq m|z_L(x_+) - \mathbf{b}|. \tag{2.16}$$

Now we claim that

$$\min\big\{|z_L(x) - \mathbf{c}|, |z_L(x) - \mathbf{d}|, \quad x \in [-L, L]\big\} > \delta$$

for L sufficiently large. Suppose this is not true, without loss of generality we assume that there exists $x_0 > 0$ such that $|z_L(x_0) - \mathbf{c}| = \delta$. Now define a test function

$$h(x) = \begin{cases} \mathbf{c} + (z_L(x_0) - \mathbf{c})e^{\sqrt{\lambda_1}(x - x_0)}, & x < x_0 \\ z_L(x), & x \in [x_0, x_+], \\ \mathbf{b} + (z_L(x_+) - \mathbf{b})e^{-\sqrt{\lambda_1}(x - x_+)}, & x > x_+. \end{cases}$$

As in the proof of (**??**), we obtain

$$E_1(h;[x_+,\infty)) \le C|z_L(x_+)-\mathbf{b}|^2,$$

and

$$E_1(h;(-\infty,x_0]) \le C|z_L(x_0)-\mathbf{c}|^2,$$

for some positive constant C.

Therefore

$$\begin{aligned} E_1(z_L,[x_0,x_+]) =& E_1(h;\mathbb{R}) - E_1(h;[x_+,\infty)) - E_1(h;(-\infty,x_0])) \\ &\ge e_1 - C|z_L(x_+)-\mathbf{b}|^2 - C|z_L(x_0)-\mathbf{c}|^2, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} E_1(z_L,[-L,L]) &\ge 2E_1(z_L,[x_0,x_+]) \\ &\ge 2e_1 - 2C|z_L(x_+)-\mathbf{b}|^2 - 2C|z_L(x_0)-\mathbf{c}|^2 \\ &\ge 2e_1 - 2C(\epsilon(\beta)^2+\delta^2) \ge \frac{3}{2}e_1 \end{aligned} \tag{2.18}$$

for β and δ sufficiently small and L sufficiently large.

This is a contradiction which proves

$$\min\{|z_L(x)-\mathbf{c}|, \quad x\in[-L,L]\} > \delta.$$

Similarly, we can prove

$$\min\{|z_L(x)-\mathbf{d}|, \quad x\in[-L,L]\} > \delta.$$

Therefore the claim is proven.

In summary, if $\delta > 0$ is sufficiently small, for L large enough any minimizer of $E_1(\cdot;(-L,L))$ under conditions

$$\min\{|z(x)-\mathbf{c}|,|z(x)-\mathbf{d}|\} \ge \delta, \quad \forall x\in[-L,L], \quad \text{and}\, \gamma_{\mathbf{ab}}z = z\circ\gamma_{\mathbf{ab}}$$

must not saturate the constraints and therefore satisfies Neumann boundary conditions and the Euler-Lagrange equation

$$-z_L'' + DW(z_L)^T = 0.$$

Let us extend z_L to all of $\mathbb{R}$ by letting

$$z_L(x) = \mathbf{b} + (z_L(L)-\mathbf{b})e^{-(x-L)\sqrt{\lambda_1}} \text{ for } x\ge L, \tag{2.19}$$

keeping the symmetry condition $\gamma_{\mathbf{ab}}\circ z_L = z_L\circ\gamma_{\mathbf{ab}}$. It is plain from the above analysis that $z_L(L)-\mathbf{b}$ tends to 0 as L tends to infinity; the extended z_L verifies

$$e_1 \le E_1(z_L) \le e_1 + C_4e^{-2\kappa_0 L} + O(|z_L(L)-\mathbf{b}|^2).$$

Therefore, the sequence $(z_L)_L$ is a minimizing sequence for the energy E_1; the proof of estimate (**??**) clearly extends to (x_+,∞); it is therefore possible to extract from the sequence of the z_L an uniformly convergent subsequence whose derivative converges strongly in $L^2(\mathbb{R})^3$; the limit of this subsequence is either $z_{\mathbf{ab},\mathbf{c}}$ or $z_{\mathbf{ab},\mathbf{d}}$.

If $z_L(x_+)$ is close to $\mathbf{a}$, we flip z_L by $\gamma_{\mathbf{ab}}$ and the conclusion still holds.

This proves the lemma. □

Remark 2.8. In view of corollary **??** and the convergence of z_L, we note that δ in the proof can indeed be chosen as any positive number smaller than $1/2$.

It is basically a corollary of Lemmas **??** and **??** that $z_L - z_{\mathbf{ab},\mathbf{c}} 1_{[-L,L]}$ or $z_L - z_{\mathbf{ab},\mathbf{d}} 1_{[-L,L]}$ converges exponentially fast to 0 in $H^2(-L,L)^3$ norm as well as in the maximum norm, and the energy of z_L tends to e_1 exponentially fast:

Theorem 2.9. *Assume that z_L, extended according to (**??**), converges to $z = z_{\mathbf{ab},\mathbf{c}}$, then there exists a constant C_6 such that*

(2.20)
$$\|z_L - z1_{[-L,L]}\|_{L^2} \le C_6 e^{-\kappa_0 L} \text{ and } E_1(z_L, [-L,L]) \ge e_1 - C_6 e^{-2\kappa_0 L}.$$

Consequently, the L^2 norm in the above estimate can be replaced by H^2 norm or the maximum norm L^∞.

Proof. Define a function $\tilde{z}_L$ such that $\gamma_{\mathbf{ab}} \circ \tilde{z}_L(x) = \tilde{z}_L(-x)$ by

$$\tilde{z}_L(x) = z(x) - z'(L)\chi(L-x) \quad \text{for } 0 \le x \le L,$$

where χ is a smooth cut-off function with support in $(-\infty, 1]$ and satisfying $\chi'(0) = -1$. The function $\tilde{z}_L$ satisfies Neumann boundary conditions at $\pm L$ and the ordinary differential equation

(2.21)
$$\begin{aligned} &- \tilde{z}_L'' + DW(\tilde{z}_L)^T \\ &= -z'(L)\chi''(L-x) - z'(-L)\chi''(x+L) + DW(\tilde{z}_L)^T - DW(z_L)^T. \end{aligned}$$

Define ρ to be the right-hand side of (**??**). Thanks to (**??**), we have the estimate

$$\|\rho\|_{L^2} \le C_7 e^{-\kappa_0 L}.$$

We define $y = z_L - \tilde{z}_L$; it satisfies Neumann boundary conditions at $x = -L, L$ and the ordinary differential equation

$$-y'' + DW(z_L)^T - DW(\tilde{z}_L)^T = -\rho; \tag{2.22}$$

rewrite (**??**) so as to make visible the operator A_L:

$$A_L y + DW(z_L)^T - DW(\tilde{z}_L)^T - D^2W(\tilde{z})y = -\rho. \tag{2.23}$$

We multiply (**??**) by y and integrate over $[-L, L]$; the expression $DW(z_L) - DW(\tilde{z}_L) - D^2W(z)(z_L - \tilde{z}_L)$ is a quadratic term that can be rewritten

$$\int_0^1 \big(D^2W(\tilde{z}_L+ty)-D^2W(z)\big)y\,dt = \int_0^1\int_0^1 D^3W(z+s(\tilde{z}_L+ty))y\otimes y\,dtds;$$

therefore, there exists a constant C_7 such that

$$\|DW(z_L)^T - DW(\tilde{z}_L)^T - D^2W(\tilde{z})y\|_{L^2} \leq C_8\|y\|_{L^\infty}\|y\|_{L^2}. \tag{2.24}$$

For L large enough, the lower bound of the spectrum of A_L is at least equal to $\nu^2/2$; therefore

$$\left(\frac{\nu^2}{2} - C_8\|y\|_{L^\infty}\right)\|y\|_{L^2} \leq |\rho|_{L^2}.$$

Since $z_L - z1_{[-L,L]}$ converges to 0 in maximum norm, it is now clear that the first estimate in (**??**) holds; the second estimate can be immediately inferred from the following facts: $\|y''\|_{L^2}$ satisfies the same exponential estimate as y; therefore, $z_L - z1_{[-L,L]}$ converges to zero exponentially fast in maximum norm; the second estimate of (**??**) is obtained by a Taylor expansion of the energy, using the assumption that W is of class C^3:

$$\begin{aligned} E_1(z_L, [-L,L]) = \int_{-L}^{L}\Big[&\frac{|\tilde{z}_L'|^2 + 2\tilde{z}_L'\cdot(z_L' - \tilde{z}_L') + |z_L'|^2}{2} \\ &+ W(\tilde{z}_L) + DW(\tilde{z}_L)(z_l - \tilde{z}_L) + \frac{D^2W(\tilde{z}_L)(z_L - \tilde{z}_L)^{\otimes 2}}{2} \\ &+ O(|z_L - \tilde{z}_L|^3)\Big]\,dx. \end{aligned}$$

The term of order 0 is $E_1(z, [-L, L])$, up to the correction due to χ for $L - 1 \leq |x| \leq L$; thus it is of order $e^{-2\kappa_0 L}$; the term of order 1 is integrated by parts, and it vanishes, up to the correction due to χ, and it is equal to

$$2\int_{L-1}^{L}(-\tilde{z}_L'' + DW(\tilde{z}_L))(z_L - \tilde{z}_L)\,dx$$

which is also of order $e^{-2\kappa_0 L}$. The term of order 2 is quadratic in y; therefore it is of the same order $e^{-2\kappa_0 L}$; finally, the term of order 3 is negligible relative to the other ones.

□

Remark 2.10. The energy estimate in the above theorem does not need hypothesis **(H4)**. The reader may follow the proof of Theorem 2.8 and Theorem 3.3 in [**?**] to obtain it directly.

2.4. **One dimensional connections with end points constrained on planes.** In this subsection, we are concerned with minimization problems of E_1 in interval $[-L, L]$ with boundary values constrained on the two planes of symmetry. To be more precise, we define

$$\mathcal{H}_L := \big\{u \in H^1([-L, L])^3 : \gamma_{\mathbf{ab}} u = u \circ \gamma_{\mathbf{ab}} \quad \text{and } u(-L) \in \mathcal{P}\mathbf{oad}\big\}.$$

For any given $0 < \delta < 1/2$ we consider

$$\min\big\{E_1(u, [-L, L]) : u \in \mathcal{H}_L, \quad |u(x) - \mathbf{d}| \geq \delta, \ \forall x \in [-L, L]\big\}. \tag{2.25}$$

This minimization problem is the three dimensional counterpart of problem (3.4) in [**?**]. Because of the extra well $\mathbf{d}$ of W, we have to impose the extra constraint $|u(x) - \mathbf{d}| \geq \delta, \forall x \in [-L, L]$. We shall prove the following theorem, which is the counterpart of Theorem 3.6 in [**?**].

Theorem 2.11. *There exists a solution $z^s_{L,\delta}$ to the minimization problem (**??**). The solution is independent of δ when L is large enough, and therefore may be written as z^s_L. Moreover, z^s_L converges to either $z_{\mathbf{ab},\mathbf{c}}$ or $z_{\mathbf{ab},\mathbf{d}}$ (denoted by z), and*

$$\begin{aligned} &\|z^s_L - z 1_{[-L,L]}\| \leq C e^{-\kappa_0 L} \text{ and} \\ &|E_1(z^s_L, [-L, L]) - e_1| < C e^{-2\kappa_0 L} \end{aligned} \tag{2.26}$$

for some positive constant C.

Proof. We shall prove the theorem by reducing (**??**) to an equivalent problem with one more constraint $|u(x) - \mathbf{c}| \geq \delta, \forall x \in [-L, L]$. This can be done with the following transformation. For any $z \in \mathcal{H}_L$ we define a transformation

$$\tilde{z}(x) := \begin{cases} \gamma_{\mathbf{cb}} z(L - x), & \text{if } x \in [0, L]; \\ \gamma_{\mathbf{ca}} z(-L - x), & \text{if } x \in [-L, 0]. \end{cases} \tag{2.27}$$

See Figure **??** for an illustration of the transformation.

Since $\gamma_{\mathbf{bc}} z(-L) = z(-L), \gamma_{\mathbf{ac}} z(L) = z(L)$, we obtain $\gamma_{\mathbf{ac}} z(-L) = \gamma_{\mathbf{bc}} z(L)$ using the geometry of the equilateral triangle. Hence $\tilde{z}$ is continuous at $x = 0$, and consequently belongs to $\mathcal{H}_L$.

It is easy to see that $E_1(\tilde{z}) = E_1(z)$. By using a proper test function, we know that the minimum energy in (**??**) is less than $e_1 + Ce^{-2\kappa_0 L}$ for some constant C. For any minimizing sequence $\{z_n\}$ of (**??**), we have $E_1(z_n, [-L, L]) < 3e_1/2$ for sufficiently large n and L. Therefore, if $\delta > 0$ is sufficiently small, following the proof of (**??**) we have either $\min(|z_n - \mathbf{b}|, x \in [0, L]) > \delta$, or $\min(|z_n - \mathbf{c}|, x \in [0, L]) > \delta$. (Since otherwise the energy would be bigger than $3e_1/2$.) If the latter is not true, the former must hold. We replace z_n by $\tilde{z}_n$, and then it

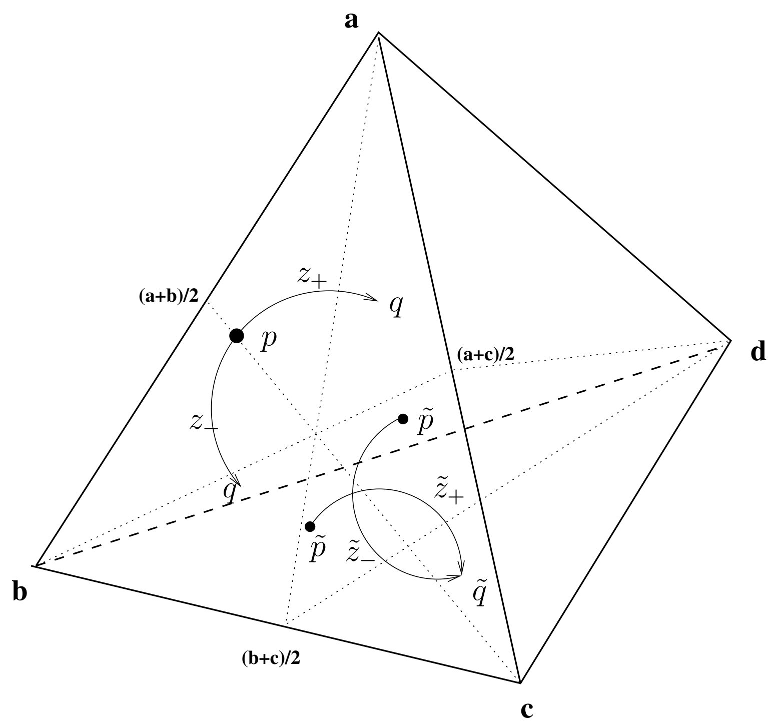

FIGURE 6. z_- denotes the curve $z(x), -L < x < 0$, while z_+ denotes the curve $z(x), 0 < x < L$. The point p is mapped to $\tilde{p}$, the points $\tilde{q}$ are mapped to q.

becomes $\min(|\tilde{z}_n - \mathbf{c}|, x \in [0, L]) > \delta$. Therefore, there always exists a minimizing sequence $\{z_n\}$ for (**??**) such that $\min(|z_n - \mathbf{c}|, |z_n - \mathbf{d}|, x \in [-L, L]) > \delta$ for L, n sufficiently large. With the extra constraint, the rest of the proof then follows exactly that of Lemma **??** and Theorem **??**.

□

We note that $\tilde{z}^s_L$ is also a minimizer to (**??**). Similar to Propositions 3.8 and 3.9 in [**?**], we have

Corollary 2.12. *The there exists a constant $\beta > 0$ such that*

$$|z^s_L(x) - \tilde{z}^s_L(x)| \geq \beta, \quad \forall x \in [-L, L] \tag{2.28}$$

for sufficiently large L.

Corollary 2.13. *There exists a function $\gamma(\epsilon, L)$, which tends to 0 as $\epsilon \to 0, L \to \infty$, such that for any $v \in \mathcal{H}_L$ satisfying $|v(x) - \mathbf{d}| \geq \delta$, $\forall x \in [-L, L]$, and*

$$E_1(v, [-L, L]) \leq e_1 + \epsilon$$

we have either

$$||v - z_L^s||_{C([-L,L])} \le \gamma(\epsilon, L) \quad or$$
$$||v - \tilde{z}_L^s||_{C([-L,L])} \le \gamma(\epsilon, L). \tag{2.29}$$

3. Triple junction and transitional layer solutions

In this section, we shall construct a triple junction solution on entire plane $\mathbb{R}^2$ with target space Er^3. We shall describe very precisely the asymptotic behavior of this triple junction solution at infinity with three phases $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The domain space may be regarded as the plane spanned by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ which are indeed in the target space, and therefore we may use $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{c})$ for group actions in both the target and domain spaces without distinguishing them. We then use the asymptotic behavior to obtain an estimate of the energy of the triple junction solution restricted to large equilateral triangles. In similar manner, we shall obtain an energy lower bound estimate for $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{c})$- equivariant functions in large equilateral triangles provided that the functions are away from $\mathbf{d}$. Another interesting solutions defined in entire $\mathbb{R}^2$ are transitional layer solutions, which connect two heteroclinic solutions in x_1 variable along the direction of x_2. This type of solutions appears in the quadruple junction structure in a subtle but essential way. We shall study the asymptotic behavior and energy estimates of such solutions as well.

3.1. **The triple junction solutions in entire space and its behavior at infinity.** The construction of the triple junction in full space follows closely the strategy developed in [**?**]. Therefore, we give details only in the places where the difference is significant.

We define a $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{c})$-equivariant covering of $\mathbb{R}^2$ as follows: we let $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ be respectively the orthogonal projection of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ on the plane through 0 which is parallel to $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$. The region $\mathcal{B}_{\mathbf{ab}}$ is the truncated cone defined by

$$\mathcal{B}_{\mathbf{ab}} = \left\{ x \in \bigcup_{\lambda \ge 0} \lambda[\mathbf{a}', \mathbf{b}'] : \, x \cdot (\mathbf{a}' + \mathbf{b}') \ge 1 \right\}.$$

The region $\mathcal{B}^+_{\mathbf{ab}}$ is defined for some small $\alpha > 0$ by

$$\mathcal{B}^+_{\mathbf{ab}} = \{x \in \mathbb{R}^2 : d(x, \mathcal{B}_{\mathbf{ab}})\} \le \alpha.$$

The regions $\mathcal{B}_{\mathbf{ab}}$ and $\mathcal{B}^+_{\mathbf{ab}}$ are pictured in Figure **??**.

The group $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{c})$ operates in a straightforward fashion on $\mathbb{R}^2$: for instance $\gamma_{\mathbf{ab}}$ operates as the reflection exchanging $\mathbf{a}'$ and $\mathbf{b}'$, and keeping $\mathbf{c}'$ invariant, and the composition $\gamma_{\mathbf{cb}}\gamma_{\mathbf{ab}}$ is the rotation mapping $\mathbf{a}'$

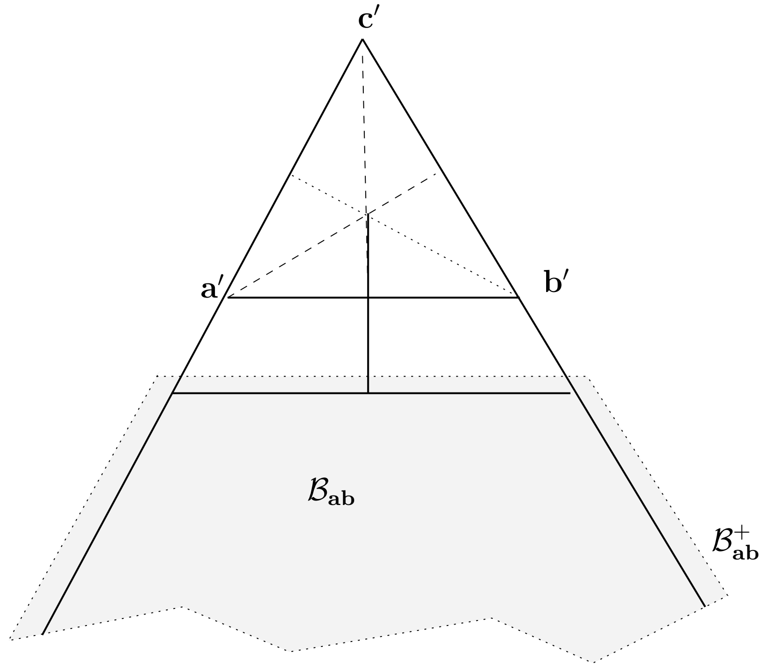

FIGURE 7. The regions $\mathcal{B}_{\mathbf{ab}}$ and $\mathcal{B}^+_{\mathbf{ab}}$.

to $\mathbf{c}'$, $\mathbf{c}'$ to $\mathbf{b}'$ and $\mathbf{b}'$ to $\mathbf{c}'$. Thus we define

$$\mathcal{B}_{\mathbf{bc}} = \gamma_{\mathbf{ac}}\mathcal{B}_{\mathbf{ab}}, \quad \mathcal{B}_{\mathbf{ca}} = \gamma_{\mathbf{bc}}\mathcal{B}_{\mathbf{ab}}.$$

Finally we define the region

$$\mathcal{S} = \{x \in \mathbb{R}^2 :\, x \cdot (\mathbf{a}' + \mathbf{b}') \leq 1,\, x \cdot (\mathbf{b}' + \mathbf{c}') \leq 1,\, x \cdot (\mathbf{c}' + \mathbf{a}') \leq 1\},$$

and

$$\mathcal{S}^+ = \{x \in \mathbb{R}^2 : d(x, \mathcal{S}) \leq \alpha\}.$$

It is not difficult to produce a partition of the unity $\{\phi_{\mathbf{ab}}, \phi_{\mathbf{bc}}, \phi_{\mathbf{ca}}, \phi_0\}$ which has the following properties:

$$\text{supp } \phi_{\mathbf{xy}} \subset \mathcal{B}^+_{\mathbf{xy}}, \quad \text{supp } \phi_0 \subset \mathcal{S}^+;$$

$$\phi_{\mathbf{xy}} \circ \gamma_{\mathbf{xy}} = \phi_{\mathbf{xy}} \stackrel{\text{def}}{=} \phi_{\mathbf{yx}}, \quad \phi_{\mathbf{xy}} \circ \gamma_{\mathbf{yz}} = \phi_{\mathbf{xy}}$$

With these definitions, we are now able to define a test function

$$\begin{aligned} u_0(x) = {} & \phi_{\mathbf{ab}}(x) z_{\mathbf{ab},\mathbf{d}} \left(\frac{x \cdot (\mathbf{b} - \mathbf{a})}{|\mathbf{b} - \mathbf{a}|} \right) + \phi_{\mathbf{bc}}(x) z_{\mathbf{bc},\mathbf{d}} \left(\frac{x \cdot (\mathbf{c} - \mathbf{b})}{|\mathbf{c} - \mathbf{b}|} \right) \\ & + \phi_{\mathbf{ca}}(x) z_{\mathbf{ca},\mathbf{d}} \left(\frac{x \cdot (\mathbf{a} - \mathbf{c})}{|\mathbf{a} - \mathbf{c}|} \right) + \phi_0(x)\rho(x), \quad x \in \mathbb{R}^2 \end{aligned} \tag{3.1}$$

where ρ is any $\Gamma(\mathbf{a},\mathbf{b},\mathbf{c})$-equivariant smooth function from $\mathbb{R}^2$ to $\mathbb{R}^3$ that we care to choose. Note $\mathbf{x}'-\mathbf{y}'=\mathbf{x}-\mathbf{y}$ for $\mathbf{x},\mathbf{y}\in\{\mathbf{a},\mathbf{b},\mathbf{c}\}$.

We also define an equilateral triangle T_L of edge length $2L$ as

$$T_L=\{x\in\mathbb{R}^2:\, x\cdot(\mathbf{a}'+\mathbf{b}')\le\frac{L}{3},\, x\cdot(\mathbf{b}'+\mathbf{c}')\le\frac{L}{3},\, x\cdot(\mathbf{c}'+\mathbf{a}')\le\frac{L}{3}\},$$

A straightforward calculation gives

$$E_2(u_0,T_L)\le C_0+Le_1\sqrt{3}. \tag{3.2}$$

The construction of the triple junction proceeds as follows:

Step 1: We minimize the energy

$$E_2(u,T_L)=\int_{T_L}\mathcal{L}_2(u)\,dx$$

over the set of functions u from $\mathbb{R}^2$ to $\mathbb{R}^3$ which are $\Gamma(\mathbf{a},\mathbf{b},\mathbf{c})$-equivariant and coincide with u_0 over ∂T_L. The minimizer u_L satisfies

$$E_2(u_L,T_L)\le C_0+Le_1\sqrt{3}. \tag{3.3}$$

Step 2: For fixed large enough M and $L\ge M$, we prove the following estimate from below on the energy in a triangular annulus:

$$E_2(u_L,T_L\setminus T_M)\ge(L-M)e_1\sqrt{3}-C_1; \tag{3.4}$$

Step 3: Obtain the following energy estimate for $E_2(u_L,T_M)$

$$E_2(u_L,T_M)\le e_1\sqrt{3}M+C; \tag{3.5}$$

where the constant C is independent of $L\ge M$, and use it to show the existence and asymptotic behavior of a triple junction solution.

Step 1 is performed exactly as in [**?**]; Step 2 is more delicate compared to the case in [**?**]. Below we show the proof in details. For convenience, we may choose the coordinates of $\mathbb{R}^2$ so that

$$\mathbf{a}'=(-1,1/\sqrt{3}),\ \mathbf{b}'=(1,1/\sqrt{3}),\ \mathbf{c}'=(0,-2/\sqrt{3}),$$

and concentrate on the section S_L of T_L with

$$S_L=\{x=(x_1,x_2)\in T_L:-\sqrt{3}x_2\le x_1\le\sqrt{3}x_2,\quad x_2>0\}. \tag{3.6}$$

For $0<\epsilon<1$, we define

$$\mathcal{M}=\left\{x_2>0:\ E_1\left(u_L(\cdot,x_2),[-\sqrt{3}x_2,\sqrt{3}x_2]\right)\ge(2-\epsilon)e_1.\right\}$$

We have an obvious estimate of the Lesbegue measure of $\mathcal{M}$ as below

$$m(\mathcal{M})\le\frac{\sqrt{3}L}{3(2-\epsilon)}+C. \tag{3.7}$$

Following the proof of (**??**), it is easy to see that there exists $\delta > 0$ such that

$$E_1(v(x_1)) \geq (2-\epsilon)e_1, \quad v \in H^1([a,b])^3 \tag{3.8}$$

whenever $d(v, \mathbf{x}) \leq \delta$ for at least three different points $\mathbf{x}$ in $\mathbf{X} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$. Therefore, for any $x_2 \notin \mathcal{M}$ the image of $u_L(x_1, x_2)$ cannot be within δ distance of three different points in $\mathbf{X}$. Furthermore, the Lesbegue measure of the set where $u_L(x_1, x_2)$ is within δ distance of $\mathbf{X}$ must be large in the following sense

$$m\big\{x_1 \in [-\sqrt{3}x_2, \sqrt{3}x_2] : \quad d\left(u_L(x_1, x_2), \mathbf{X}\right) \leq \delta\big\} \geq 2\sqrt{3}x_2 - \frac{C}{\delta^2}$$

for some constant $C > 0$ independent of L and δ.

Now we define

$$\mathcal{N} = \big\{x_2 : d\left(u_L(x_1, x_2), \mathbf{d}\right) \geq \delta, \, \forall x_1 \in [-\sqrt{3}x_2, \sqrt{3}x_2]\big\}\big\}.$$

By (**??**) in Theorem **??**, we know that for $x_2 \in \mathcal{N}$

$$E_1\big(u_L(\cdot, x_2), [-\sqrt{3}x_2, \sqrt{3}x_2]\big) \geq e_1 - C_\delta e^{-2\kappa_0\sqrt{3}x_2} \tag{3.9}$$

We define

$$\tau = \sup\{\mathcal{M}^c \cap \mathcal{N}^c\}.$$

We shall prove that τ has an upper bound independent of L.

For any $x_2 > \tau$, we have $x_2 \in \mathcal{M} \cup \mathcal{N}$ and hence (**??**) holds.

Then, by symmetry,

$$\begin{aligned}
\frac{1}{3}E_2(u_L, T_L) = & \int_0^{\frac{\sqrt{3}}{3}L} E_1(u_L(\cdot, x_2), [-\sqrt{3}x_2, \sqrt{3}x_2])dx_2 \\
\geq & (2-\epsilon)e_1 m(\mathcal{M}) + \int_{x_2 \in \mathcal{N} \setminus \mathcal{M}} E_1(u_L(\cdot, x_2), [-\sqrt{3}x_2, \sqrt{3}x_2])dx_2 \\
\geq & (2-\epsilon)e_1 m(\mathcal{M}) + (\frac{\sqrt{3}}{3}L - \tau - m(\mathcal{M}))e_1 - C.
\end{aligned}$$

In view of (**??**), we have

$$(2-\epsilon)m(\mathcal{M}) + \big(\frac{\sqrt{3}}{3}L - \tau - m(\mathcal{M})\big) - C \leq \frac{\sqrt{3}}{3}L + C. \tag{3.10}$$

Hence

$$m(\mathcal{M}) \leq \frac{\tau}{1-\epsilon} + C. \tag{3.11}$$

We claim that there exists $\tau_1 \in \mathcal{N} \setminus \mathcal{M}$ such that

$$\tau < \tau_1 \leq \frac{2-\epsilon}{1-\epsilon}\tau + C. \tag{3.12}$$

This is obviously true if

$$\frac{L}{\sqrt{3}} - \tau > \frac{\tau}{1-\epsilon} + C. \tag{3.13}$$

If (**??**) is not true, we just choose τ_1 to be $\frac{L}{\sqrt{3}}$ thanks to the Dirichlet boundary condition of u_L which is given by (**??**). Hence (**??**) still holds. Note that for $x_2 = \tau_1$ we have

$$\text{meas}\Big(\big\{x_1 : d\left(u_L(x_1, \tau_1) - \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}\right) < \delta\big\}\Big) \geq 2\sqrt{3}\tau_1 - \frac{C}{\delta^2}(2-\epsilon)e_1.$$

On the other hand, u_L must be close to $\mathbf{d}$ in a significant set when $x_2 = \tau$, thanks to the energy estimates. To be more precise, we have the estimate of the measure of the set as below

$$\text{meas}(\{x_1 : |u_L(x_1, \tau) - \mathbf{d}| \leq \delta\}) \geq 2\sqrt{3}\tau - \frac{C}{\delta^2}(2-\epsilon)e_1.$$

Now we estimate the energy of u_L in $T_{\sqrt{3}\tau_1} \setminus T_{\sqrt{3}\tau}$ in two ways. Integrating in the x_2 direction first we obtain

$$\begin{aligned}
E_2(u_L, T_{\sqrt{3}\tau_1} \setminus T_{\sqrt{3}\tau}) =& \int_{T_{\sqrt{3}\tau_1} \setminus T_{\sqrt{3}\tau}} \frac{1}{2}|\nabla u_L|^2 + W(u_L) dx \\
\geq& 3 \int_{-\sqrt{3}\tau}^{\sqrt{3}\tau} [\int_{\tau}^{\tau_1} \frac{1}{2}|\nabla u_L|^2 + W(u_L) dx_2] dx_1 \\
\geq& 3(1-\epsilon)e_1[2\sqrt{3}\tau - 2\frac{C}{\delta^2}(2-\epsilon)e_1]
\end{aligned}$$

Using (**??**) and (**??**) we can also obtain

$$\begin{aligned}
E_2(u_L, T_{\sqrt{3}y} \setminus T_{\sqrt{3}\tau}) \leq& E_2(u_L, T_L) - E_2(u_L, T_L \setminus T_{\sqrt{3}\tau_1}) \\
\leq& \sqrt{3}e_1 L + C - 3\int_{\tau_1}^{\frac{\sqrt{3}}{3}L} E_1(u_L(\cdot, x_2)) dx_2 \\
\leq& 3e_1\tau_1 + C \leq 3\frac{2-\epsilon}{1-\epsilon}e_1\tau + C
\end{aligned}$$

By choosing ϵ and accordingly δ sufficiently small we conclude

$$\tau \leq C \tag{3.14}$$

Then the energy lower bound (**??**) follows immediately for all $0 < x_2 < L/\sqrt{3}$, since (**??**) holds for $\tau \leq x_2 \leq L/\sqrt{3}$ due to the definition of τ.

Therefore

$$\begin{aligned} &E_2\big(u_L; T_L \setminus T_M\big) \\ &\quad = 3\int_{M/\sqrt{3}}^{L/\sqrt{3}} \int_{-x_2\sqrt{3}}^{x_2\sqrt{3}} \Big[\frac{1}{2}\left|\frac{\partial u_L}{\partial x_1}\right|^2 + \frac{1}{2}\left|\frac{\partial u_L}{\partial x_2}\right|^2 + W(u_L)\Big]\,dx_1\,dx_2, \\ &\quad \geq 3\int_{M/\sqrt{3}}^{L/\sqrt{3}} \big(e_1 - C_6 e^{-2\kappa_0\sqrt{3}x_2}\big)\,dx_2 \geq e_1\sqrt{3}(L-M) - C. \end{aligned}$$

Then (**??**) follows immediately.

Step 3 is completely straightforward. We may extract from the sequence of the u_L a subsequence which converges uniformly on all compact sets to a certain u, which solves the Euler-Lagrangian equation in the entire plane.

$$-\Delta u + DW(u)^T = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}^3. \tag{3.15}$$

The energy estimate (**??**) yields a similar estimate for u

$$E_2(u, T_M) \leq \sqrt{3}e_1 M + C.$$

Using the same arguments as in Theorem 4.7 of [**?**], we can show that $u(\cdot, x_2)$ converges uniformly to $z_{\mathbf{ab},\mathbf{c}}$ or to $z_{\mathbf{ab},\mathbf{d}}$, which we denote by z in both cases. Therefore u is a triple junction solution.

Consequently, there exist constants δ and R_0 independent of L such that

$$|u(x) - \mathbf{d}| > \delta, \quad \text{if } |x| \geq R_0, \quad x \in \mathbb{R}^2. \tag{3.16}$$

Under the nondegeneracy hypothesis **(H4)**, the asymptotic behavior of u can be accurately described as exponentially close to the one dimensional heteroclinic solution z at infinity along x_2-axis. To be more precise, we state the following theorem.

Theorem 3.1. *There exists a triple junction solution u to* (**??**) *which is equivariant under the group action $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Moreover, the convergence of $u(x_1, x_2)$ to $z_{\mathbf{ab},\mathbf{c}}(x_1)$ or $z_{\mathbf{ab},\mathbf{d}}(x_1)$ is uniform and exponentially fast as x_2 tends to positive infinity.*

Proof. We shall prove the exponential convergence of u. Define indeed

$$g(x_2) = \int_{-\infty}^{\infty} |u(x_1, x_2) - z(x_1)|^2\,dx_1, \tag{3.17}$$

where z denotes the limit of u as x_2 tends to $+\infty$. When we differentiate g, the derivative of g is given by

$$g'(x_2) = 2\int_{-\infty}^{\infty} \frac{\partial u}{\partial x_2}(x_1, x_2) \cdot (u(x_1, x_2) - z(x_1))\,dx_1.$$

A second differentiation yields

$$g''(x_2) = 2\int_{-\infty}^{\infty} \Big[\Big|\frac{\partial u}{\partial x_2}\Big|^2 + \Big(-\frac{\partial^2 u}{\partial x_1^2} + DW(u)^T\Big)\cdot(u-z)\Big]\,dx_1.$$

We use now the results of Lemma 2.6 in §2.2: we observe that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\Big[-\frac{\partial^2 u}{\partial x_1^2} + DW(u)^T\Big]\cdot(u-z)\,dx_1\\
&\quad= \int_{-\infty}^{\infty}\Big(-\frac{\partial^2 u}{\partial x_1^2} + z''\Big)\cdot(u-z) + D^2W(z)(u-z)^{\otimes 2}\,dx_1\\
&\quad+ \int_{-\infty}^{\infty}\big(DW(u) - DW(z)\big)(u-z) - D^2W(z)(u-z)^{\otimes 2}\\
&\geq \frac{\nu^2}{2} g
\end{aligned}$$

when x_2 is sufficiently large. Therefore we have for some M_0 large enough

$$g''(x_2) - \frac{\nu^2}{2} g(x_2) \geq 0, \quad x_2 \geq M_0 \tag{3.18}$$

Using comparison function $h(t) = Ce^{-\frac{\nu}{\sqrt{2}}t}$ and the maximum principle, we obtain that

$$g(x_2) \leq h(x_2) = Ce^{-\frac{\nu}{\sqrt{2}}x_2}, \quad x_2 \geq M_0$$

for some constant C large enough. Then the theorem follows from the standard elliptic theory. $\square$

From the definition of u as the limit of u_L, we know that u is a local minimizer in the following sense:

(3.19)

$$E_2(u,\Omega) = \min\{E_2(v,\Omega) : v \in u + H_0^1(\Omega)^3, \gamma\circ v = v\circ\gamma, \forall\gamma\in\Gamma(\mathbf{a},\mathbf{b},\mathbf{c})\}.$$

The proof follows the minimizing property of u_L and is the same as in [**?**].

Now we define the renormalized energy of the triple junction solution u by

$$e_2 = \lim_{L\to\infty}\big(E_2(u, T_L) - e_1\sqrt{3}L\big). \tag{3.20}$$

The definition is valid due to the exponential convergence of u to $z_{\mathbf{ab}}$ as x_2 tends to infinity.

Indeed, we have an exponential asymptotic estimate of the energy of u_L.

Corollary 3.2. *There exist constants κ and C such that*

$$|E_2(u,T_L) - e_2 - \sqrt{3}e_1 L| \le Ce^{-\kappa L}. \tag{3.21}$$

Proof. Theorem **??** implies that there exists constants κ and C such that

$$\begin{aligned} &||u(\cdot,x_2) - z_{\mathbf{ab}}||_{C^1([-\sqrt{3}x_2,\sqrt{3}x_2])} \le Ce^{-\kappa x_2}, \\ &||\frac{\partial u}{\partial x_2}(\cdot,x_2)||_{C^1([-\sqrt{3}x_2,\sqrt{3}x_2])} \le Ce^{-\kappa x_2}. \end{aligned} \tag{3.22}$$

Then for any $M \ge L$,

$$\begin{aligned} &|\big(E_2(u,T_M) - (e_2+\sqrt{3}e_1 M)\big) - \big(E_2(u,T_L) - (e_2+\sqrt{3}e_1 L)\big)| \\ =&|E_2(u,T_M\setminus T_L) - \sqrt{3}e_1(M-L)| \le Ce^{-\kappa L}. \end{aligned} \tag{3.23}$$

Letting M tend to infinity, we obtain (**??**). □

Finally, we state the asymptotic energy formula for u_L, assuming the boundary condition (**??**) of u_L is given by $z_{\mathbf{ab},\mathbf{c}}$ which is also the limit of u at infinity.

Corollary 3.3. *There exists constants κ and C such that*

$$|E_2(u_L,T_L) - e_2 - \sqrt{3}e_1 L| \le Ce^{-\kappa L}. \tag{3.24}$$

Proof. Using the boundary condition and the same arguments as in the proof of Theorem **??** we can obtain asymptotic behavior of u_L:

$$\begin{aligned} &||u_L(\cdot,x_2) - z_{\mathbf{ab}}||_{C^1([-\sqrt{3}x_2,\sqrt{3}x_2])} \le Ce^{-\kappa x_2}, \\ &||\frac{\partial u_L}{\partial x_2}(\cdot,x_2)||_{C^1([-\sqrt{3}x_2,\sqrt{3}x_2])} \le Ce^{-\kappa x_2}. \end{aligned} \tag{3.25}$$

with the constants C and κ independent of L, x_2. Then using the minimizing property of both u_L and u and constructing exponentially small test functions connecting u and u_L (here the boundary condition is important), we can estimate the energy difference of u and u_L as exponentially small. Therefore, (**??**) follows directly from (**??**). □

3.2. **Transition layer solution between $z_{\mathbf{ab},\mathbf{c}}(x_1)$ and $z_{\mathbf{ab},\mathbf{d}}(x_1)$.** In this section, we are going to construct a solution $u : \mathbb{R}^2 \to \mathbb{R}^3$ which connects $z_{\mathbf{ab},\mathbf{c}}(x_1)$ and $z_{\mathbf{ab},\mathbf{d}}(x_1)$. To be more precise, we choose a smooth function $\eta(t)$ with value 0 and 1 for $t < -1$ and $t > 1$ respectively, and define

$$\phi_0(x) = z_{\mathbf{ab},\mathbf{c}}(x_1)\eta(x_2) + z_{\mathbf{ab},\mathbf{d}}(x_1)(1-\eta(x_2)). \tag{3.26}$$

It is convenient to assume that $\eta - \frac{1}{2}$ is an odd function.

We then consider a solution u of the Euler-Lagrange equation of E_2 which satisfies

$$\lim_{|x_2|\to\infty} \sup_{x_1\in\mathbb{R}} \{|u(x_1,x_2) - \phi_0(x_1,x_2)|\} = 0, \tag{3.27}$$

The construction of such a solution u is essentially the same as in [?]. Here we just outline the strategy: First we minimize $E_2(u)$ in the set $\phi_0 + \big(H_0^{1,s}(S_{-L,L})\big)^3$, where $S_{L,M} = \{x : M < x_2 < L\}$ is an infinite strip and $H_0^{1,s}(S_{L,M})$ consists functions in $H_0^1(S_{L,M})$ which are $\gamma_{\mathbf{ab}}$ equivariant, i.e. $u(\gamma_{\mathbf{a'b'}}x) = \gamma_{\mathbf{ab}} \circ u(x)$. The minimizer is denoted by u_L.

Second, we obtain an energy estimate of u_L for $M < N < K < L$

$$(N-K)e_1 \le E_2(u_L, S_{N,K}) \le (N-K)e_1 + C \tag{3.28}$$

where $C > 0$ is independent of L, M, N, K.

Third, we may translate u_L in x_2 -axis so that $u_L(0,0)$ belongs to the plane $\mathcal{P}\mathbf{oab}$. Then the boundedness of u_L in $C^2(S_{-L,L})^3$ guarantees the existence of limit $u_{\mathbf{ab}}$ of u_L in a subsequence as L tends to ∞.

Finally, we use the above energy estimate and the minimizing property of u_L to conclude that the limit solution $u_{\mathbf{ab}}$ satisfies (**??**) and $u_{\mathbf{ab}}(0,0)$ belongs to the plane $\mathcal{P}\mathbf{oab}$.

We can also minimize $E_2(u)$ in the set $\phi_0 + \big(H_0^{1,s}(S_{-L,L})\big)^3$ and with further constraint that $u(x_1,-x_2) = \gamma_{\mathbf{cd}} \circ u(x_1,x_2)$ and denote the minimizer by $u_{L,s}$. The above procedure then leads to the existence of a transitional layer solution $u^s_{\mathbf{ab}} : \mathbb{R}^2 \to \mathbb{R}^3$ satisfying (**??**) with one more symmetry $u^s_{\mathbf{ab}}(x_1,-x_2) = \gamma_{\mathbf{cd}} \circ u^s_{\mathbf{ab}}(x_1,x_2)$.

We summarize the above discussion as follows.

Theorem 3.4. *There exists a transition layer solution $u^s_{\mathbf{ab}}$ to* (**??**) *and* (**??**) *satisfying the symmetry condition $u^s_{\mathbf{ab}}(x_1,-x_2) = \gamma_{\mathbf{cd}} \circ u^s_{\mathbf{ab}}(x_1,x_2)$ and $u^s_{\mathbf{ab}}(-x_1,x_2) = \gamma_{\mathbf{ab}} \circ u^s_{\mathbf{ab}}(x_1,x_2)$. Moreover, $u^s_{\mathbf{ab}}$ converges exponentially to $z_{\mathbf{ab},c}(x_1)$ in $C^1(\mathbb{R})^3$ as x_2 tends to ∞.*

We note that $u^s_{\mathbf{ab}}$ is a local minimizer (in the sense of compact perturbation in $\mathbb{R}^2$) in the corresponding symmetric space.

Question 3.5. *Can we show that $u_{\mathbf{ab}}$ and $u^s_{\mathbf{ab}}$ are indeed the same?*

Now we define

$$\tilde{e}_2 = \lim_{L\to\infty} \big(E_2(u_{\mathbf{ab}}, S_{-L,L}) - 2Le_1\big). \tag{3.29}$$

and

$$e'_2 = \lim_{L\to\infty} \big(E_2(u^s_{\mathbf{ab}}, S_{-L,L}) - 2Le_1\big) \tag{3.30}$$

It is clear that

$$0 < \tilde{e}_2 \le e_2'.$$

We also have the exponential asymptotic formula for the renormalized energy

$$|E_2(u^s_{\mathbf{ab}}, S_{-L,L}) - e_2' - 2e_1 L| \le Ce^{-\kappa L}. \tag{3.31}$$

The proof is similar to that of (**??**).

3.3. **Possible two triple junction solutions $u_{\mathbf{c}}$ and $u_{\mathbf{d}}$.** As shown in Theorem **??**, the solution u should be exponentially close to $z_{\mathbf{ab},\mathbf{c}}$ or $z_{\mathbf{ab},\mathbf{d}}$ as x_2 goes to infinity. We denote the solution by $u_{\mathbf{c}}$ or $u_{\mathbf{d}}$ respectively, according to it's limit.

Similar to the minimization procedure in §3.1, we may choose a different test function for boundary condition than (**??**)

$$\begin{aligned} u_1(x) = {} & \phi_{\mathbf{ab}}(x) z_{\mathbf{ab},\mathbf{c}}\left(\frac{x\cdot(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|}\right) + \phi_{\mathbf{bc}}(x) z_{\mathbf{bc},\mathbf{a}}\left(\frac{x\cdot(\mathbf{c}-\mathbf{b})}{|\mathbf{c}-\mathbf{b}|}\right) \\ & + \phi_{\mathbf{ca}}(x) z_{\mathbf{ca},\mathbf{b}}\left(\frac{x\cdot(\mathbf{a}-\mathbf{c})}{|\mathbf{a}-\mathbf{c}|}\right) + \phi_0(x)\rho(x) \end{aligned} \tag{3.32}$$

and carry out the same analysis. If both minimization procedure end up with the same triple junction solution $u_{\mathbf{c}}$ (or $u_{\mathbf{d}}$), then we do not need to worry about possible two different energy levels for triple junction solutions. We note that even though $u_{\mathbf{c}}$ (or $u_{\mathbf{d}}$) may not be unique, they have the same asymptotic behavior $z_{\mathbf{ab},\mathbf{c}}$ (or $z_{\mathbf{ab},\mathbf{d}}$) and the same energy level, thanks to the minimizing property of the limiting sequences. If each procedure yields a triple junction solution with different asymptotic limit, then we define the renormalized energies of the triple junctions as below

$$e_{\mathbf{c},2} = \lim_{L\to\infty}\big(E_2(u_{\mathbf{c}}, T_L) - \sqrt{3}e_1 L\big). \tag{3.33}$$

and

$$e_{\mathbf{d},2} = \lim_{L\to\infty}\big(E_2(u_{\mathbf{d}}, T_L) - \sqrt{3}e_1 L\big). \tag{3.34}$$

According to §3.2, the renormalized energy connecting u_1 and u_0 is basically $3e_2'$. From the construction of $u_{\mathbf{c}}$ and $u_{\mathbf{d}}$ we can then obtain

$$|e_{\mathbf{c},2} - e_{\mathbf{d},2}| \le 3e_2',$$

since otherwise the two procedure would yield the same solution. We leave the straightforward details to the reader.

In any case, we have a triple junction solution with the minimum energy level. Without loss of generality, we just call it u with renormalized energy

$$e_2 = \min\{e_{\mathbf{c},2}, e_{\mathbf{d},2}\}.$$

We would like to understand more about the triple junction solutions. In particular, we have the following open question.

Question 3.6. *Does the image of the triple junction solution $u_{\mathbf{c}}$ lie entirely inside the cone* **Coabc***? Does $u_{\mathbf{d}}$ really exist?*

3.4. **The triple junction with Neumann boundary conditions.** We shall consider a triple junction solution in a triangle T_L with Neumann boundary condition. To be more precise, we consider the following minimizing problem:

$$\min E_2(u, T_L) \tag{3.35a}$$

under the conditions

$$u \circ \gamma = \gamma \circ u, \quad \forall \gamma \in \Gamma(\mathbf{a}, \mathbf{b}, \mathbf{c}) \tag{3.35b}$$

$$|u - \mathbf{d}| \geq \frac{\delta}{2}, \quad \text{for} \quad |x| \geq R_0,\ x \in T_L \tag{3.35c}$$

where $\delta > 0$ and R_0 are as in (**??**).

We shall prove the existence of a triple junction solution with Neumann boundary condition and obtain an asymptotic formula for it's energy.

Theorem 3.7. *The solution u_L^N to the minimization problem* (**??**) *does not saturate* (**??**) *for L sufficiently large, and hence satisfies the Euler-Lagrange equation as in* (**??**) *with Neumann boundary condition on ∂T_L. Furthermore, we have the following energy estimate*

$$|E_2(u_L^N, T_L) - e_2 - \sqrt{3}e_1 L| \leq Ce^{-\kappa L}. \tag{3.36}$$

for some constant C and κ independent of L.

Proof. The strategy of proof is the same as in Theorem **??**:

First, it is standard to show the existence of a minimizer u_L^N to (**??**). Since the triple junction u in $\mathbb{R}^2$ is an admissible test function for (**??**) due to (**??**), we can bound from above the energy of u_L^N on T_L

$$E_2(u_L^N, T_L) \leq E_2(u, T_L) \leq e_2 + \sqrt{3}e_1 L + Ce^{-\kappa L}. \tag{3.37}$$

for some constant C and κ independent of L.

Second, we note that the energy lower bound (**??**) is always true for $u_L^N(\cdot, x_2$, thanks to Theorem **??** and the constraint (**??**). Then energy estimates (**??**) and (**??**) hold for u_L^N. Moreover, for any $\epsilon > 0$ there exists M_0 large enough (independent of L) such that when $M \geq M_0$

$$|E_2(u_L^N, T_{M+1} \setminus T_M) - \sqrt{3}e_1| \leq \epsilon. \tag{3.38}$$

Therefore for some $x_2 \in [M, M+1]$ we have either

$$||u_L^N(\cdot, x_2) - z_L^s||_{C^1[-\sqrt{3}x_2, \sqrt{3}x_2])} \leq c(\epsilon) \tag{3.39}$$

or

$$||u_L^N(\cdot, x_2) - \tilde{z}_L^s||_{C^1[-\sqrt{3}x_2, \sqrt{3}x_2])} \leq c(\epsilon), \tag{3.40}$$

where $c(\epsilon)$ is a function which goes to zero as ϵ goes to 0.

By (**??**), it is easy to see that (**??**) or (**??**) happen simultaneously for all M large enough, since otherwise the energy in T_L would violate estimate (**??**). (See proof of Theorem 4.7 in [?] for details.) If (**??**) happened, we would have

$$||u_L^N(\cdot, x_2) - \tilde{z}_L^s||_{C^1([-\sqrt{3}x_2, \sqrt{3}x_2])} \leq c(\epsilon), \quad \forall x_2 \geq M_0.$$

Note that $\tilde{z}_L^s$ converges to $\mathbf{c}$ locally in $\mathbb{R}$ as L tends to infinity, by integrating in the directions $\mathbf{b}' - \mathbf{c}'$ and $\mathbf{a}'$ instead of x_1, x_2, we obtain

$$E_L(u_L^N, T_L) \geq 2\sqrt{3}e_1 L - C \tag{3.41}$$

which contradicts (**??**). Therefore, (**??**) holds for $M \geq M_0$. By constructing a test function connecting u_L^N and u near $x_2 \in [M, M+1]$ and using the minimizing property of u, we obtain for $M \geq M_0$

$$|E_2(u_L^N, T_M) - \sqrt{3}e_1 M - e_2| \leq \gamma(\epsilon)$$

where $\gamma(\epsilon)$ tends to 0 as ϵ goes to 0. In particular, if we follow the procedure in §3.1 and use the same notations, for $x_2 \geq \max\{M/\sqrt{3}, \tau\}$ we have $x_2 \notin \mathcal{M}$ and therefore $x_2 \in \mathcal{N}$.

In other words, if we adjust R_0 properly according to M_0 when needed (note M_0 is independent of R_0 in above discussion), we have

$$|u_L^N(x) - \mathbf{d}| > \delta, \quad \text{for } |x| \geq R_0, \quad x \in T_L. \tag{3.42}$$

Therefore u_L^N does not saturate the constraint (**??**), and satisfies the Euler-Lagrange equation in (**??**). The convergence of u_L^N to u as L tends to infinity follows immediately.

Finally we use Lemma **??** with g given by (**??**); it is easy to see that $g'(L/\sqrt{3})$ vanishes; the argument of Theorem **??** applies here. Then we conclude that on the boundary of T_L, the distance of u_L to the heteroclinic connection z is exponentially small in L; moreover, the bound extends inside the triangle and leads to the following exponential bound:

$$
\begin{aligned}
&||u_L^N(\cdot, x_2) - z_{\mathbf{ab}}||_{C^1([-\sqrt{3}x_2,\sqrt{3}x_2])} \le Ce^{-\kappa x_2}, \\
&||\frac{\partial u_L^N}{\partial x_2}(\cdot, x_2)||_{C^1([-\sqrt{3}x_2,\sqrt{3}x_2])} \le Ce^{-\kappa x_2}.
\end{aligned} \tag{3.43}
$$

We extend u_L^N continuously to the entire plane so that it equals u outside T_{L+1} and is a essentially a linear function of x_2 in $S_{L+1} \setminus S_L$ except near the end, where both u_L^N and u are exponentially close to $\mathbf{a}$ or $\mathbf{b}$ and their derivatives are exponentially close to 0. (Note that S_L is defined in (**??**).) The renormalized energy of this extension is then larger than the renormalized energy of the triple junction solution u. The desired estimate (**??**) follows from (**??**) and (**??**) immdediately.

□

3.5. **The transition layer solution in a rectangle with Neumann boundary condition.** In this section, we fix $0 < \alpha < 1$ and define the rectangle Π_L as follows

$$\Pi_L = \{x = (x_1, x_2) : |x_1| \le L, |x_2| \le \alpha L\}.$$

We consider a minimization problem

$$
\begin{aligned}
\min\Big\{E_2(u, \Pi_L) :&u \in H^1(\Pi_L)^3, \quad , u(-x_1, x_2) = \gamma_{\mathbf{ab}} \circ u(x_1, x_2), \\
&u(x_1, -x_2) = \gamma_{\mathbf{cd}} \circ u(x_1, x_2)\Big\}.
\end{aligned} \tag{3.44}
$$

The minimizer $\tilde{u}_L$ exists and satisfies the Euler-Lagrange equation and Neumann boundary condition. We note that $\gamma_{\mathbf{cd}} \circ \tilde{u}_L, \gamma_{\mathbf{ab}} \circ \tilde{u}_L$ are also solutions. In the discussion below we always mean a properly chosen one of them.

We shall prove the asymptotic behavior and an energy formula for $\tilde{u}_L$ in Π_L.

Theorem 3.8. *The solution $\tilde{u}_L$ satisfies*

$$
\begin{aligned}
&||\tilde{u}_L(\cdot, x_2) - z_{\mathbf{ab}}||_{C^1([-L,L])} \le Ce^{-\kappa x_2}, \\
&||\frac{\partial \tilde{u}_L}{\partial x_2}(\cdot, x_2)||_{C^1([-L,L])} \le Ce^{-\kappa x_2}.
\end{aligned} \tag{3.45}
$$

Furthermore, $\tilde{u}_L$ converges uniformly to $u^s_{\mathbf{ab}}$ as L tends to infinity, and the following energy estimate holds.

$$|E_2(\tilde{u}_L, \Pi_L) - e_2' - 2\alpha e_1 L| \le Ce^{-\kappa L}. \tag{3.46}$$

for some constant C and κ independent of L.

Proof. Using $u^s_{\mathbf{ab}}$ as a test function, we easily obtain the upper bound

$$E_2(\tilde{u}_L, \Pi_L) \le e_2' + 2\alpha e_1 L + Ce^{-\kappa L}. \tag{3.47}$$

We claim that, for a fixed small ϵ, there exists a $\delta > 0$ such that for $x_2 \in [-\alpha L, \alpha L]$ either

$$E_1(\tilde{u}_L(\cdot, x_2), [-L, L]) \ge (2 - \epsilon)e_1 \tag{3.48}$$

or

$$d(\tilde{u}_L(x_1, x_2) - \{\mathbf{c}, \mathbf{d}\}) \ge \delta, \quad \forall x_1 \in [-L, L] \tag{3.49}$$

when L is large enough.

Suppose that there exists $\tau_2 \in [-L, L]$ such that both the above statements are not true. Obviously we can assume $\tau_2 > 0$ and $|\tilde{u}_L(x_1, x_2) - \mathbf{c}| \le \delta$, since $\tilde{u}_L(x_1, 0) \in \mathcal{P}\mathbf{oab}$ by the symmetry of $\tilde{u}_L$ with respect of $\gamma_{\mathbf{ab}}$. Observe, by symmetry of $\tilde{u}_L$ with respect to $\gamma_{\mathbf{ab}}$, that $\tilde{u}_L(x_1, \tau_2)$ cannot be close to both $\mathbf{a}, \mathbf{c}$ due to the above energy restriction (**??**). (See the arguments for (**??**).) Then we have

$$\text{meas}\Big\{x_1 : |d(\tilde{u}_L(x_1, \tau_2) - \{\mathbf{c}, \mathbf{d}\})| \le \delta\Big\} \ge 2L - \frac{C}{\delta^2}$$

Use the symmetry of $\tilde{u}_L$ with respect $\gamma_{\mathbf{cd}}$, we can estimate the energy of $\tilde{u}_L$ by integrating in x_2 first as follows

$$E_2(\tilde{u}_L, \Pi_L) \ge \int_{-L}^{L} \int_{-\tau_2}^{\tau_2} \mathcal{L}_2(\tilde{u}_L) dx_2 dx_1 \ge (e_1 - \epsilon)(2L - \frac{C}{\delta^2}).$$

This contradicts (**??**) when $\epsilon > 0$ is sufficiently small and L is sufficiently large (note $\alpha < 1$). The claim is proven. Therefore we have

$$E_1(\tilde{u}_L(\cdot, x_2), [-L, L]) \ge e_1 - Ce^{-\kappa L} \quad , \forall x_2 \in [-\alpha L, \alpha L]. \tag{3.50}$$

Therefore, for any fixed $M < \alpha L$,

$$E_2(\tilde{u}_L, [-L, L] \times [-M, M]) \le 2Me_1 + e_2' + CLe^{-\kappa L}.$$

Then it is easy to see that $\tilde{u}_L$ converges uniformly in any compact set in $\mathbb{R}^2$ to a transition layer solution $u^s_{\mathbf{ab}}$. Using similar arguments as in above section, we can prove (**??**) and (**??**) immediately.

□

3.6. The triple junction with symmetric boundary condition.

We consider the following minimization problem with symmetric boundary condition

$$\min\Big\{E_2(u, T_L) : \quad u \in H^1(T_L),\ u \circ \gamma = u \circ \gamma,\ \forall \gamma \in \Gamma(\mathbf{a}, \mathbf{b}, \mathbf{c}), \\ u(x_1, L/\sqrt{3})) \in \mathcal{P}\mathbf{oab},\ \forall x_1 \le [-L, L]\Big\} \tag{3.51}$$

It is easy to see that there exists a minimizer u_L^s to the above minimization problem.

Define a smooth function $\eta(t)$ which equals 1 when $t \geq 1$ and 0 when $t \leq 0$. We fix a positive constant $\alpha < 1/2$, and choose a $\Gamma(\mathbf{a}, \mathbf{b}, \mathbf{c})$-equivariant test function ψ which is basically the gluing of the triple junction u and the transition layer solution $u^s_{\mathbf{ab}}$. Namely,

(3.52)
$$\psi(x) = \begin{cases} (1-\eta(\sqrt{3}x_2+x_1))\mathbf{a} + \eta(\sqrt{3}x_2+x_1)u^s_{\mathbf{ab}}(x_1, x_2 - \dfrac{L}{\sqrt{3}}), \\ \qquad \text{if } \ x_1 \leq 0, \ \text{ and } \ x \in S_L \setminus S_{(1-\alpha)L}, \\ u(x), \quad \text{if } \ x \in S_{\alpha L}, \\ \big[u(x)\big((1-\alpha)L/\sqrt{3} - x_2\big) + (x_2 - \alpha L/\sqrt{3})u^s_{\mathbf{ab}}\big] \cdot \dfrac{\sqrt{3}}{(1-2\alpha)L}, \\ \qquad \text{if } \ x \in S_{(1-\alpha)L} \setminus S_{\alpha L}. \end{cases}$$

where S_L is defined in (**??**).

Then it is easy to obtain an upper bound of the energy of u_L^s

$$E_2(u_L^s, T_L) \leq e_1\sqrt{3}L + e_2 + \frac{1}{2}e_2' + Ce^{-\kappa L}. \tag{3.53}$$

Following the procedure of §3.1, we can prove (**??**) and consequently (**??**). We need to make the following modification when applying the boundary condition to conclude (**??**): Proceeding exactly as in §3.1 to get (**??**), then consider

Case 1: If $\tau < \frac{1-\epsilon}{2-\epsilon}(\frac{L}{\sqrt{3}} - C)$, nothing needs to be modified in order to prove (**??**). In particular, we obtain (**??**) for all $x_2 \geq 0$. Therefore, (**??**) holds. Following exactly the same arguments as in [**?**] and §3.1, we can prove that u_L^s converges to u in $C^2_{loc}(\mathbb{R}^2)$. Repeating the arguments in the Neumann boundary case, we can also obtain (**??**) for u_L^s.

Case 2: If $\tau \geq \frac{1-\epsilon}{2-\epsilon}(\frac{L}{\sqrt{3}} - C)$, we have to argue using the boundary condition to get a contradiction for L sufficiently large.

By the symmetry of W, it is easy to see that

$$\min\big\{E_1(z, [a,b]) : \quad z \in H^1([a,b])^3, \ |z(a) - \mathbf{d}| < \delta, \ z(b) \in \mathcal{P}\mathbf{oab}\big\} \geq (\frac{1}{2} - \epsilon)e_1$$

for some $\delta > 0$ sufficiently small. Then

$$\begin{aligned} E_2(u_L^s, S_L) \geq & \int_{-\sqrt{3}\tau}^{\sqrt{3}\tau} \big[\int_{\tau}^{L/\sqrt{3}} \mathcal{L}_2(u_L^s) dx_2\big] dx_1 \\ \geq & (\frac{1}{2} - \epsilon) 2\sqrt{3}\tau e_1 \end{aligned}$$

On the other hand, (**??**) implies

$$E_2(u_L^s, S_L) \le L/\sqrt{3}e_1 + C$$

Then we obtain

$$\tau \le \frac{L}{3(1-2\epsilon)} + C \le \frac{2L}{5} \tag{3.54}$$

for L sufficiently large and $\epsilon > 0$ sufficiently small.

Furthermore, we know that for some $\tau_2 \ge \frac{2L}{5}$

$$E_1\big(u_L^s(\cdot,\tau_2), [-\sqrt{3}\tau_2, \sqrt{3}\tau_2]\big) \le \frac{E_2(u_L^s, S_L)}{\frac{L}{\sqrt{3}} - \frac{2L}{5}} \le C$$

for some constant C independent of L. Let us define

$$\begin{aligned} \mathcal{K}_1 &= \big\{x_1 : |u_L^s(x_1,\tau_2) - \mathbf{d}| \le \delta\big\}, \\ \mathcal{K}_2 &= \big\{x_1 : d\big(u_L^s(x_1,\tau_2)), \{\mathbf{a},\mathbf{b},\mathbf{c}\}\big) \le \delta\big\}. \end{aligned} \tag{3.55}$$

Then

$$\text{meas}(\mathcal{K}_1) + \text{meas}(\mathcal{K}_2) \ge 2\sqrt{3}\tau_2 - \frac{C}{\delta^2}.$$

Now we define

$$\begin{aligned} &\Omega_1 = \{x : x_1 + \sqrt{3}(\tau_2 - x_2) \in \mathcal{K}_2, \quad x_1 > 0, \quad \tau \le x_2 \le \tau_2\} \\ &\Omega_2 = \{x = (-x_1, x_2) : x \in \Omega_1\} \end{aligned}$$

and use new variable

$$x' = (x_1', x_2') = (\frac{\sqrt{3}}{2}x_1 + \frac{1}{2}x_2, -\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2).$$

See Figure **??** for the geometry of the definitions.

We now estimate the energy of u_L^s by decomposing the S_L into several domains as follows

$$\begin{aligned} E_2(u_L^s, S_L) \ge &\int_{\mathcal{K}_1} \big[\int_{\tau_2}^{L/\sqrt{3}} \mathcal{L}_2(u_L^s) dx_2\big] dx_1 + \int_{\Omega_1 \cup \Omega_2} \mathcal{L}_2(u_L^s) dx_1 dx_2 \\ \ge &(\frac{1}{2} - \epsilon) e_1 \,\text{meas}(\mathcal{K}_1) + (1-\epsilon) e_1 \frac{1}{2} \,\text{meas}(\mathcal{K}_2) \\ \ge &(\frac{1}{2} - \epsilon) e_1 (2\sqrt{3}\tau_2 - \frac{C}{\delta^2}) \ge \frac{(2-4\epsilon)\sqrt{3}}{5} e_1 L - \frac{C}{\delta^2} \\ > &\frac{1}{\sqrt{3}} e_1 L + C \ge E_2(u_L^s, S_L) \end{aligned}$$

when ϵ and δ fixed sufficiently small and L large sufficiently large. This is a contradiction, which implies that Case 2 cannot happen.

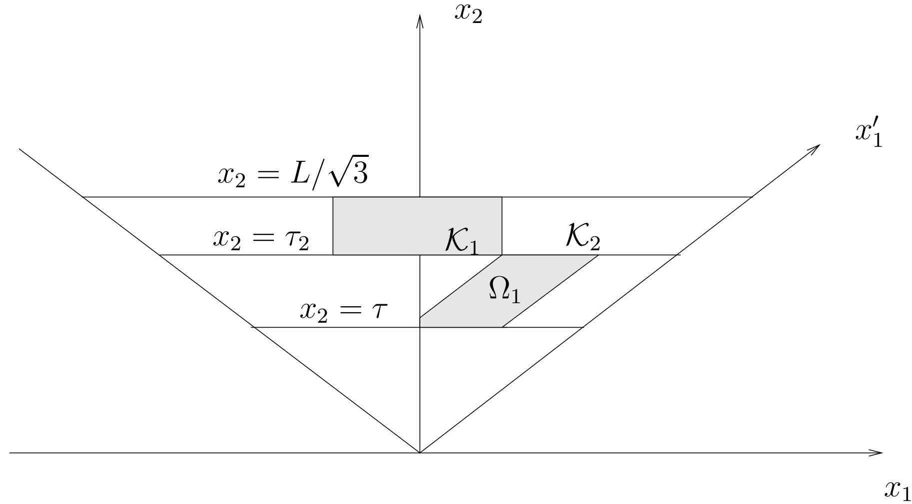

FIGURE 8. The decomposition of T_L to several domains for energy estimation

Now we fix any positive constant $0 < \alpha < 1/2$. By Theorems **??** and **??**, we obtain

(3.56)
$$\begin{aligned}
&E_2(u_L^s, T_L) \\
&\geq E_2(u_L^s, T_{\alpha L}) + E_2(u_L^s, T_{(1-\alpha)L} \setminus T_{\alpha L}) + E_2(u_L^s, T_L \setminus T_{(1-\alpha)L}) \\
&\geq e_1\sqrt{3}\alpha L + e_2 - Ce^{-\kappa\alpha L} + e_1\sqrt{3}\big((1-\alpha)L - \alpha L\big) - CLe^{-\kappa\alpha L} \\
&+ \frac{1}{2}e_2' + e_1(L - (1-\alpha)L) - Ce^{-\kappa\alpha L} \\
&\geq e_1\sqrt{3}L + e_2 + \frac{1}{2}e_2^s - Ce^{-\kappa\alpha L/2}.
\end{aligned}$$

and hence

$$E_2(u_L^s, T_{(1-\alpha)L} \setminus T_{\alpha L}) \leq e_1\sqrt{3}(1-2\alpha)L + Ce^{-\kappa\alpha L/2}. \tag{3.57}$$

As a consequence, we have

(3.58)
$$|E_1(u_L^s(\cdot, x_2), [\sqrt{3}x_2, \sqrt{3}x_2]) - e_1| \leq Ce^{-\kappa L}, \quad \forall x_2 \in [\alpha L, (1-\alpha L)].$$

after we make a possible adjustment of κ.

Hence we have proved

Theorem 3.9. *The solution u_L^s satisfies*

$$|E_2(u_L^s, T_L) - e_1\sqrt{3}L - e_2 - \frac{1}{2}e_2'| \leq Ce^{-\kappa L}. \tag{3.59}$$

Moreover, we have

(3.60)
$$||u^s_L(\cdot,x_2)-z_{\mathbf{ab}}||_{C^1[-\sqrt{3}x_2,\sqrt{3}x_2])}\leq Ce^{-\kappa L},\quad \forall x_2\in[\alpha L,(1-\alpha L)].$$

4. Quadruple junction

4.1. **Estimates from above.** The construction of the quadruple junction is performed through a limiting process: in the first step, we define a test function which will enable us to bound the energy of minimizers from above.

We shall use the same copy of $\mathbb{R}^3$ for both domain and target space. Therefore we may not distinguish them. We start with a partition of the unity which is Γ-equivariant. The first step is to define conical regions: the conical region $\hat{\mathcal{C}}_{\mathbf{bcd}}$ is the cone $\mathbb{R}[\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3]$ with vertex 0 generated by the triangle pictured on Figure **??**: the vertices of the triangle are

$$\mathbf{x}_1=\frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}+\frac{1}{2}\Big(\mathbf{b}-\frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}\Big),\quad \mathbf{x}_2=\frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}+\frac{1}{2}\Big(\mathbf{c}-\frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}\Big),$$
$$\mathbf{x}_3=\frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}+\frac{1}{2}\Big(\mathbf{d}-\frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}\Big).$$

The conical region $\hat{\mathcal{C}}_{\mathbf{bd}}$ is the cone with vertex 0 and generated by the polyhedral region pictured at Figure **??**; the vertices of that polyhedron are 0, $\mathbf{x}_1$, $\mathbf{x}_3$ and

$$\mathbf{x}_5=\frac{\mathbf{b}+\mathbf{a}+\mathbf{d}}{3}+\frac{1}{2}\Big(\mathbf{a}-\frac{\mathbf{b}+\mathbf{a}+\mathbf{d}}{3}\Big),\quad \mathbf{x}_6=\frac{\mathbf{b}+\mathbf{a}+\mathbf{d}}{3}+\frac{1}{2}\Big(\mathbf{d}-\frac{\mathbf{b}+\mathbf{a}+\mathbf{d}}{3}\Big).$$

We now let

$$\mathcal{C}_{\mathbf{bcd}}=\hat{\mathcal{C}}_{\mathbf{bcd}}\setminus\mathcal{T},\quad \mathcal{C}_{\mathbf{bd}}=\hat{\mathcal{C}}_{\mathbf{bd}}\setminus\mathcal{T}.$$

Note that the $\mathcal{T}$ is the regular tetrahedron with vertices $\mathbf{a},\mathbf{b},\mathbf{c}$ and $\mathbf{d}$, which has side length 2 and is centered at the origin.

We also define larger regions

$$\mathcal{C}^+_{\mathbf{bcd}}=\{x\in\mathbb{R}^3 : d(x,\mathcal{C}_{\mathbf{bcd}}\leq\beta\},$$

with identical definitions for $\mathcal{C}^+_{\mathbf{bd}}$ and $\mathcal{T}^+$. By suitable transformations belonging to Γ, we define regions $\mathcal{C}_{\mathbf{xyz}}$ and $\mathcal{C}_{\mathbf{xy}}$ for all values of the letters $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ taken in $\{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}\}$. Then it is clear that there is a partition of the unity subordinate to the covering

$$\bigcup\mathcal{C}^+_{\mathbf{xyz}}\cup\bigcup\mathcal{C}^+_{\mathbf{xy}}\cup\mathcal{T}^+.$$

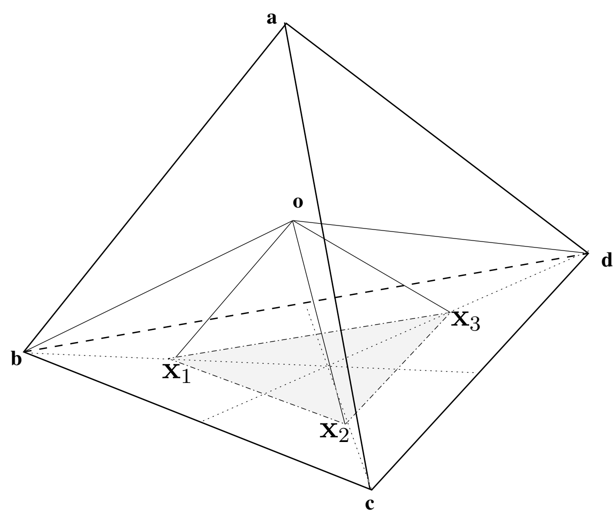

FIGURE 9. The triangle $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ spans the conical region whose truncation by a plane parallel to the plane of the triangle will be the region $\mathcal{C}_{\mathbf{abc}}$.

The test function is made up as follows:

$$U_0(x) = \sum_{\mathbf{xyz}} \phi_{\mathbf{xyz}} u_{\mathbf{xyz}} \quad \text{\% triple junction relative to } \mathbf{xyz}$$
$$+ \sum_{\mathbf{xy}} \phi_{\mathbf{xy}} u^s_{\mathbf{xy}} \quad \text{\% transition layer solution relative to } \mathbf{xy}$$
$$+ \phi_{\mathcal{T}} u \quad \text{\% any function that is symmetric and smooth.}$$

Figure **??** shows the pyramidal region which will give $\mathcal{C}_{\mathbf{bd}}$.

Let $\mathcal{T}_L$ be the tetrahedron $L\mathcal{T}_1$ with side length $2L$. We will use coordinates (x_1, x_2, x_3) for $\mathcal{C}_{\mathbf{xyz}}$, with x_3 pointing from the origin to the center of the triangle $\mathbf{xyz}$; while for $\mathcal{C}_{\mathbf{xy}}$ we use the rectangle coordinates (x'_1, x'_2, x'_3) with x'_3-axis pointing from the origin to the center of the line segment $\mathbf{xy}$.

Then, by decomposing $\mathcal{T}_L$ into $\mathcal{C}_{\mathbf{xyz}}, \mathcal{C}_{\mathbf{xy}}, \mathcal{T}$ and using (**??**), (**??**) we can obtain a upper energy estimate as follows.

First, we have

$$\int_{T_L} \mathcal{L}_3(U_0)\, dx = \sum_{\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbf{X}} \int_{\mathcal{C}_{\mathbf{xyz}}\cap\mathcal{T}_L} \mathcal{L}_3(U_0) dx + \sum_{\mathbf{x},\mathbf{y}\in\mathbf{X}} \int_{\mathcal{C}_{\mathbf{xy}}\cap\mathcal{T}_L} \mathcal{L}_3(U_0) dx + \int_{\mathcal{T}} \mathcal{L}_3(U_0) dx,$$

where

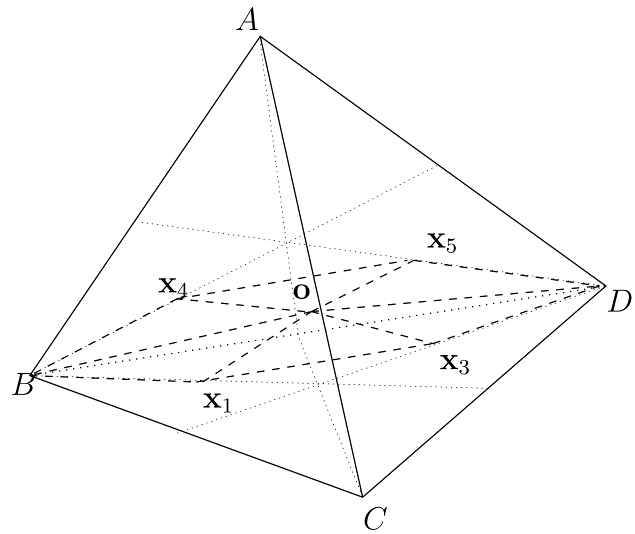

FIGURE 10. The convex set spanned by $\mathbf{x}_1, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{b}$ and $\mathbf{d}$ generates the pyramidal region which will give $\mathcal{C}_{\mathbf{bd}}$.

$$
\begin{aligned}
&\int_{\mathcal{C}_{\mathbf{xyz}} \cap \mathcal{T}_L} \mathcal{L}_3(U_0)dx \\
&\leq \int_{-\frac{L}{\sqrt{6}}}^{-\frac{1}{\sqrt{6}}} [\int_{T_{\frac{\sqrt{6}}{2}|x_3|}} \frac{1}{2}|\frac{\partial U_0}{\partial x_3}|^2 + \mathcal{L}_2(U_0(\cdot, x_3))dx_1dx_2]dx_3 \\
&\leq \int_{-\frac{L}{\sqrt{6}}}^{-\frac{1}{\sqrt{6}}} [C|x_3|e^{-2\kappa\sqrt{6}|x_3|} + e_2 + \sqrt{3}(\frac{\sqrt{6}}{2}|x_3|)e_1 + Ce^{-\kappa\sqrt{6}|x_3|}]dx_3 + C. \\
&\leq \frac{\sqrt{2}}{8}L^2e_1 + \frac{\sqrt{6}}{6}Le_2 + C
\end{aligned}
$$

Using Figure **??** above, which gives the side view of the center cross section of $\mathcal{C}_{\mathbf{bc}}$ in the tetrahedron $\mathcal{T}_L$, we can estimate the energy

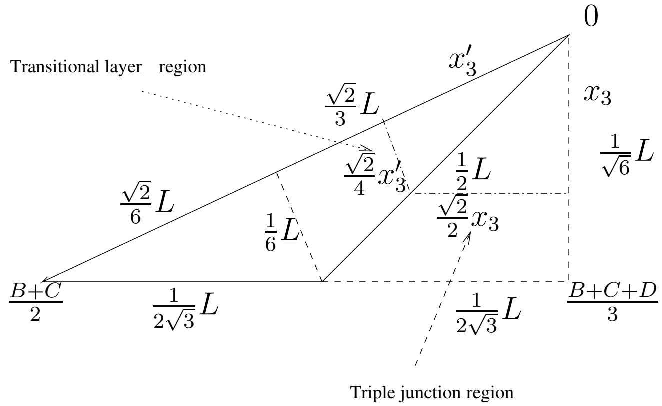

FIGURE 11. The side view of the region connecting the triple junction and transitional layer .

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{C}_{\mathbf{xy}}\cap\mathcal{T}_L}\mathcal{L}_3(U_0)dx\\
&\le \int_0^{\frac{L}{\sqrt{2}}}[\int_{\Pi_{\sqrt{6}|x_3'|}} +\frac{1}{2}|\frac{\partial U_0}{\partial x_3'}|^2+\mathcal{L}_2(U_0(\cdot,x_3'))dx_1'dx_2']dx_3'\\
&\le \int_0^{\frac{L}{\sqrt{2}}} C(x_3'+1)e^{-2\kappa\sqrt{x}_3'}dx_3'+\int_0^{\frac{\sqrt{2}L}{3}}[\int_{\Pi_{\sqrt{6}|x_3|}}\mathcal{L}_2(U_0(\cdot,x_3')dx_1'dx_2']dx_3'\\
&\quad+\int_{\frac{\sqrt{2}L}{3}}^{\frac{\sqrt{2}L}{2}}[\int_{\Pi_{\sqrt{6}|x_3'|}}\mathcal{L}_2(U_0(\cdot,x_3')dx_1'dx_2']dx_3',\\
&\le \big(\frac{\sqrt{2}}{36}L^2e_1+\frac{\sqrt{2}}{6}e_2'L\big)+\big(\frac{\sqrt{2}}{72}L^2e_1+\frac{\sqrt{2}}{12}e_2'L\big)+C
\end{aligned}$$

Here $\Pi_{\sqrt{6}|x_3'|}$ denotes the cross section of the solid region $\mathcal{C}_{\mathbf{xy}}$ at x_3' . We have also used

$$\begin{aligned}
&\int_0^{\frac{\sqrt{2}L}{3}}[\int_{\Pi_{\sqrt{6}|x_3|}}\mathcal{L}_2(U_0(\cdot,x_3')dx_1'dx_2']dx_3'\\
&\le \int_0^{\frac{\sqrt{2}L}{3}}\big[\frac{\sqrt{2}}{4}x_3'e_1+\frac{e_2'}{2}+C(1+|x_3'|^2)e^{-\kappa\frac{3\sqrt{2}}{2}x_3'}\big]dx_3'\\
&\le \frac{\sqrt{2}}{36}L^2e_1+\frac{\sqrt{2}}{6}e_2'L+C
\end{aligned}$$

and

$$\int_{\frac{\sqrt{2}L}{3}}^{\frac{\sqrt{2}L}{2}} [\int_{\Pi_{\sqrt{6}|x_3'|}} \mathcal{L}_2(U_0(\cdot, x_3')dx_1'dx_2']dx_3'$$
$$\leq \int_{\frac{\sqrt{2}L}{3}}^{\frac{\sqrt{2}L}{2}} \big[(\frac{L}{2} - \frac{\sqrt{2}}{2}x_3')e_1 + \frac{e_2'}{2} + C(1+|x_3'|^2)e^{-2\kappa\frac{3\sqrt{2}}{2}x_3' - \frac{1}{2}}\big]dx_3'$$
$$\leq \frac{\sqrt{2}}{72}L^2 e_1 + \frac{\sqrt{2}}{12}e_2' L + C.$$

Hence we obtain

$$\int_{T_L} \mathcal{L}_3(U_0)\, dx \leq \delta_1 e_1 L^2 + \delta_2 e_2 L + \delta_2' e_2' L + C. \tag{4.1}$$

where δ_1, δ_2 and δ_2' are the geometric constants of tetrahedron defined in §1.3, and C is a constant independent of L.

Now we solve the minimization problem with Dirichlet boundary condition in $\mathcal{T}_L$

(4.2)
$$\min\big\{E_3(U) : U \in \big(H^1(\mathcal{T}_L)\big)^3, \gamma \circ U = U \circ \gamma, \forall \gamma \in \Gamma, U = U_0 \,\mathrm{on}\, \partial\mathcal{T}_L\big\}.$$

The existence of such a minimizer U_L is standard. For example, one may use a modification of Theorem 5 in Chapter 8.2 of [**?**] by adding the equi-variance constraint. Then we have the upper energy estimate for U_L

$$E_3(U_L, \mathcal{T}_L) \leq \delta_1 e_1 L^2 + \delta_2 e_2 L + \delta_2' e_2' L + C. \tag{4.3}$$

4.2. **Estimates from below.** First, we have to estimate from below the energy of U_L on a tetrahedral annulus $\mathcal{T}_L \setminus \mathcal{T}_M$; see Figure **??** for a quarter of the annulus. We proceed differently in different regions. In the region $(\mathcal{T}_L \setminus \mathcal{T}_M) \cap \mathcal{C}_{\mathbf{abc}}$, we perform a comparison with the triple junction energy; in the region $(\mathcal{T}_L \setminus \mathcal{T}_M) \cap \mathcal{C}_{\mathbf{ab}}$ we perform a comparison with the energy of transition layer solution which has the same underlining rectangles. (See Figures 10-12.)

First we decompose the integral as

$$\int_{\mathcal{T}_L \setminus \mathcal{T}_M} \mathcal{L}_3(U_L)\, dx = \sum_{\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbf{X}} \int_{\mathcal{C}_{\mathbf{xyz}} \cap (\mathcal{T}_L \setminus \mathcal{T}_M)} \mathcal{L}_3(U_L) dx$$
$$+ \sum_{\mathbf{x},\mathbf{y},\in \mathbf{X}} \int_{\mathcal{C}_{\mathbf{xy}} \cap (\mathcal{T}_L \setminus \mathcal{T}_M)} \mathcal{L}_3(U_L) dx$$

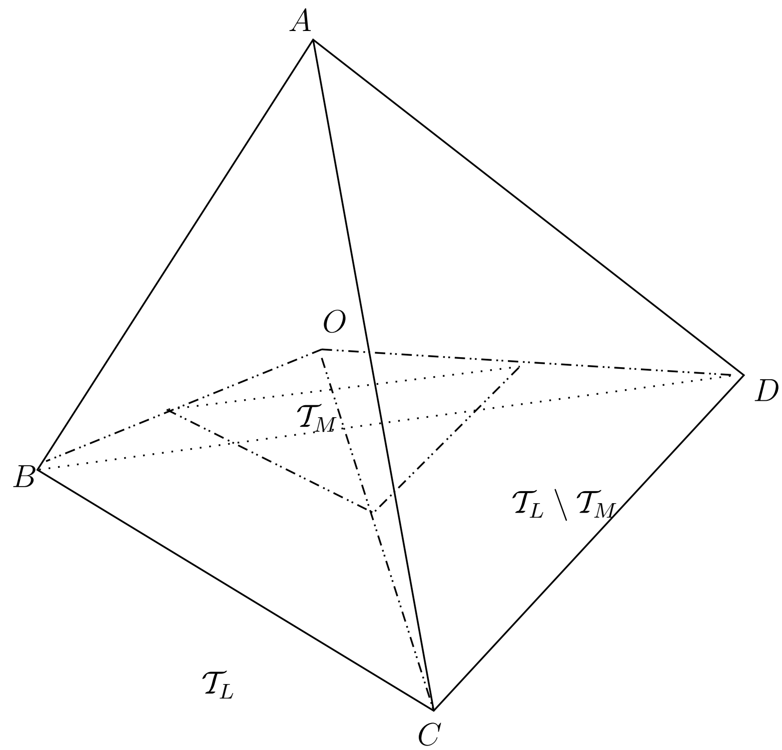

FIGURE 12. One quarter of the tetrahedric annulus

By using the lower energy estimates (**??**) and (**??**), we derive

$$\begin{aligned}
\int_{\mathcal{C}\mathbf{xyz}\cap(\mathcal{T}_L\setminus\mathcal{T}_M)} \mathcal{L}_3(U_L)dx &= \int_{-\frac{L}{\sqrt{6}}}^{-\frac{M}{\sqrt{6}}} [\int_{T_{\frac{\sqrt{6}}{2}|x_3|}} \mathcal{L}_2(U_L(\cdot, x_3))dx_1dx_2]dx_3 \\
&\geq \int_{-\frac{L}{\sqrt{6}}}^{-\frac{M}{\sqrt{6}}} [e_2 + \sqrt{3}(\frac{\sqrt{6}}{2}|x_3|)e_1 - Ce^{-\kappa\sqrt{6}|x_3|}]dx_3 \\
&\geq \frac{3\sqrt{2}}{8}(L^2 - M^2)e_1 + \frac{\sqrt{6}}{6}(L-M)e_2 - C
\end{aligned}$$

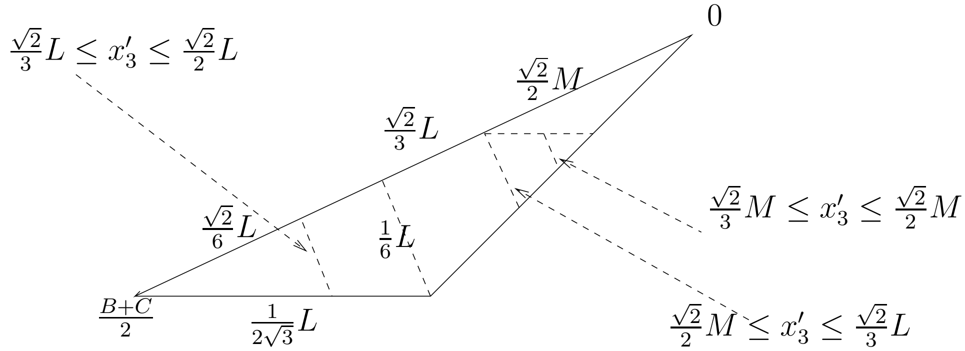

FIGURE 13. The side view of the cross section S for various x_3' .

and

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{C}_{\mathbf{xy}}\cap(\mathcal{T}_L\setminus\mathcal{T}_M)}\mathcal{L}_3(U_L)dx &= \int_{\frac{M}{\sqrt{2}}}^{\frac{\sqrt{2}L}{3}}[\int_{\Pi_{\sqrt{6}|x_3|}}\mathcal{L}_2(U(\cdot,x_3'))dx_1'dx_2']dx_3' \\ &+ \int_{\frac{\sqrt{2}L}{3}}^{\frac{\sqrt{2}L}{2}}[\int_{\Pi_{\frac{L}{2}-\frac{\sqrt{2}}{2}x_3'}}\mathcal{L}_2(U(\cdot,x_3'))dx_1'dx_2']dx_3' \\ &+ \int_{\frac{\sqrt{2}M}{3}}^{\frac{\sqrt{2}M}{2}}[\int_{\Pi_{\frac{\sqrt{2}}{4}x_3'}\setminus\Pi_{\frac{M}{2}-\frac{\sqrt{2}}{2}x_3'}}\mathcal{L}_2(U(\cdot,x_3'))dx_1'dx_2]dx_3' \\ &= I + II + III\end{aligned}$$

where Π represents the cross section of $\mathcal{C}_{\mathbf{xy}}$ at x_3' at various value of x_3'. See Figure **??** for the decomposition of the integrals.

Further we estimate

$$\begin{aligned} I &\geq \int_{\frac{M}{\sqrt{2}}}^{\frac{\sqrt{2}L}{3}}[\frac{\sqrt{2}}{4}x_3'e_1 + \frac{e_2'}{2} - Ce^{-\kappa\frac{3\sqrt{2}}{2}x_3'}]dx_3' \\ &\geq \frac{\sqrt{2}}{8}(\frac{2}{9}L^2 - \frac{1}{2}M^2)e_1 + \frac{e_2'}{2}(\frac{\sqrt{2}}{3}L - \frac{\sqrt{2}}{2}M) - C,\end{aligned}$$

and

$$\begin{aligned} II &\geq \int_{\frac{\sqrt{2}L}{3}}^{\frac{\sqrt{2}L}{2}}[(\frac{L}{2} - \frac{\sqrt{2}}{2})x_3'e_1 + \frac{e_2'}{2} - Ce^{-\kappa\frac{3\sqrt{2}}{2}x_3' - \frac{L}{2}}]dx_3' \\ &\geq \frac{\sqrt{2}}{72}L^2e_1 + \frac{\sqrt{2}}{12}Le_2' - C,\end{aligned}$$

and

$$III \geq \int_{\frac{\sqrt{2}M}{3}}^{\frac{\sqrt{2}M}{2}} [\frac{\sqrt{2}}{4} x_3' e_1 - (\frac{M}{2} - \frac{\sqrt{2}}{2} x_3') e_1 - Ce^{-\kappa \frac{3\sqrt{2}}{2} x_3'}] dx_3$$
$$\geq \frac{\sqrt{2}}{8} (\frac{1}{2} M^2 - \frac{2}{9} M^2) e_1 - \frac{\sqrt{2}}{72} M^2 e_1 - C$$

Hence we obtain

(4.4)
$$\int_{\mathcal{T}_L \setminus \mathcal{T}_M} \mathcal{L}_3(U_L) dx \geq \delta_1 e_1 (L^2 - M^2) + \delta_2 e_2 (L - M) + \delta_2' e_2' (L - M) - C.$$

where $\delta_1, \delta_2, \delta_2'$ are universal constants defined in §1.3 , and C is a constant independent of L, M and may differ from line to line.

Therefore, combining (**??**) and (**??**) we have

$$\int_{\mathcal{T}_M} \mathcal{L}_3(U_L)\, dx \leq \delta_1 e_1 M^2 + \delta_2 e_2 M + \delta_2' e_2' M + C \tag{4.5}$$

where C is independent of L and M.

By the standard elliptic theory, we know that $||U_L||_{C^{2,\alpha}(\mathcal{T}_L)} \leq C < \infty$, where C is independent of L. Now letting L go to infinity, after taking a subsequence U_L converges in $C^2_{loc}(\mathbb{R}^3)$ to a solution U of the Euler Lagrange equation

$$\Delta U - \big(DW(U)\big)^T = 0, \quad U : \mathbb{R}^3 \to \mathbb{R}^3. \tag{4.6}$$

Furthermore, both (**??**) and (**??**) hold with U_L replaced by U.

We conclude that U is the desired symmetric quadruple junction solution. Indeed, a simple estimate of the energy of the trivial solution $V_0 \equiv 0$ shows that

$$E_3(V_0; \mathcal{T}_M) = CW(0)M^3,$$

which establishes U as a nontrivial solution. The asymptotic behavior in the next subsection shall give an accurate description of the quadruple junction structure of the solution.

4.3. **Behavior at infinity.** We shall study the asymptotic behavior of the solution U which is obtained in the previous section. From the energy estimates (**??**) and (**??**) with U_L replaced by U, we can obtain

$$\int_{\mathcal{C}_{\mathbf{xyz}}} |\frac{\partial U}{\partial x_3}|^2 dx \leq C \tag{4.7}$$

and

$$\int_{\mathcal{C}_{\mathbf{xy}}} |\frac{\partial U}{\partial x_3'}|^2 dx \leq C. \tag{4.8}$$

Similar to [**?**], we can prove that for any sequence $\{x_3^n\}$ tending to infinity, $U(x_1,x_2,x_3)$ converges to a triple junction solution $u(x_1,x_2)$ with minimum normalized energy e_2 along a subsequence of $\{x_3^n\}$. We can also prove that for any sequence $\{(x_3')^n\}$ tending to infinity, U converges to a transition layer solution with minimum normalized energy e_2' along a subsequence of $\{(x_3')^n\}$. To be more precise, let us define

$$\mathcal{K} = \big\{u_{\mathbf{abc}} \in C^2(\mathbb{R}^2 \to \mathbb{R}^3) : \quad , u_{\mathbf{abc}} \circ \gamma = \gamma \circ u_{\mathbf{abc}}, \forall \gamma \in \Gamma(\mathbf{a},\mathbf{b},\mathbf{c}), \\ u_{\mathbf{abc}} \text{ is a triple junction solution with renormalized energy } e_2\big\}$$

and

$$\mathcal{K}_L = \big\{u_{\mathbf{abc}}|_{T_L \to \mathbb{R}^3} : \quad u_{\mathbf{abc}} \in \mathcal{K}\big\}.$$

Similarly, we can define

$$\mathcal{S} = \big\{u^s_{\mathbf{ab}} \in C^2(\mathbb{R}^2 \to \mathbb{R}^3) : \quad u^s_{\mathbf{ab}}(-x_1,x_2) = \gamma_{\mathbf{ab}} \circ u^s_{\mathbf{ab}},\, u^s_{\mathbf{ab}}(x_1,-x_2) = \gamma_{\mathbf{cd}} \circ u^s_{\mathbf{ab}}, \\ u^s_{\mathbf{ab}} \text{ is a transition layer solution with renormalized energy } e_2'\big\}$$

and

$$\mathcal{S}_L = \big\{u^s_{\mathbf{ab}}|_{\Pi_{L/\alpha} \to \mathbb{R}^3} : \quad u^s_{\mathbf{ab}} \in \mathcal{S}\big\}.$$

From the standard theory of elliptic equations and systems, we know that $\mathcal{K}$ and $\mathcal{S}$ are compact sets in $C^2(\mathbb{R}^2 : \mathbb{R}^3)$. We also know that U is bounded in $C^2(\mathbb{R}^3 : \mathbb{R}^3)$. Then, by (**??**), (**??**) and (**??**) we derive

$$|\int_{\mathcal{C}_{\mathbf{abc}} \cap \{x_3 = t\}} \mathcal{L}_2(U(\cdot,t))dx_1x_2 - (\frac{3\sqrt{2}}{2}e_1 t + e_2)| \to 0, \quad \text{as } t \to \infty \tag{4.9}$$

and

$$|\int_{\mathcal{C}_{\mathbf{ab}} \cap \{x_3' = t\}} \mathcal{L}_2(U(\cdot,t))dx_1'x_2' - (\frac{\sqrt{2}}{2}e_1 t + e_2')| \to 0, \quad \text{as } t \to \infty. \tag{4.10}$$

Therefore, we conclude

$$\lim_{x_3 \to \infty} d\big(U(\cdot,x_3), \mathcal{K}_{\frac{\sqrt{6}}{2}|x_3|}\big) = 0. \tag{4.11}$$

and

$$\lim_{x_3' \to \infty} d\big(U(\cdot,x_3'), \mathcal{S}_{\frac{\sqrt{2}}{4}|x_3'|}\big) = 0. \tag{4.12}$$

In particular, if there are only finitely many triple junction solutions $u_{\mathbf{xyz}}$ for $\{\mathbf{x},\mathbf{y},\mathbf{z}\}$, then U converges to $u_{\mathbf{xyz}}$ as x_3 goes to infinity. Similarly, if there are only finitely many transition layer solution $u^s_{\mathbf{xy}}$ for $\{\mathbf{x},\mathbf{y}\}$, then U converges to $u^s_{\mathbf{xy}}$ as x_3' goes to infinity.

Further asymptotic behaviors of U restricted to rays of line segments and rays of points follow immediately from the asymptotic behavior of $u_{\mathbf{xyz}}$ and $u^s_{\mathbf{xy}}$. For example, for $0 < k < \sqrt{2}$ we have

$$\lim_{x_3\to\infty} \sup_{|x_1|\le\sqrt{3}kx_3} |U(x_1, kx_3, x_3) - z_{\mathbf{ab}}(x_1)| = 0. \tag{4.13}$$

For any vector $\mathbf{x} \in \mathbb{R}^3$ with $\mathbf{x}\cdot(\mathbf{a}-\mathbf{b}) > 0$, $\mathbf{x}\cdot(\mathbf{a}-\mathbf{c}) > 0$, $\mathbf{x}\cdot(\mathbf{a}-\mathbf{d}) > 0$ we have

$$\lim_{t\to\infty} |U(t\mathbf{x}) - \mathbf{a}| = 0. \tag{4.14}$$

The symmetry of U also yields the corresponding limits of U relative to other wells $\mathbf{b}, \mathbf{c}, \mathbf{d}$ and other heteroclinic solutions $z_{\mathbf{xy}}$ in different regions.

With the above limits, the quadruple structure of U is hence clearly displayed.

In summary, we have proven the main theorem of this article.

Theorem 4.1. *Under the assumptions* **(H1)-(H3)** *for the potential W and* **(H4)** *for the linearized operator A, there exists a solution U in $C^2(\mathbb{R}^3 : \mathbb{R}^3)$ to the Euler-Lagrange equation (**??**) with a symmetric quadruple structure. Namely, U satisfies $U \circ \gamma = \gamma \circ U$ for $\gamma \in \Gamma$ and (**??**), (**??**), (**??**) and (**??**). Furthermore, if we assume that $\mathcal{K}$ and $\mathcal{S}$ have only finite elements, then*

$$\lim_{x_3\to\infty} U(x_1, x_2, x_3) = u_{\mathbf{abc}}(x_1, x_2), \quad (x_1, x_2) \in \mathbb{R}^2 \tag{4.15}$$

and

$$\lim_{x'_3\to\infty} U(x'_1, x'_2, x'_3) = u^s_{\mathbf{ab}}(x'_1, x'_2), \quad (x'_1, x'_2) \in \mathbb{R}^2 \tag{4.16}$$

where $u_{\mathbf{abc}}$ and $u^s_{\mathbf{ab}}$ are a triple junction solution and a transition layer solution respectively.

The structure of the quadruple junction solution may be illustrated by Figure **??**.

5. Appendix: The generic feature of the nondegeneracy condition

In this appendix, we shall show that hypothesis **(H4)** holds generically. To this end, we define

$$\mathcal{Z} := \big\{ z : z \text{ is a heteroclinic connection of } \mathbf{a} \text{ and } \mathbf{b} \text{ with } E_1(z) = e_1, \quad \text{and } \gamma_{\mathbf{ab}} z = z \circ \gamma_{\mathbf{ab}} \big\} \tag{5.1}$$

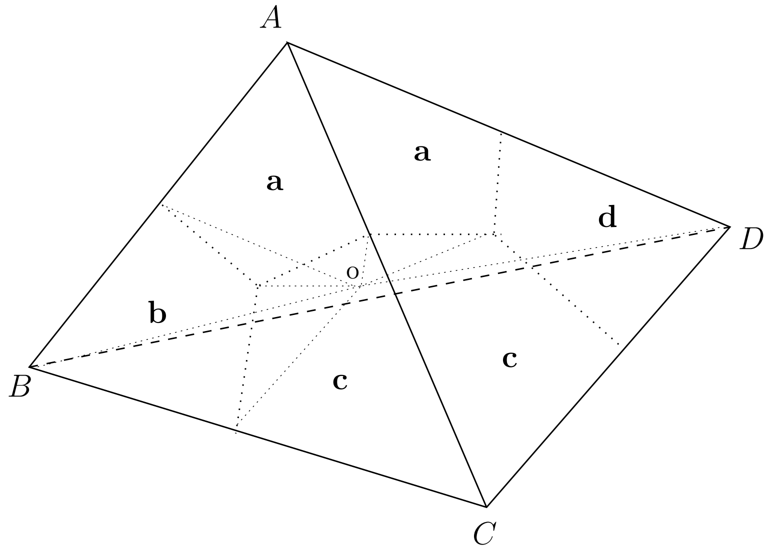

FIGURE 14. The quadruple junction structure with triple junction on each face

From §2.1, we know that $\mathcal{Z}$ is compact in $C^2(\mathbb{R} : \mathbb{R}^3)$ and

$$\mathcal{Z} = \mathcal{Z}_{\mathbf{c}} \cup \mathcal{Z}_{\mathbf{d}} \cup \mathcal{Z}_0$$

where $\mathcal{Z}_{\mathbf{c}}, \mathcal{Z}_{\mathbf{d}}$ are the subsets of elements of $\mathcal{Z}$ whose image lie entirely inside the cone **Coabc** and **Coabd** respectively, and $\mathcal{Z}_0$ is the set of elements of $\mathcal{Z}$ whose image lies entirely on the plane $\mathcal{P}$**oab**. By Corollary 2.5, we can further conclude that the image of elements in $\mathcal{Z}_{\mathbf{c}}$ belongs entirely in the interior of the cone **C** which has vertex at the origin **o** and the base triangle $[\mathbf{a}, \mathbf{b}, \frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}]$.

We may choose the coordinates (u_1, u_2, u_3) of target space $\mathbb{R}^3$ so that $\mathbf{c} - \mathbf{d}$ is the direction of u_3 axis, $\mathbf{a} - \mathbf{b}$ is the direction of u_1 axis, and $\mathbf{a} + \mathbf{b}$ is the direction of u_2 axis.

Select a function $z = (z_1, z_2, z_3) \in \mathcal{Z}_{\mathbf{c}} \cup \mathcal{Z}_0$. We claim $z_1'(0) \neq 0$. Indeed, by integrating the dot product of equation (1.5) and z', and using (2.8) we obtain

$$-\frac{1}{2}|z'(t)|^2 + W(z(t)) = 0, \quad t \in \mathbb{R} \tag{5.2}$$

By symmetry of z, we always have $z_2'(0) = z_3'(0) = 0$. Then $W(u(0)) = 0$ if $z_1'(0) = 0$. This is impossible and proves the claim. Now by the inverse function theorem we can find even functions η_2 and η_3 on a non empty interval $(-\delta_1, \delta_1)$ such that

$$z_2(t) = \eta_2(z_1(t)), \quad z_3(t) = \eta_3(z_1(t)), \quad |t| < \delta_1.$$

Moreover by Lemma 2.1, there is no other minimizer (up to translation) which satisfies this relation. Since z is of class C^3, η is of class C^3.

Choose $\delta_2 \in (0, \delta_1)$ such that $\{z(t) : |t| \le \delta_2\}$ is in the interior of $\mathbf{C} \cup \gamma_{\mathbf{cd}}\mathbf{C}$. Now we choose a nonnegative function ρ so that it satisfies $\rho \circ \gamma_{\mathbf{cd}} = \rho$ in $\mathbf{C} \cup \gamma_{\mathbf{cd}}\mathbf{C}$ and it's support lies in the interior of $\mathbf{C} \cup \gamma_{\mathbf{cd}}\mathbf{C}$. We can adjust ρ so that η_2, η_3 are well defined in the support of ρ.

Given any positive constant δ, we can define $\tilde{W}$ in $\mathbf{C} \cup \gamma_{\mathbf{cd}}\mathbf{C}$ by

$$\tilde{W}(u) = W(u) + \delta\rho(u)[(u_2 - \eta_2(u_1))^2 + (u_3 - \eta_3(u_1))^2].$$

Then $\tilde{W}$ can be defined over all of $\mathbb{R}^3$ by Γ invariance. The corresponding energy is denoted

$$\tilde{E}_1(u) = \int_{\mathbb{R}} \left[\frac{1}{2}|u'|^2 + \tilde{W}(u) \right].$$

It is easy to check $E_1(z) = \tilde{E}_1(z)$, and that for all $v \in \mathcal{Z}_{\mathbf{c}} \setminus \{u\}$, $\tilde{E}_1(v) > \tilde{E}_1(z) = E_1(z)$. Moreover, for all $v \in \mathcal{Z}_{\mathbf{c}} \cup \mathcal{Z}_0$, $\tilde{E}_1(v) \ge E_1(v)$. This shows that generically either $\mathcal{Z} = \mathcal{Z}_0$ has only one minimizer of E_1 or $\mathcal{Z} = \mathcal{Z}_{\mathbf{c}} \cup \mathcal{Z}_{\mathbf{d}}$ has exactly two minimizers of E_1 which are reflections of each other by $\gamma_{\mathbf{cd}}$.

We pass now to the spectral analysis of the linearized operator around z. This operator is an unbounded operator in $L^2(\mathbb{R})^3$ defined by

$$D(A) = H^2(\mathbb{R})^3, \quad Av = -v'' + (D^2W(z)v).$$

From the point of view of linear algebra, $v \mapsto D^2W(z(t))v$ is a linear symmetric mapping form $\mathbb{R}^3$ to itself which is uniformly bounded in t ; thus A is a self-adjoint operator in $L^2(\mathbb{R})^3$; by a generalization of Persson's theorem to the self-adjoint vector case (see [**?**] Theorem 3.12), we have

(5.3)

$$\begin{aligned} \inf \sigma_{\text{ess}}(A) &= \sup_{L>0} \inf \{ \int_{-L}^{L} \left[|z'|^2 + D^2W(u) z \otimes z \right] dx : |z|_{L^2} = 1, \} \\ &= \sup_{L>0} \inf \{ D^2W(u(x))\zeta \otimes \zeta : \zeta \in \mathbb{R}^2, |\zeta| = 1, |x| \ge L \} \\ &= \lambda_1. \end{aligned}$$

If we differentiate (1.5), we can see that

$$-z''' + D^2W(z)^T z' = 0.$$

The function z' satisfies $(z \circ \gamma_{\mathbf{ab}})' = -\gamma_{\mathbf{ab}} \circ z'$. Since z' and z'' are square integrable, z''' is square integrable too ; this means that z' is an eigenfunction of A corresponding to the eigenvalue 0. Moreover, 0 is the lower bound of the spectrum of $A:$ we know that for all $t \in \mathbb{R}$ and all $v \in H^1(\mathbb{R})^3$,

$$E_1(z + tv) \geq E_1(z).$$

Hence,

$$(Av, v) \geq 0, \quad \forall v \in H^1(\mathbb{R})^3.$$

Since $v(t)$ is a function with values in $\mathbb{R}^3$, there are at most three independent solutions of the ordinary differential system

$$-v'' + D^2W(z)v = 0, \quad v : \mathbb{R} \to \mathbb{R}^3 \tag{5.4}$$

which tend to zero at $-\infty$; therefore, the eigenvalue 0 has at most multiplicity 3. In particular, if v is restricted to the symmetric space

$$\mathbf{L}^2_s(\mathbb{R}) = \{v \in L^2(\mathbb{R})^3 : v \circ \gamma_{\mathbf{ab}} = \gamma_{\mathbf{ab}} \circ v\},$$

the multiplicity of eigenvalue $\lambda = 0$ is at most 2.

We show now that "generically", the kernel of A restricted to $\mathbf{L}^2_s(\mathbb{R})$ is reduced to $\{0\}$.

Define a space $\mathbf{H}^1_s(\mathbb{R})$ by

$$\mathbf{H}^1_s(\mathbb{R}) = \mathbf{L}^2_s(\mathbb{R}) \cap H^1(\mathbb{R})^3.$$

Define $\tilde{A}$ as the linearized operator at z associated to $\tilde{W}$, i.e.

$$\tilde{A}(v) := -v'' + D^2\tilde{W}(z)v, \quad v \in \mathbf{H}^1_s(\mathbb{R}).$$

The lower bound of the spectrum of $\tilde{A}$ in $\mathbf{L}^2_s(\mathbb{R})$ is given by

$$\inf \sigma(\tilde{A}) = \inf\{(\tilde{A}v, v) : v \in \mathbf{H}^1_s(\mathbb{R}), |v|_{L^2} = 1\}.$$

Let P be the orthogonal projection on the orthogonal of the kernel of A restricted to $\mathbf{L}^2_s(\mathbb{R})$; we have

$$(Av, v) \geq \nu^2 |Pv|^2_{L^2}, \forall v \in \mathbf{H}^1_s(\mathbb{R}).$$

Note that ν^2 is the infimum of positive spectrum of A. Therefore,

$$(\tilde{A}v, v) \geq \nu^2 |Pv|^2 + \int (D^2\tilde{W}(z) - D^2W(z)) v^{\otimes 2} \, dt.$$

If the kernel of A in $\mathbf{L}^2_s(\mathbb{R})$ is reduced to zero, we are done. If it contains an element ϕ, then

$$(\tilde{A}\phi, \phi) \geq \nu^2 |P\phi|^2 + \int (D^2\tilde{W}(z) - D^2W(z)) \phi^{\otimes 2} \, dt.$$

so that, according to the definition of $\tilde{W}$,

$$(\tilde{A}\phi, \phi) \geq \nu^2 |P\phi|^2 + 2\delta \int \rho(z) [|\phi_1(t)\eta_2'(z_1(t)) - \phi_2(t)|^2 + |\phi_1(t)\eta_3'(z_1(t)) - \phi_3(t)|^2] \, dt.$$

This shows that $\tilde{A} - A$ is positive in the sense of quadratic forms, and the kernel of $\tilde{A}$ will be strictly larger than $\{\phi \equiv 0\}$ iff

$$(5.5) \quad \int \rho(z)[|\phi_1(t)\eta_2'(z_1(t)) - \phi_2(t)|^2 + |\phi_1(t)\eta_3'(z_1(t)) - \phi_3(t)|^2] = 0$$

Then, for $|t| \leq \delta_2$

$$(5.6) \quad \phi_1(t)\eta_2'(z_1(t)) = \phi_2(t), \quad \phi_1(t)\eta_3'(z_1(t)) = \phi_3(t)$$

By symmetry of ϕ :

$$\phi_1(0) = 0, \quad \phi_2'(0) = 0, \quad \phi_3'(0) = 0.$$

From (**??**), we also have

$$\phi_2(0) = 0, \quad \phi_3(0) = 0.$$

Since ϕ and z' both are in the kernel of A, i.e., solutions of (**??**), we have

$$(5.7) \quad (\phi \cdot z'' - \phi' \cdot z')' = 0, \quad t \in \mathbb{R}$$

Therefore, by using the asymptotic behavior of z and ϕ we obtain

$$(5.8) \quad \phi \cdot z'' - \phi' \cdot z' = 0, \quad t \in \mathbb{R}.$$

In particular, letting $t = 0$ and using $z_1'(0) \neq 0$, we conclude

$$\phi_1'(0) = 0.$$

Since ϕ is the solution of a differential equation of the second order, ϕ has to vanish identically. This is a contradiction. Therefore we have proved

Theorem 5.1. *Assume that $\mathcal{Z}$ has more than two elements or that $\ker(A) \cap \mathbf{L}_s^2(\mathbb{R}) \neq \{0\}$; then, there is a non empty interval $(0, \delta_0)$ and a function $\hat{W} \in C_0^3$ such that for $\tilde{W} = W + \delta\hat{W}$, $\delta \in (0, \delta_0)$ either the corresponding $\tilde{\mathcal{Z}}$ has only one element which is in $\mathcal{Z}_0$, or $\tilde{\mathcal{Z}}$ has exactly two elements which are reflections of each other by $\gamma_{\mathbf{cd}}$. Furthermore, the kernel of the corresponding $\tilde{A}$ intersected with $\mathbf{L}_s^2(\mathbb{R})$ is reduced to zero.*

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References

[1] Stanley Alama, Lia Bronsard and Changfeng Gui, Stationary layered solutions in R^2 for an Allen-Cahn systems with multiple-well potentials, *Calculus of Variation and PDE,* **5**, *359-390 (1997)*

[2] L. Bronsard and F. Reitich, On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation, *Arch. Rational Mech. Anal.* , **124**, pp 355-379 (1993).

[3] Lia Bronsard, Changfeng Gui and Michelle Schatzman, A three layered solution in R^2 for a variational problem with a symmetric three well potential, *Communications on Pure and Applied Mathematics,* **49**, *pp 677-715 (1996).*

[4] H.L. Cycon, R.G. Froese, W. Kirsch and B. Simon, Schrodinger Operators, with Applications to Quantum Mechanics and Global Geometry, Berlin, Springer-Verlag, 1987.

[5] Lawrence C. Evans, Partial Differential Eqautions, AMS, 1998.

[6] V. Y. Gertsman, Coincidence site lattice theory of multicrystalline ensembles, *Acta Crystallographica, Section A* **57**, *pp 649-655 (2001).*

[7] A. H. King, The geometric and thermodynamic properties of grain boundary junctions , *Interface Science* , **7** pp 251-271 (1999).

[8] Bryan W. Reed, Roger W. Minich, Robert E. Rudd and Mukul Kumar, The structure of the cubic coincident site lattice rotation group, *Acta Crystallographica, Section A* **60**, *pp 263-277 (2004).*

[9] P. Sternberg, Vector-valued local minimizers of nonconvex variational problems, *Rocky Mt. J. Math* **21** *(2), pp 799-807 (1991).*

[10] P. Sternberg and W. Ziemer, Local minimizers of a three-phase partition problem with triple junction, *Proc. Roy. Soc. Edin.* **124A** *pp 1059-1073 (1994).*

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