

HAMILTONIAN IDENTITIES FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

CHANGFENG GUI

ABSTRACT. New identities for elliptic partial differential equations are obtained. Several applications are discussed. In particular, Young's Law for the contact angles in triple junction formation is proven rigorously. Structure of level curves of saddle solutions to Allen-Cahn equation are also carefully analyzed.

1. INTRODUCTION AND THE STATEMENT OF HAMILTONIAN TYPE IDENTITY

Given a $C^{1,\alpha}$ potential function $H(p), p \in \mathbb{R}^m$, and consider a solution $p(t)$ to a system of second order ordinary differential equation

$$(1.1) \quad -p''(t) + \nabla_p H(p(t)) = 0, \quad t \in \mathbb{R}$$

we always have the Hamiltonian identity

$$(1.2) \quad \frac{1}{2}|p'(t)|^2 - H(p(t)) \equiv C, \quad \text{in } \mathbb{R}.$$

Another way of writing the above equation is in the form of first order Hamiltonian system

$$\begin{cases} p' = H_q(p, q), & t \in \mathbb{R} \\ q' = -H_p(p, q), & t \in \mathbb{R} \end{cases}$$

where $H(p, q) = \frac{1}{2}|q|^2 - H(p)$. It is a basic and fundamental fact that $H(p, q)$ remains constant in the orbits of the solutions.

On the other hand, consider the case of $m = 1$ and a bounded entire solution $u(x)$ of the second order elliptic equation

$$(1.3) \quad -\Delta u(x) + H'(u(x)) = 0, \quad x \in \mathbb{R}^n$$

Modica proved a point wise gradient estimate

$$(1.4) \quad \frac{1}{2}|\nabla u|^2 - H(u) \leq 0, \quad x \in \mathbb{R}^n.$$

This inequality may be regarded as a generalization of the second order Hamiltonian to higher spatial dimensions in the case of single equation. It plays an important role in the study of entire solutions, and leads properties such as monotonicity formula. However, it is only an inequality. This makes one wonder if any identity could be regarded as a more natural generalization of Hamiltonian identity to partial differential equations in higher dimensions. In particular, we would ask the following questions

- is there any identity for partial differential equations which may be a generalization of (1.2)?
- how about systems of partial differential equations?

It is the intention of this article to provide a version of such generalization, which may be called Hamiltonian identity in higher dimensions, and to show some examples of its applications. It would be interesting to see other types of the generalizations.

We first state a Hamiltonian identity for partial differential equations on two dimensional planes, which can be generalized to higher dimensional spaces. However, due to its simpler formulation and applications, we present it separately.

Consider an entire solution $u \in C^2(\mathbb{R}^2, \mathbb{R}^m)$ to the system of partial differential equations

$$(1.5) \quad -\Delta u + \nabla_u H(u(x)) = 0, \quad x \in \mathbb{R}^2.$$

Theorem 1.1. *If u is bounded and $u(x_1, x_2)$ converges to $a(x_2), b(x_2)$ respectively as x_1 tends ∞ and $-\infty$, then the following Hamiltonian identity holds for u*

$$(1.6) \quad \int_{-\infty}^{\infty} \left[\frac{1}{2} \left(\left| \frac{\partial u}{\partial x_1} \right|^2 - \left| \frac{\partial u}{\partial x_2} \right|^2 \right) + H(u(x)) \right] dx_1 = C, \quad \forall x_2 \in \mathbb{R}.$$

provided that the integral is finite for at least one value of x_2 . In general, the identity holds whenever the integral is finite for $x_2 \in \mathbb{R}$ and the limit in the right hand side of (1.9) below is zero as N, M go to ∞ .

Proof. Let us define

$$(1.7) \quad \rho_{N,M}(x_2) = \int_{-M}^N \left[\frac{1}{2} \left(\left| \frac{\partial u}{\partial x_1} \right|^2 - \left| \frac{\partial u}{\partial x_2} \right|^2 \right) + H(u(x)) \right] dx_1.$$

Then, using the equation and integration by parts, we have

$$(1.8) \quad \begin{aligned} \rho'_{N,M}(x_2) &= \int_{-M}^N (u_{x_1} \cdot u_{x_1 x_2} - u_{x_2} \cdot u_{x_2 x_2} + \nabla_u H(u) \cdot u_{x_2}) dx_1 \\ &= \int_{-M}^N [u_{x_1} \cdot u_{x_1 x_2} + u_{x_1 x_1} \cdot u_{x_2}] dx_1 \\ &= (u_{x_1} \cdot u_{x_2})(N, x_2) - (u_{x_1} \cdot u_{x_2})(-M, x_2). \end{aligned}$$

Without loss of generality, we may assume that the value of x_2 for which the integral in (1.6) is finite is $x_2 = 0$. We can rewrite the above equality as

$$(1.9) \quad \rho_{N,M}(x_2) - \rho_{N,M}(0) = \int_0^{x_2} [(u_{x_1} \cdot u_{x_2})(N, s) - (u_{x_1} \cdot u_{x_2})(-M, s)] ds$$

Since u is bounded and $H(u)$ is C^2 , by the standard elliptic theory we know that u is bounded in $C^3(\mathbb{R}^2, \mathbb{R}^m)$. Furthermore, $u(x_1 + N, x_2)$ converges in $C^3_{loc}(\mathbb{R}^2, \mathbb{R}^m)$ to a solution $u_1(x)$ and $u_1(x_1, x_2) = a(x_2)$. Similarly, $u(x_1 - M, x_2)$ converges in $C^3_{loc}(\mathbb{R}^2, \mathbb{R}^m)$ to a solution $u_2(x)$ and $u_2(x_1, x_2) = b(x_2)$. Therefore $u_{x_1}(x_1, x_2)$ converges to 0 uniformly in any compact set of x_2 as x_1 goes to infinity. The Hamiltonian identity follows immediately by letting N, M in (1.9) go to ∞ .

In general, if the right hand side of (1.9) has zero limit, then the identity (1.6) holds. Therefore, we may write the Hamiltonian identity formally, and verify the limiting procedure in each application. □

The following identity may be regarded as the Hamiltonian identity for higher dimensional spaces.

Write $x = (x', x_n) \in \mathbb{R}^n$ and consider an entire solution $u \in C^2(\mathbb{R}^n, \mathbb{R}^m)$ to the system of partial differential equations

$$(1.10) \quad -\Delta u + \nabla_u H(u(x)) = 0, \quad x \in \mathbb{R}^n.$$

Theorem 1.2. *The following Hamiltonian identity holds for u*

$$(1.11) \quad \int_{\mathbb{R}^{n-1}} \left[\frac{1}{2} (|\nabla_{x'} u|^2 - |u_{x_n}|^2) + H(u(x)) \right] dx' = C, \quad \forall x_n \in \mathbb{R}.$$

provided that the integral is finite for at least one value of x_n and the right hand side of (1.14) below tends to zero as R goes to infinity along a sequence.

Proof. Let us define

$$(1.12) \quad \rho_R(x_n) = \int_{B_R(0)} \left[\frac{1}{2} (|\nabla_{x'} u|^2 - |u_{x_n}|^2) + H(u(x)) \right] dx'$$

Then, using the equation and integration by parts, we have

$$(1.13) \quad \begin{aligned} \rho'_R(x_n) &= \int_{B_R(0)} [\nabla_{x'} u \cdot \nabla_{x'} u_{x_n} - u_{x_n} \cdot u_{x_n x_n} + \nabla_u H(u(x)) \cdot u_{x_n}] dx' \\ &= \int_{B_R(0)} [\nabla_{x'} u \cdot \nabla_{x'} u_{x_n} + \Delta_{x'} u \cdot u_{x_n}] dx' \\ &= \int_{\partial B_R(0)} \left[\frac{\partial u}{\partial \nu_{x'}} \cdot u_{x_n} \right] dS_{x'} \end{aligned}$$

We may assume that the integral in (1.11) is finite for $x_n = 0$. We can rewrite the above equality as

$$(1.14) \quad \rho_R(x_n) - \rho_R(0) = \int_0^{x_n} \int_{\partial B_R(0)} \left[\frac{\partial u}{\partial \nu_{x'}}(x', s) \cdot u_{x_n}(x', s) dS_{x'} \right] ds$$

The formal identity becomes rigorous, by taking the limit of the above equality as R tends to infinity, under the condition that the limit goes to zero. \square

As a special case, the Hamiltonian identity holds with $C = 0$ when the solution belongs to a Sobolev space H^1 .

Corollary 1.3. *Assume H is C^2 and $u \in H^1(\mathbb{R}^n, \mathbb{R}^m)$ is a solution to (1.10). Then the following Hamiltonian identity holds for any choice of cartesian coordinates*

$$(1.15) \quad \int_{\mathbb{R}^{n-1}} \left[\frac{1}{2} (|\nabla_{x'} u|^2 - |u_{x_n}|^2) + H(u(x)) \right] dx' = 0, \quad \forall x_n \in \mathbb{R}.$$

where H is chosen so that $H(0) = 0$.

Proof. We note that u is also a classical solution and $u(x) \rightarrow 0$ uniformly as $x \rightarrow \infty$, according to the standard theory of elliptic equations. Hence $\nabla H(0) = 0$. Then the integral in (1.15) is finite for at least a sequence of x_n which goes to infinity, since u belongs to $H^1(\mathbb{R}^n, \mathbb{R}^m)$. The same fact also guarantees that the limit condition in Theorem 1.2 holds true and therefore (1.11) is valid. On the other hand, we know that $\rho(x_n)$ tends to 0 at least along a sequence of x_n tending to infinity. Therefore $C = 0$. \square

A typical example of a H^1 solution is the unique positive radial solution of

$$(1.16) \quad -\Delta u + u - u^p = 0, \quad x \in \mathbb{R}^2, \quad p > 1$$

when $H(u) = \frac{1}{2}u^2 - \frac{1}{p+1}u^{p+1}$.

We shall see below that Pohazaev identity can be derived immediately from the above identity.

Integrating (1.15) in \mathbb{R} with respect to x_n , we obtain

$$(1.17) \quad \int_{\mathbb{R}^n} \left[\frac{1}{2} (|\nabla_{x'} u|^2 - |u_{x_n}|^2) + H(u(x)) \right] dx = 0.$$

Replacing x_n with x_i , we shall obtain $n - 1$ similar identities. Sum up all these identities, we derive

$$(1.18) \quad \int_{\mathbb{R}^n} \left[\frac{n-2}{2} |\nabla u|^2 + nH(u(x)) \right] dx = 0.$$

This is indeed Pohazaev identity in the entire space. We believe that identity (1.15) is a fundamental property of solutions, which gives more detailed information in a lower dimension space and applies to a general class of problems in the whole space.

When a solution u is not in $H^1(\mathbb{R}^n)$, we may still have Hamiltonian identity (1.15) even though Pohazaev identity (1.18) may not hold. A typical example is a solution u of degree $d \geq 1$ to the following two dimensional Ginzburg-Landau equation

$$(1.19) \quad \Delta u + u(1 - |u|^2) = 0, \quad u : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \approx \mathbb{C}.$$

with

$$(1.20) \quad \int_{\mathbb{R}^2} H(u) dx = \frac{1}{4} \int_{\mathbb{R}^2} (1 - |u|^2)^2 dx = \frac{1}{2} \pi d^2 < \infty.$$

Indeed, we can prove

Theorem 1.4. *The solution u of (1.19) and (1.20) satisfies*

$$(1.21) \quad \int_{\mathbb{R}} \left[\frac{1}{2} \left(\left| \frac{\partial u}{\partial x_1} \right|^2 - \left| \frac{\partial u}{\partial x_2} \right|^2 \right) + \frac{1}{4} (1 - |u(x)|^2)^2 \right] dx_1 = 0, \quad \forall x_2 \in \mathbb{R}.$$

The identity basically follows from (1.6) and the following asymptotic behavior of u at infinity.

Proposition 1.5. ([8], [23]) *Suppose u is a solution to (1.19) and (1.20). Then there exists $R_0 > 0$ such that $u(x) = f(x)e^{i(d\theta + \psi(x))}$, $\forall x \in B_{R_0}^c$ and*

$$(i) \quad f(x) = 1 - \frac{d^2}{2|x|^2} + o\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \rightarrow \infty$$

$$(ii) \quad |\nabla f(x)| = \frac{d^2}{2|x|^3} + o\left(\frac{1}{|x|^3}\right), \quad \text{as } |x| \rightarrow \infty$$

$$(iii) \quad \lim_{|x| \rightarrow \infty} \psi(x) = \theta_0, \quad \int_{|x| \geq R_0} |\nabla \psi(x)|^2 dx < \infty.$$

Proof of Theorem 1.4. We note that Proposition 1.5 leads to

$$(1.22) \quad |\nabla u|^2 \leq 2|\nabla f|^2 + C(|\nabla\theta|^2 + |\nabla\psi|^2) \leq C\left(\frac{1}{|x_2|^2 + 1} + |\nabla\psi|^2\right), \quad x \in \mathbb{R}^2.$$

Therefore, the integral in (1.21) is finite for almost all $x_2 \in \mathbb{R}$. It is also easy to see that $u(x_1, x_2) \rightarrow e^{i\theta_0}$ as $x_1 \rightarrow \infty$ and $u(x_1, x_2) \rightarrow e^{i(d\pi + \theta_0)}$ as $x_1 \rightarrow -\infty$ for any fixed x_2 . Therefore, (1.6) holds. By (1.22) and Proposition 1.5, there exists at least a sequence $\{s_n\}$ such that $\lim_{n \rightarrow \infty} s_n = \infty$ and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |\nabla u(x_1, s_n)|^2 dx_1 = 0.$$

It is obvious from (i) of Proposition 1.5 that

$$\lim_{x_2 \rightarrow \infty} \int_{\mathbb{R}} (1 - |u(x)|^2)^2 dx_1 = 0.$$

Hence (1.21) holds. The theorem is proven.

We note that the solution u to (1.19) and (1.20) does not belong in $H^1(\mathbb{R}^2)$ when $d \geq 1$.

In next sections, more applications of the Hamiltonian identity and its modifications shall be discussed. The applications are less obvious and need more analysis. In particular, §2 deals with solutions to the vector-valued Allen-Cahn equation in \mathbb{R}^2 , which needs some preliminaries in the formulation of the problem. §3 and §4 deal with sign changing solutions to the scalar Allen-Cahn equation, which is conceptually easier to understand than §2, but contain technically harder analysis. It is arranged that §3 consists of the main ideas with simple formulation and §4 is devoted to some technical details. The reader may choose either §2 or §3 to start with.

2. TRIPLE JUNCTIONS AND THE YOUNG'S LAW

In the study of multiple phase separation, a vector-valued Allen-Cahn model was proposed by Bronsard and Reitich in [10]. In this model, a physical state of material of multiple phases is represented by an order parameter (vector-valued function) $v \in \mathbb{R}^2$. The dynamics of the physical state may be modeled by an Allen-Cahn type system of partial differential equation

$$(2.1) \quad v_t = \epsilon \Delta v - \frac{1}{\epsilon} \nabla_v W(v), \quad x \in \Omega, t > 0$$

where $W \in C^{1,\alpha}(\mathbb{R}^2 \rightarrow \mathbb{R})$ is a triple well potential satisfying

(H1): there exist three points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$ such that $W(\mathbf{a}) = W(\mathbf{b}) = W(\mathbf{c}) = 0$ and $W(u) > 0$ for $u \in \mathbb{R}^2 \setminus \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, and $D^2W(\mathbf{a}), D^2W(\mathbf{b})$ and $D^2W(\mathbf{c})$ are positive definite;

(H2): there exists $R_0 > 0$ such that $\nabla W(u) \cdot u \geq 0$ when $|u| \geq R_0$.

Choose any two wells $\mathbf{x}, \mathbf{y} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, we may consider the minimization problem

$$(2.2) \quad \mathbf{e}_{\mathbf{xy}} = \min \left\{ \int_{\mathbb{R}} \frac{1}{2} |v'|^2 + W(v) dt : v \in H_{\text{loc}}^1(\mathbb{R})^2, v(-\infty) = \mathbf{x}, v(\infty) = \mathbf{y} \right\}.$$

It can be shown that $\mathbf{e}_{\mathbf{xy}} > 0$ (see, e.g., [25]). It is also shown in [1] that there is at least one minimizer $\mathbf{u}_{\mathbf{xy}}$ for (2.2) as long as the following partial wetting condition holds

$$(2.3) \quad \mathbf{e}_{\mathbf{xy}} < \mathbf{e}_{\mathbf{xz}} + \mathbf{e}_{\mathbf{yz}}, \quad z \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \setminus \{\mathbf{x}, \mathbf{y}\}.$$

The minimizer is a heteroclinic solution. See also [3] for more detailed discussion regarding the existence of heteroclinic solutions.

To make our arguments more transparent, we assume in this section that

(H3): the wetting condition (2.3) holds and $\mathbf{u}_{\mathbf{x}\mathbf{y}}$ is unique up to translation for all \mathbf{x}, \mathbf{y} .

We say that a triple well potential W is of symmetry of an equilateral triangle if it satisfies

(H4): three wells $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form an equilateral triangle and the potential W is equivariant under the group action of the isometry group Γ of the triangle.

An example of a triple well potential which satisfies (H1), (H2) and (H4) is

$$W(u) = |u^3 - 1|^2, \quad u \in \mathbb{R}^2 \approx \mathbb{C}.$$

A special feature of multiple phase separation is the formation of triple junctions, which is analyzed formally in [10]. The finer structure of triple junctions may be demonstrated by an entire solution u to the following system of elliptic equations (vector-valued equation)

$$(2.4) \quad -\Delta u + \nabla_u W(u) = 0, \quad u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

with u asymptotically close to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in three separate sectors of \mathbb{R}^2 .

Under the conditions (H1), (H2) and (H4), Bronsard, Gui and Schatzman proved rigorously in [9] the existence of such a triple junction solution. To be more precise, a simple version of the main result of [9] may be stated as follows.

Theorem 2.1. ([9]) *Suppose W satisfies (H1), (H2) and (H4). Then*

- (1) *there exists a nontrivial bounded Γ -equivariant solution \mathbf{U} in $C^3(\mathbb{R}^2, \mathbb{C})$ to (2.4). If we identify $x = (x_1, x_2)$ with $x_1 + ix_2 = re^{i\theta}$, then for any $\eta \in (0, \pi/3)$, $\mathbf{U}(re^{i\theta})$ converges to \mathbf{a} uniformly with respect to $\theta \in [(\pi/2) + \eta, (7\pi/6) - \eta]$ as r tends to infinity.*
- (2) *the solution $\mathbf{U}(x_1, x_2)$ converges to $\mathbf{u}_{\mathbf{ab}}$ uniformly on \mathbb{R} as x_2 goes to infinity.*

In other words, \mathbf{U} is a solution with a triple junction structure, i.e., \mathbf{U} has three transition layers separating the regions where \mathbf{U} is close to \mathbf{a}, \mathbf{b} or \mathbf{c} respectively. (See Figure 1.)

It is natural to ask if there is any other solution to (2.4) which is not necessarily symmetric with respect to Γ , but still displays a triple junction structure. This question seems very difficult to answer now. It would be interesting to ask whether a triple junction solution should be asymptotically symmetric. If we call the angles between the interfaces contact angles, the question would be whether the contact angles of any triple junction solutions must be the same. In physics theory regarding the interfaces of materials, the contact angles near a triple junction are determined by the tensions at the interfaces between the different materials according the Young's Law (see [26]):

$$(2.5) \quad \frac{k_1}{\sin \theta_1} = \frac{k_2}{\sin \theta_2} = \frac{k_3}{\sin \theta_3},$$

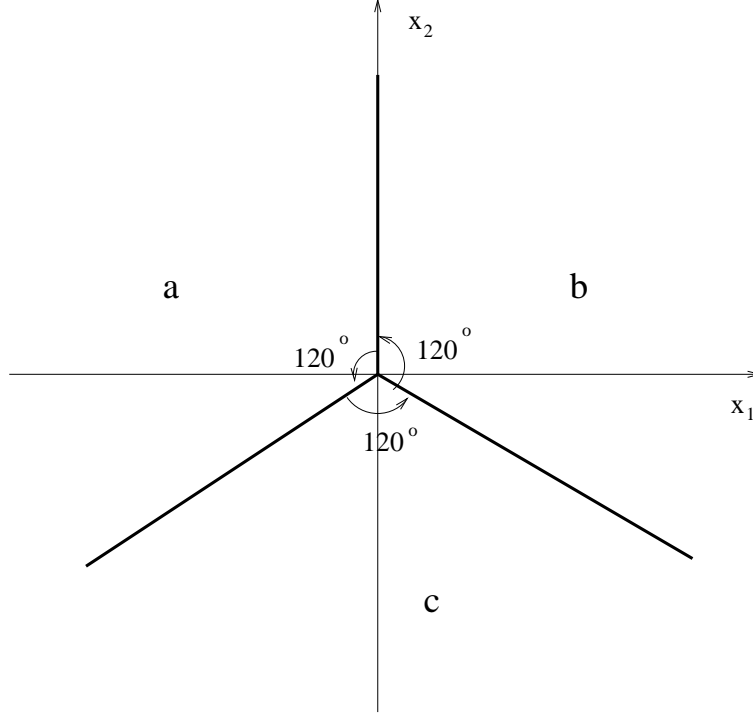


FIGURE 1. The triple junction solution u with regions close to **a**, **b**, **c** respectively

where k_i are the surface tension between two materials and θ_i are contact angle of the corresponding two materials. Regarding the limiting problem of (2.1), which is a geometric evolution problem, a formal analysis leads to Young's Law, with $\mathbf{e}_{\mathbf{x}\mathbf{y}}$ being the surface tension between the phases represented by \mathbf{x}, \mathbf{y} . See [10] and [16]. We shall show rigorously below the counterpart of Young's Law for system (2.4). In particular, we answer positively the above question of equal angles for symmetric triple well potential.

Theorem 2.2. *Suppose W satisfies (H1)–(H4) and $u \in C^3(\mathbb{R}^2, \mathbb{R}^2)$ is a solution to (2.4) with the following triple junction structure:*

- (1) *if $\mathbb{R}^2 \approx \mathbb{C}$ is divided into three sections $S_1 := \{x = re^{i\theta} \mid 0 < \theta < \theta_{\mathbf{ca}}\}$, $S_2 := \{x = re^{i\theta} \mid \theta_{\mathbf{ca}} < \theta < \theta_{\mathbf{ca}} + \theta_{\mathbf{ab}}\}$, $S_3 := \{x = re^{i\theta} \mid \theta_{\mathbf{ca}} + \theta_{\mathbf{ab}} < \theta < \theta_{\mathbf{ca}} + \theta_{\mathbf{ab}} + \theta_{\mathbf{bc}} = 2\pi\}$, then $u(re^{i\theta})$ converges to **b** in S_1 as the distance $d(x)$ to the boundary of S_1 goes to infinity. Similar statements holds for S_2 and S_3 with limits to **c**, **a** respectively;*
- (2) *for any sufficiently small $\delta > 0$, $u(x_1, x_2)$ converges to $\mathbf{u}_{\mathbf{ab}}(x_2)$ uniformly in $S_{13}^\delta := \{x = re^{i\theta} \mid \theta \in [-\theta_{\mathbf{bc}} + \delta, \theta_{\mathbf{ca}} - \delta]\}$ as x_1 goes to infinity. Similar statements holds for $S_{12}^\delta, S_{23}^\delta$ with limiting transitions $\mathbf{u}_{\mathbf{ac}}$ and $\mathbf{u}_{\mathbf{cb}}$ respectively.*

Then the following Young's Law holds

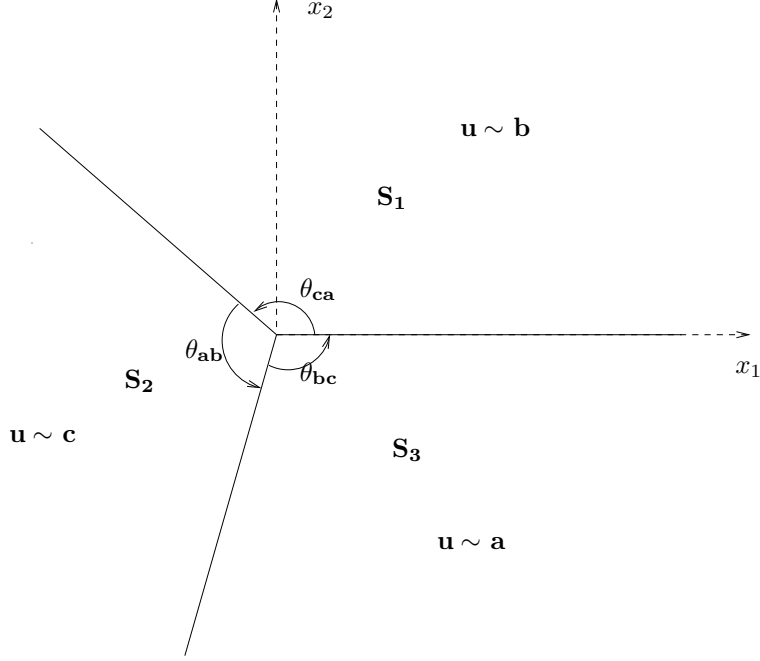


FIGURE 2. The three sectors where the value of u is close to $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively

$$(2.6) \quad \frac{\mathbf{e}_{\mathbf{ab}}}{\sin \theta_{\mathbf{ab}}} = \frac{\mathbf{e}_{\mathbf{bc}}}{\sin \theta_{\mathbf{bc}}} = \frac{\mathbf{e}_{\mathbf{ca}}}{\sin \theta_{\mathbf{ca}}}.$$

Proof. We shall use the Hamiltonian identity in Theorem 1.1 to prove this theorem. We shall first show that in S_1 the solution u is exponentially close to \mathbf{b} in terms of $d(x)$ as $d(x)$ goes to infinity. Similar estimate can be proven for u in S_2, S_3 . Since u goes to \mathbf{b} as $d(x)$ goes to infinity in S_1 , and $D^2W(\mathbf{b})$ is positive definite, then

$$(u - \mathbf{b}) \cdot (\nabla_u W(u) - \nabla_u W(\mathbf{b})) \geq \mu |u - \mathbf{b}|^2, \quad \text{when } d(x) \geq D_0$$

for some positive constants μ and D_0 .

Then, from (2.4) we obtain

$$(2.7) \quad \begin{aligned} \Delta |u - \mathbf{b}|^2 &\geq 2\Delta(u - \mathbf{b}) \cdot (u - \mathbf{b}) \\ &\geq (u - \mathbf{b}) \cdot (\nabla_u W(u) - \nabla_u W(\mathbf{b})) \\ &\geq \mu |u - \mathbf{b}|^2, \quad \forall d(x) > D_0, \quad x \in S_1. \end{aligned}$$

Choose exponential function $Ce^{-2\alpha d(x)}$ as a comparison function and apply the maximum principle to the above inequality for $|u - \mathbf{b}|^2$ (see e.g. [12], [13]). We obtain

$$(2.8) \quad |u(x) - \mathbf{b}| \leq Ce^{-\alpha d(x)}, \quad x \in S_1$$

for some positive constant C, α . By the standard theory for elliptic equations, we can obtain

$$(2.9) \quad |\nabla u(x)| \leq Ce^{-\alpha d(x)}, \quad x \in S_1$$

Then, u satisfies the condition of Theorem 1.1, and we can apply (1.6) to u (with x_1 and x_2 switched) to obtain

$$(2.10) \quad \rho(x_1) := \int_{-\infty}^{\infty} \left[\frac{1}{2} \left(\left| \frac{\partial u}{\partial x_2} \right|^2 - \left| \frac{\partial u}{\partial x_1} \right|^2 \right) + W(u(x)) \right] dx_2 = C, \quad \forall x_1 \in \mathbb{R}.$$

In the case that all angles $\theta_{\mathbf{ab}}, \theta_{\mathbf{bc}}, \theta_{\mathbf{ca}}$ are in $(\pi/2, \pi)$, by using Assumption (2) in Theorem 2.2 it is easy to see that $u(x_1 + s, x_2)$ converges to $\mathbf{u}_{\mathbf{ab}}(x_2)$ in $C_{loc}^1(\mathbb{R}^2)$. Therefore, we derive (see Figure 2)

$$\lim_{x_1 \rightarrow \infty} \rho(x_1) = \mathbf{e}_{\mathbf{ab}}.$$

On the other hand, by Assumption (2) in Theorem 2.2 we also have

$$(2.11) \quad \begin{cases} \|u(x_1, x_2) - \mathbf{u}_{\mathbf{bc}}(-x_1 \sin(\theta_{\mathbf{ca}}) - x_2 \cos(\theta_{\mathbf{ca}}))\|_{C^1(\mathbb{R}^+)} \rightarrow 0, \\ \|u(x_1, x_2) - \mathbf{u}_{\mathbf{ca}}(x_1 \sin(\theta_{\mathbf{bc}}) - x_2 \cos(\theta_{\mathbf{bc}}))\|_{C^1(\mathbb{R}^-)} \rightarrow 0 \end{cases}$$

as $x_1 \rightarrow -\infty$.

Then, in view of the exponential convergence of u to \mathbf{b}, \mathbf{c} in S_2, S_3 respectively we have

$$(2.12) \quad \begin{aligned} & \lim_{x_1 \rightarrow -\infty} \int_0^{\infty} \left[\frac{1}{2} \left(\left| \frac{\partial u}{\partial x_2} \right|^2 - \left| \frac{\partial u}{\partial x_1} \right|^2 \right) + W(u(x)) \right] dx_2 \\ &= \int_{-\infty}^{\infty} \left[\frac{1}{2} \left| \frac{\partial}{\partial x_2} \mathbf{u}_{\mathbf{bc}}(-x_1 \sin(\theta_{\mathbf{ca}}) - x_2 \cos(\theta_{\mathbf{ca}})) \right|^2 \right. \\ & \quad \left. - \frac{1}{2} \left| \frac{\partial}{\partial x_1} \mathbf{u}_{\mathbf{bc}}(-x_1 \sin(\theta_{\mathbf{ca}}) - x_2 \cos(\theta_{\mathbf{ca}})) \right|^2 \right. \\ & \quad \left. + W(\mathbf{u}_{\mathbf{bc}}(-x_1 \sin(\theta_{\mathbf{ca}}) - x_2 \cos(\theta_{\mathbf{ca}}))) \right] dx_2 \\ &= \frac{1}{2} \mathbf{e}_{\mathbf{bc}} \left[\frac{\cos^2(\theta_{\mathbf{ca}})}{\cos(\theta_{\mathbf{ca}})} - \frac{\sin^2(\theta_{\mathbf{ca}})}{\cos(\theta_{\mathbf{ca}})} + \frac{1}{\cos(\theta_{\mathbf{ca}})} \right] \\ &= \mathbf{e}_{\mathbf{bc}} \cos(\theta_{\mathbf{ca}}) \end{aligned}$$

Similarly, we have

$$(2.13) \quad \lim_{x_1 \rightarrow -\infty} \int_{-\infty}^0 \left[\frac{1}{2} \left(\left| \frac{\partial u}{\partial x_2} \right|^2 - \left| \frac{\partial u}{\partial x_1} \right|^2 \right) + W(u(x)) \right] dx_2 = \mathbf{e}_{\mathbf{ca}} \cos(\theta_{\mathbf{bc}})$$

Therefore we obtain

$$(2.14) \quad \mathbf{e}_{\mathbf{ab}} = \mathbf{e}_{\mathbf{bc}} \cos(\theta_{\mathbf{ca}}) + \mathbf{e}_{\mathbf{ca}} \cos(\theta_{\mathbf{bc}}).$$

If we change the coordinates so that the x_1 -axis becomes the direction of $\bar{S}_1 \cap \bar{S}_2$ and $\bar{S}_2 \cap \bar{S}_3$ respectively and apply the Hamiltonian identity as above, we can also obtain

$$(2.15) \quad \begin{cases} \mathbf{e}_{\mathbf{bc}} = \mathbf{e}_{\mathbf{ca}} \cos(\theta_{\mathbf{ab}}) + \mathbf{e}_{\mathbf{ab}} \cos(\theta_{\mathbf{ca}}) \\ \mathbf{e}_{\mathbf{ca}} = \mathbf{e}_{\mathbf{ab}} \cos(\theta_{\mathbf{bc}}) + \mathbf{e}_{\mathbf{bc}} \cos(\theta_{\mathbf{ab}}). \end{cases}$$

In view of $\theta_{\mathbf{ab}} + \theta_{\mathbf{bc}} + \theta_{\mathbf{ca}} = 2\pi$, we derive (2.6) from (2.14) and (2.15) immediately.

Using the above procedure, we can indeed prove that

$$(2.16) \quad \frac{\pi}{2} < \theta_{\mathbf{ab}}, \quad \theta_{\mathbf{bc}}, \quad \theta_{\mathbf{ca}} < \pi$$

This finishes the proof. \square

An immediate corollary of Theorem 2.2 is that a triple junction solution for (2.4) with symmetric potential W must have equal contact angles.

3. SADDLE SOLUTIONS TO ALLEN-CAHN EQUATION IN \mathbb{R}^2

Allen-Cahn equation is a well-known model for bi-phase transition. Its stationary equation in entire space is

$$(3.1) \quad -\Delta u + F'(u) = 0, \quad |u| < 1, \quad x \in \mathbb{R}^n$$

where $F(u)$ is a double well potential with equal depths at $u = 1, -1$, and the scalar function u represents the physical state of a mixture of two materials, with $u \equiv \pm 1$ being two pure phases. A typical double well potential is $F(u) = \frac{1}{4}(1 - u^2)^2$. An entire solution to (3.1) represents a local structure of phase transition near interface or singularities. Regarding monotone solutions of (3.1), i.e., $u_{x_n}(x', x_n) > 0$ in \mathbb{R}^n , De Giorgi conjectured that all such solutions must depend on one direction when $n \leq 8$. The conjecture has been proved for $n = 2$ in [12] and $n = 3$ in [4]. For dimensions up to 8, the conjecture is essentially proved in [20], provided that u satisfies the limiting condition

$$(3.2) \quad \lim_{x_n \rightarrow \pm\infty} u(x', x_n) = \pm 1, \quad \forall x' \in \mathbb{R}^{n-1}$$

Related results can also be found in [5], [7], [21], [22], etc. Therefore, all monotone entire solutions to Allen-Cahn equation for $n = 2, 3$ or $4 \leq n \leq 8$ with (3.2) are like $g(x \cdot \nu + a)$ for some $a \in \mathbb{R}^n$ and $\nu \in S^{n-1}$, where g is the unique solution (up to translation) to the corresponding ordinary differential equation

$$(3.3) \quad -g''(t) + F'(g(t)) = 0, \quad g'(t) > 0, \quad t \in \mathbb{R}.$$

We may fix g so that $g(0) = 0$. This solution can also be regarded as a minimizer of

$$(3.4) \quad \min\{E(v) = \int_{-\infty}^{\infty} \frac{1}{2}|v'(t)|^2 + F(v(t))dt : v \in H_{loc}^1(\mathbb{R}), \lim_{t \rightarrow \pm\infty} v = \pm 1\}$$

with minimum energy

$$(3.5) \quad \mathbf{e} = \mathbf{e}_F := \int_{-1}^1 \sqrt{2F(u)} du.$$

When the potential $F(u)$ is an even function, it is obvious that g is odd.

There are also other types of solutions to (3.1) which are not monotone. In particular, saddle solutions are shown to exist in [11] for some *even* potential F . Indeed, the following slightly more general existence theorem can be proven. For simplicity, below we will only discuss the two dimensional case $n = 2$ and assume that F is a C^2 function satisfying

$$(3.6) \quad \begin{cases} F(1) = F(-1) = 0, & F(u) > 0, \quad \forall u \in (-1, 1), \\ F'(-1) = F'(1) = 0, & F''(-1) > 0, F''(1) > 0, \\ F(u) \text{ has only one critical point in } & (-1, 1). \end{cases}$$

We define $Q^1 := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ and similarly we can define Q^2, Q^3, Q^4 .

Proposition 3.1. *If we assume that F is an even function and satisfies (3.6), then there exists a saddle solution \mathbf{u} to (3.1) such that*

$$(3.7) \quad \begin{cases} \mathbf{u}(x_1, x_2) = -\mathbf{u}(x_1, -x_2) = -\mathbf{u}(-x_1, x_2), & \forall x \in \mathbb{R}^2; \\ \mathbf{u}(x) > 0, & \forall x \in Q^1 \cup Q^3; \quad \mathbf{u}(x) < 0, & \forall x \in Q^2 \cup Q^4. \end{cases}$$

It is easy to see that \mathbf{u} is unique and has another symmetry:

$$(3.8) \quad \mathbf{u}(x_1, x_2) = \mathbf{u}(x_2, x_1) = \mathbf{u}(-x_2, -x_1), \quad x \in \mathbb{R}^2$$

The reader may use the direct variational method or the super-sub solution method to solve the boundary value problem in $Q_R^1 = \{x = (x_1, x_2) : x_1 > 0, x_2 > 0, |x| \leq R\}$ with 0 boundary value on both axes and $u = 1$ on the remaining boundary, and hence obtain the desired solution as the limit by taking R to infinity. It can be easily proven that the limiting solution is not trivial by constructing a positive sub solution.

Remark 3.2. In [11], it is assumed that F satisfies an additional condition

$$\frac{F'(u)}{u} \text{ is increasing in } (0, 1).$$

This condition can be dropped for both the existence and uniqueness of \mathbf{u} , as in Proposition 3.1. See Corollary 3.9 below or [14] for a more detailed proof.

Definition 3.3. We may call a solution of (3.1) a *saddle solution* if its 0-level set consists of exactly two non self-intersecting C^1 curves which intersect each other at most once.

There are two natural questions regarding saddle solutions:

- Does there exist any saddle solution to (3.1) when F is not even;
- Are there any saddle solutions other than \mathbf{u} (and its rotation and translation) when F is even?

If the answer to the second question is affirmative, can we classify all saddle solutions? or can we show some properties of the solutions such as symmetry?

Regarding the first question, it is claimed in [24] that a saddle solution with 0-level set being the two axes *does* exist. However, existence of such a saddle solution is very counter intuitive. There has been doubt of this result among researchers of Allen-Cahn equation, even though there is no counter example or argument to disprove it. Here we give a rigorous proof that the result is indeed wrong, by using the Hamiltonian identity (1.6). To be more precise, we have proved the following necessary condition for the existence of the above mentioned saddle solutions.

Theorem 3.4. *Suppose F satisfies (3.6) and u is a solution to (3.1) satisfying*

$$(3.9) \quad u(x) > 0, \quad \forall x \in Q^1 \cup Q^3; \quad u(x) < 0, \quad \forall x \in Q^2 \cup Q^4.$$

Then $F'(0) = 0$ and

$$(3.10) \quad \int_0^1 \sqrt{F(u)} du = \int_{-1}^0 \sqrt{F(u)} du.$$

Proof. Let

$$F''(1) = \lambda_1^2, \quad F''(-1) = \lambda_2^2.$$

For any $\epsilon > 0$, by using comparison functions of the form $Ce^{-\lambda|x_i|}$ in proper region and the maximum principle, we can obtain

$$(3.11) \quad \begin{cases} |u(x_1, x_2) - 1| \leq C_{1,\epsilon} e^{-(\lambda_1 - \epsilon) \min\{|x_1|, |x_2|\}}, & x \in Q^1 \cup Q^3 \\ |u(x_1, x_2) + 1| \leq C_{1,\epsilon} e^{-(\lambda_2 - \epsilon) \min\{|x_1|, |x_2|\}}, & x \in Q^2 \cup Q^4. \end{cases}$$

The standard gradient estimate for elliptic equations lead to

$$(3.12) \quad \begin{cases} |\nabla u| \leq C_{2,\epsilon} e^{-(\lambda_1 - \epsilon) \min\{|x_1|, |x_2|\}}, & x \in Q^1 \cup Q^3 \\ |\nabla u| \leq C_{2,\epsilon} e^{-(\lambda_2 - \epsilon) \min\{|x_1|, |x_2|\}}, & x \in Q^2 \cup Q^4 \end{cases}$$

for some constants $C_{i,\epsilon} > 0, i = 1, 2$.

Furthermore, we have

$$(3.13) \quad \begin{cases} \|u(x_1, x_2) \pm g(x_1)\|_{C^1(\mathbb{R})} \rightarrow 0 \text{ as } x_2 \rightarrow \infty, \\ \|u(x_1, x_2) \pm g(x_2)\|_{C^1(\mathbb{R})} \rightarrow 0 \text{ as } x_1 \rightarrow \mp\infty. \end{cases}$$

We also note that such a solution is unique and u satisfies (3.8).

We shall prove the (3.10) by applying the Hamiltonian identity to (3.1). For this purpose, we choose a new coordinates (y_1, y_2) so that y_1 -axis and y_2 -axis coincide with the lines $\{x : x_1 = x_2\}$ and $\{x : x_1 = -x_2\}$ respectively (see Figure 3). Now applying Theorem 1.1 (with x_1, x_2 replaced by y_2, y_1 respectively), we obtain

$$(3.14) \quad \rho(y_1) := \int_{-\infty}^{\infty} \frac{1}{2} \left(\left| \frac{\partial u}{\partial y_2} \right|^2 - \left| \frac{\partial u}{\partial y_1} \right|^2 \right) + F(u(y)) dy_2 = C, \quad \forall y_1 \in \mathbb{R}.$$

A straightforward computation as in (2.12) leads to

$$(3.15) \quad \lim_{y_1 \rightarrow \infty} \rho(y_1) = \sqrt{2} \mathbf{e}_F = 2 \int_{-1}^1 \sqrt{F(u)} du.$$

Hence

$$(3.16) \quad \rho(0) = \int_{-\infty}^{\infty} \left[\frac{1}{2} \left| \frac{\partial u}{\partial y_2} \right|^2(0, y_2) + F(u(0, y_2)) \right] dy_2 = 2 \int_{-1}^1 \sqrt{F(u)} du.$$

Note that in the above equality the derivative u_{y_1} vanishes on y_2 -axis due to (3.8). Now we modify F to get an even double potential

$$(3.17) \quad \tilde{F}(u) = \begin{cases} F(u), & u \leq 0; \\ F(-u), & u \geq 0. \end{cases}$$

It is obvious to see from Eq. (3.1) that $F'(0) = 0$. Hence \tilde{F} is also a $C^{1,\alpha}$ function and satisfies (3.6). By Theorem 3.1, there exists a saddle solution \tilde{u} to (3.1) with F replaced by \tilde{F} and \tilde{u} satisfies (3.7) and (3.8). The application of Hamiltonian identity to \tilde{u} leads to

$$(3.18) \quad \tilde{\rho}(0) = \int_{-\infty}^{\infty} \left[\frac{1}{2} \left| \frac{\partial \tilde{u}}{\partial y_2} \right|^2 + F(\tilde{u}(y)) \right] dy_2 = 2 \int_{-1}^1 \sqrt{\tilde{F}(u)} du.$$

By the uniqueness of \tilde{u} (see *Remark 3.2*), we know

$$(3.19) \quad u(x) = \tilde{u}(x), \quad \forall x \in Q^2 \cup Q^4$$

and therefore $\rho(0) = \tilde{\rho}(0)$. Then

$$(3.20) \quad \int_{-1}^1 \sqrt{F(u)} du = \int_{-1}^1 \sqrt{\tilde{F}(u)} du$$

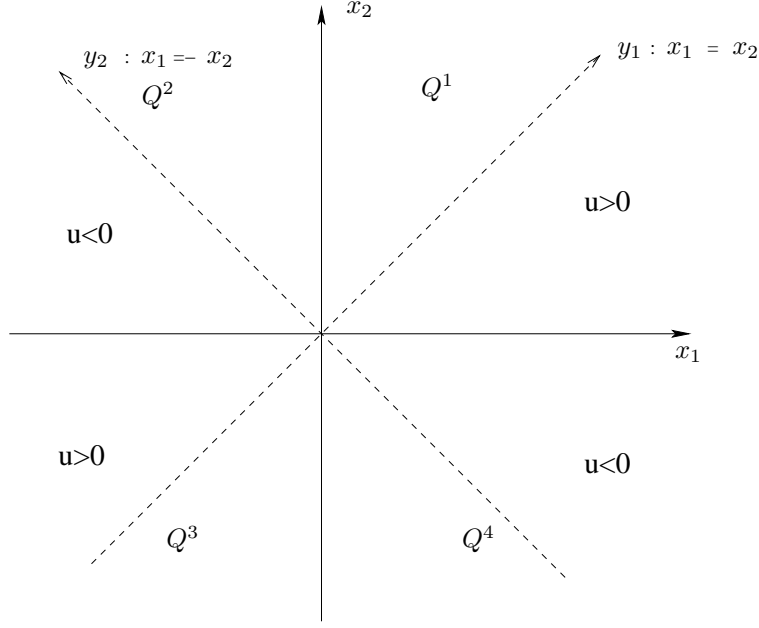


FIGURE 3. Applying Hamiltonian identity to saddle solutions

and (3.10) follows immediately from the definition of \tilde{F} . □

It remains a question whether F must be an even function in order to have a saddle solution u of (3.1) satisfying (3.9).

Now we discuss the contact angles at infinity for saddle solutions.

Definition 3.5. If the two 0-level curves are asymptotically two intersecting straight lines at infinity, we call the acute angle θ between these two lines the contact angle at infinity.

We have the following partial result.

Theorem 3.6. *Assume that F is a double well potential satisfying (3.6) and (3.10). Suppose that u is a solution to (3.1) with a contact angle θ at infinity. We further assume that u satisfies (3.8) and $u(0) = 0$. Then we have*

$$(3.21) \quad \pi/3 < \theta \leq \pi/2.$$

Proof. Without loss of generality, we may assume that the angle θ is centered at y_2 -axis and let $\theta = 2\alpha$. Following the proof of Theorem 3.4, we can obtain

$$(3.22) \quad \rho(0) = \int_{-\infty}^{\infty} \left[\frac{1}{2} \left| \frac{\partial u}{\partial y_2} \right|^2 + F(u(y)) \right] dy_2 = 2\mathbf{e}_F \sin(\alpha).$$

On the other hand, in view of $u(0) = 0$ we know

$$(3.23) \quad \rho(0) = \int_{-\infty}^{\infty} \left[\frac{1}{2} \left| \frac{\partial u}{\partial y_2} \right|^2(0, y_2) + F(u(0, y_2)) \right] dy_2 > \mathbf{e}_F.$$

Hence $\sin(\alpha) > 1/2$ and the theorem is proven. □

Lemma 3.8. *If $\alpha \geq \beta$, then the following inequality holds*

$$(3.30) \quad \mathbf{u}_\alpha(x) \geq \mathbf{u}_\beta^\lambda(x), \quad \forall x = re^{i\theta} \in S_\beta^\lambda, \quad \forall \lambda \in [-(\alpha - \beta), \alpha - \beta].$$

In particular, if the strict inequality holds if $\alpha > \beta$. In other words, if we rotate the cone S_β inside the cone S_α , the graph of \mathbf{u}_β shall always be below that of \mathbf{u}_α .

Proof. We first consider a shifted cone $S_{\beta,\mu} := \{x : (x_1 - \mu, x_2) \in S_\beta\}$ and $\mathbf{u}_{\beta,\mu}(x) := \mathbf{u}_\beta(x_1 - \mu, x_2)$, $x \in S_{\beta,\mu}$. It is clear that $\mathbf{u}_{\beta,\mu}(x)$ is a solution to (3.1) in $S_{\beta,\mu}$. Below we shall use the sliding plane method to prove $\mathbf{u}_\alpha(x) \geq \mathbf{u}_\beta(x)$ in S_β . From (3.6), there is an constant $\delta > 0$ such that $F''(u) > 0$ when $u \in (1 - \delta, 1]$. By (3.26) and (3.27), we know that when μ is large enough, $\mathbf{u}_\alpha(x) > 1 - \delta$ in $S_{\beta,\mu}$, and $\mathbf{u}_\alpha(x) \geq \mathbf{u}_{\beta,\mu}(x)$ as $x \rightarrow \infty$ in $S_{\beta,\mu}$ or as $x \rightarrow \partial S_{\beta,\mu}$. By the maximum principle, we obtain

$$(3.31) \quad \mathbf{u}_\alpha(x) \geq \mathbf{u}_{\beta,\mu}(x), \quad \forall x \in S_{\beta,\mu}$$

for μ large enough.

Then, we can decrease μ to 0 while still keep (3.31) true by the so-called sliding plane method as follows:

Let

$$\mu_0 := \min\{\mu : \text{inequality (3.31) holds}\}.$$

We claim that $\mu_0 = 0$. If this is not true, then there exist a sequence $\{\mu_n\}_1^\infty$ and a sequence of points $\{\eta_n\}_1^\infty$ such that $\mu_n \leq \mu_0$, $\lim_{n \rightarrow \infty} \mu_n = \mu_0$ and

$$\mathbf{u}_\alpha(\eta_n) < \mathbf{u}_{\beta,\mu_n}(\eta_n), \quad \forall n.$$

By the asymptotical behavior (3.27) for both \mathbf{u}_α and \mathbf{u}_β , it is easy to see that $\{\eta_n\}$ is bounded and therefore possesses a convergent sub sequence with limit η .

Then $\mathbf{u}_\alpha(\eta) \leq \mathbf{u}_{\beta,\mu_0}(\eta)$. Recall that by the definition of μ_0 , (3.31) holds with $\mu = \mu_0$. Then, the strong maximum principle implies

$$\mathbf{u}_\alpha(x) = \mathbf{u}_{\beta,\mu_0}(x), \quad \forall x \in S_{\beta,\mu_0}.$$

This is a contradiction due to the zero boundary condition for the solutions. Hence $\mu_0 = 0$ and the lemma holds with $\lambda = 0$.

Then we rotate S_β and apply the above arguments (usually called the rotating plane method) to \mathbf{u}_β^λ in S_β^λ with λ from 0 to $\alpha - \beta$ or to $-(\alpha - \beta)$. The lemma follows immediately. \square

Corollary 3.9. *It is easy to see from the above lemma that u_α is unique, by choosing $\beta = \alpha$ and exchanging the order of two possible solutions.*

Theorem 3.10. *Assume F is even and satisfies (3.6) and u is a saddle solution to (3.1) with 0-level set being two straight lines with contact angle θ . Then $\theta = \pi/2$.*

Proof. Let $\alpha = \theta/2$. We just note that if $\theta < \frac{\pi}{2}$, then $\frac{\pi}{2} - \alpha > \alpha$. By Lemma 3.8 we have

$$(3.32) \quad \mathbf{u}_{\frac{\pi}{2}-\alpha}(x) > \mathbf{u}_\alpha^{\frac{\pi}{2}-2\alpha}(x), \quad \forall x \in S_\alpha^{\frac{\pi}{2}-2\alpha}.$$

By the Hopf's lemma, we deduce

$$(3.33) \quad \frac{\partial \mathbf{u}_{\frac{\pi}{2}-\alpha}}{\partial \nu}(x) < \frac{\partial \mathbf{u}_\alpha^{\frac{\pi}{2}-2\alpha}}{\partial \nu}(x), \quad \forall x \in \partial S_\alpha^{\frac{\pi}{2}-2\alpha} \cap \partial S_{\frac{\pi}{2}-\alpha}$$

By the uniqueness of \mathbf{u}_α , we know that after a proper rotation $u(re^{i\alpha}) = \mathbf{u}_\alpha$ in S_α and $u(re^{i\alpha}) = -\mathbf{u}_{\frac{\pi}{2}-\alpha}$ in $S_{\frac{\pi}{2}-\alpha}$. Then, on $\partial S_\alpha \cap \partial S_{\frac{\pi}{2}-\alpha}$ we have

$$(3.34) \quad \frac{\partial u}{\partial \nu'}(x) > \frac{\partial u}{\partial \nu}(x)$$

where ν is the normal of $S_{\frac{\pi}{2}-\alpha}$ while ν' is normal of S_α . This is in contrast with u being a classical solution to (3.1). Therefore $\frac{\pi}{2} - \alpha = \alpha$, and hence $\theta = \pi/2$. This finishes the proof. \square

4. FURTHER STUDY OF SADDLE SOLUTIONS

In this section, we consider a saddle solution u to (3.1) satisfying the even symmetry condition (3.8). Here we just assume that F is a double well potential satisfying (3.6). We shall use the cartesian coordinates (y_1, y_2) with y_1 -axis and y_2 -axis coinciding with the lines $\{x = (x_1, x_2) : x_1 = x_2\}$ and $\{x = (x_1, x_2) : x_1 = -x_2\}$ respectively. In the new coordinates, the condition (3.8) becomes

$$(4.1) \quad u(y_1, y_2) = u(y_1, -y_2) = u(-y_1, y_2), \quad y \in \mathbb{R}^2$$

We assume further that u satisfies the following monotonicity condition

$$(4.2) \quad u_{y_1}(y) > 0, \quad \text{if } y_1 > 0; \quad u_{y_2}(y) < 0, \quad \text{if } y_2 > 0.$$

Denote $\gamma = u(0, 0)$. We can expand u near $y = (0, 0)$ as

$$u(y) = \gamma + ay_1^2 - by_2^2 + o(|y|^2)$$

where a, b are positive constants. It is easy to see that $F'(\gamma) = 2(b - a)$. Moreover, by the implicit Function Theorem the γ -level set of u near the origin in the first quadrant $Q_1 = \{y = (y_1, y_2) : y_1 > 0, y_2 > 0\}$ is a C^2 curve which can be extended to infinity. Indeed, by (4.2) we know that the γ -level set curve can be expressed as the graph of a strictly increasing C^2 function $y_2 = h(y_1)$ which has an inverse function $y_1 = k(y_2)$. In the next several lemmas we shall show that the γ -level curve is asymptotically a straight line. (See Figure 5.)

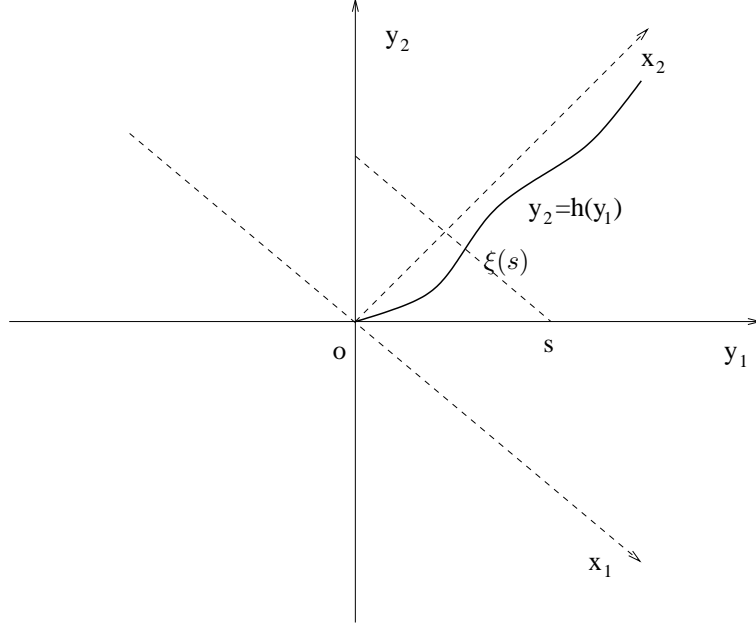
Lemma 4.1. *The function $y_2 = h(y_1)$ for the γ -level curve is defined for all $y_1 > 0$ and the following limit holds*

$$(4.3) \quad \lim_{y_1 \rightarrow \infty} h(y_1) = \infty.$$

Proof. By the monotonicity property (4.2) of u and the implicit Function Theorem, we know that the γ -level curve extends to infinity. Hence it suffices to show (4.3) when $h(y_1)$ is defined for all $y_1 > 0$. Suppose (4.3) is not true. We define $u_\infty(y) = \lim_{s \rightarrow \infty} u(y_1 + s, y_2), y \in \mathbb{R}^2$ and $A = \lim_{y_1 \rightarrow \infty} h(y_1)$. Then u_∞ is C^2 in \mathbb{R}^2 and satisfies (3.1). Furthermore, we have $\frac{\partial u_\infty}{\partial y_1}(y_1, y_2) = 0, y \in \mathbb{R}^2$ and $u_\infty(y_1, A) = \gamma$. Then $u_\infty(y_1, y_2) = g(y_2 + b), y \in \mathbb{R}^2$ for some constant b , where g is the unique solution of the ordinary differential equation (3.3). This contradicts the even symmetry (4.1) of u_∞ in y_2 . The lemma is proven. \square

Lemma 4.2. *There exists $\beta \in (0, \pi/2)$ such that*

$$(4.4) \quad \lim_{y_1 \rightarrow \infty} h'(y_1) = \tan \beta$$


 FIGURE 5. The γ -level curve of u and the coordinates

Proof. We shall use the (x_1, x_2) coordinates as well and write $\bar{u}(x_1, x_2) = u(y_1, y_2)$. Define

$$(4.5) \quad \bar{\rho}(x_2) = \frac{1}{\sqrt{2}} \int_{-x_2}^{x_2} [F(\bar{u}(x_1, x_2)) + \frac{1}{2}\bar{u}_{x_1}^2(x_1, x_2) - \frac{1}{2}\bar{u}_{x_2}^2(x_1, x_2)] dx_1.$$

Then

$$(4.6) \quad \begin{aligned} & \sqrt{2}\bar{\rho}'(x_2) \\ &= F(\bar{u}(x_2, x_2)) + \frac{1}{2}\bar{u}_{x_1}^2(x_2, x_2) - \frac{1}{2}\bar{u}_{x_2}^2(x_2, x_2) + (\bar{u}_{x_1}\bar{u}_{x_2})(x_2, x_2) \\ &+ F(\bar{u}(-x_2, x_2)) + \frac{1}{2}\bar{u}_{x_1}^2(-x_2, x_2) - \frac{1}{2}\bar{u}_{x_2}^2(-x_2, x_2) - (\bar{u}_{x_1}\bar{u}_{x_2})(-x_2, x_2) \\ &= F(u(\sqrt{2}x_2, 0)) + \frac{1}{2}u_{y_1}^2(\sqrt{2}x_2, 0) + F(u(0, \sqrt{2}x_2)) + \frac{1}{2}u_{y_2}^2(0, \sqrt{2}x_2). \end{aligned}$$

Then

$$(4.7) \quad \sqrt{2}\bar{\rho}(M/\sqrt{2}) = \int_0^M [F(u(s, 0)) + \frac{1}{2}u_{y_1}^2(s, 0)] + [F(u(0, s)) + \frac{1}{2}u_{y_2}^2(0, s)] ds.$$

On the other hand, we let $(x_1(s), s/\sqrt{2})$ be the intersection of the line $x_2 = s/\sqrt{2}$ with the level set curve $y_2 = h(y_1)$ and write its y -coordinates as $y = \xi(s) = (\xi_1(s), \xi_2(s))$. Define $u^s(y) = u(y + \xi(s))$, $y \in \mathbb{R}^2$. By the standard theory of elliptic equations, for any sequence $\{s_n\}$ there is a subsequence $\{s_{n_k}\}$ (which we will denote by $\{s_k\}$ later) such that $u_k(y) := u^{s_k}(y)$ converges to $u_\infty(y)$ in $C_{loc}^2(\mathbb{R}^2)$ as $k \rightarrow \infty$, where u_∞ is a solution of (3.1). In particular, if $s_n \rightarrow \infty$, by (4.3) we deduce $\xi_i(s_n) \rightarrow \infty$, $i = 1, 2$. Hence, by (4.2) we obtain $\frac{\partial u_\infty}{\partial y_2}(y) \geq 0$, $y \in \mathbb{R}^2$. By the strong maximum principle, we know either $\frac{\partial u_\infty}{\partial y_2} \equiv 0$ in \mathbb{R}^2 or $\frac{\partial u_\infty}{\partial y_2}(y) > 0$, $y \in \mathbb{R}^2$.

Then by Theorem 1.1 in [12] (De Giorgi conjecture for $n = 2$) we conclude that $u_\infty(y) = g(y \cdot \nu + t_0)$, $y \in \mathbb{R}^2$, where t_0 is the constant satisfying $g(t_0) = \gamma$, and $\nu \in \mathbb{R}^2$ is constant unit vector. We write $\nu = (\sin \beta, -\cos \beta)$. Fix a large positive constant M . For any small $\epsilon > 0$, we have

$$(4.8) \quad \|\bar{u}(x_1, s_k) - g((x_1 - x_1(s_k)) \sin(\pi/4 + \beta) + t_0)\|_{C^2([x_1(s_k)-M, x_1(s_k)+M])} \leq \epsilon$$

when k is sufficiently large. Moreover,

$$(4.9) \quad \begin{cases} |\bar{u}(x_1, s_k) - 1| \leq C e^{-\mu(x_1 - x_1(s_k)) \sin \pi/4}, & x_1 \geq x_1(s_k) + M, \\ |\bar{u}(x_1, s_k) + 1| \leq C e^{-\mu(x_1(s_k) - x_1) \sin \pi/4}, & x_1 \leq x_1(s_k) - M \end{cases}$$

where C, μ are positive constants independent of M, k and x_1 . The gradient estimates for elliptic equations yield

$$(4.10) \quad \begin{cases} |\nabla \bar{u}| \leq C e^{-\mu(x_1 - x_1(s_k)) \sin \pi/4}, & x_1 \geq x_1(s_k) + M, \\ |\nabla \bar{u}| \leq C e^{-\mu(x_1(s_k) - x_1) \sin \pi/4}, & x_1 \leq x_1(s_k) - M. \end{cases}$$

Using (4.5) and (4.8)-(4.10) and choosing M sufficiently large, we can obtain

$$(4.11) \quad |\bar{\rho}(s_k) - e_F \sin(\pi/4 + \beta)| \leq \epsilon$$

when k is sufficiently large. In view of (4.7), $\bar{\rho}(s)$ is increasing in s and then has a finite limit. Hence we derive that

$$(4.12) \quad \lim_{s_k \rightarrow \infty} \bar{\rho}(s_k) = e_F \sin(\pi/4 + \beta).$$

Note that the sequence $\{s_n\}$ is arbitrary and hence β in the above equality does not depend on the choice of the sequence. Therefore we conclude

$$(4.13) \quad \|u(y + \xi(s)) - g(y_1 \sin \beta - y_2 \cos \beta + t_0)\|_{C_{loc}^2(\mathbb{R}^2)} \rightarrow 0, \quad \text{as } s \rightarrow \infty.$$

Next, we show $\beta \in (0, \pi/2)$. It suffices to show $\beta > 0$, since $\beta < \pi/2$ can be proven similarly. Suppose $\beta = 0$. Following the proof of Theorem 3.4, we derive

$$\int_0^\infty [F(u(0, y_2)) + \frac{1}{2} u_{y_2}^2(0, y_2)] dy_2 = e_F.$$

By (4.7) and (4.12), we derive

$$\int_0^\infty [F(u(s, 0)) + \frac{1}{2} u_{y_1}^2(s, 0)] ds + \int_0^\infty [F(u(0, y_2)) + \frac{1}{2} u_{y_2}^2(0, y_2)] dy_2 = e_F.$$

Hence

$$\int_0^\infty [F(u(s, 0)) + \frac{1}{2} u_{y_1}^2(s, 0)] ds = 0.$$

This is a contradiction. The lemma is then proven. \square

Next we shall show that the γ -level curve is indeed asymptotically a straight line. We shall prove the following more general lemma regarding solution of (3.1) in a cone.

Lemma 4.3. *Suppose that $u(y_1, y_2)$ is a solution of (3.1) in a cone $\mathcal{C} := \{y \in \mathbb{R}^2 : |y_1| \leq y_2 \tan \alpha_0, y_2 \geq M > 0\}$ for some $0 < \alpha_0 < \pi$. For some $\gamma \in (-1, 1)$, the γ -level set of u in \mathcal{C} is given by the graph of a function $y_1 = k(y_2)$. Assume*

$$(4.14) \quad \lim_{y_2 \rightarrow \infty} k'(y_2) = 0.$$

Then there is a finite number A such that

$$(4.15) \quad \lim_{y_2 \rightarrow \infty} k(y_2) = A.$$

Proof. We shall prove the lemma in three steps. First, we show that an energy of u on a line segment $[-y_2 \tan \alpha, y_2 \tan \alpha]$, $\alpha \in (0, \alpha_0)$ is exponentially close to \mathbf{e}_F as y_2 tends to ∞ . Second, we construct an optimal approximation of u by a shift $r(y_2)$ of the one dimensional solution g , and show that the difference is exponentially small in L^2 norm as y_2 goes to infinity. Finally, we deduce that the shift $r(y_2)$ has a finite limit, and then conclude that $k(y_2)$ has a finite limit.

Step 1

Without loss of generality, we assume that $u(y_1, y_2) > \gamma$ when $y_1 > k(y_2)$ and $u(y_1, y_2) < \gamma$ when $y_1 < k(y_2)$ in \mathcal{C} .

It is easy to show by the maximum principle (see, e.g., [12], [14]) that for any fixed $\alpha \in (0, \alpha_0)$

$$(4.16) \quad \begin{cases} |u(y) - 1| \leq C_1 e^{-\kappa_1(y_1 - k(y_2))}, & \text{for } k(y_2) < y_1 < y_2 \tan \alpha \\ |u(y) + 1| \leq C_1 e^{\kappa_1(y_1 - k(y_2))}, & \text{for } -k(y_2) > y_1 > -y_2 \tan \alpha. \end{cases}$$

The standard gradient estimates of elliptic equations yield

$$(4.17) \quad \begin{cases} |\nabla u| \leq C_1 e^{-\kappa_1(y_1 - k(y_2))}, & \text{for } k(y_2) < y_1 < y_2 \tan \alpha \\ |\nabla u| \leq C_1 e^{\kappa_1(y_1 - k(y_2))}, & \text{for } -k(y_2) > y_1 > -y_2 \tan \alpha. \end{cases}$$

For any sequence $\{s_n\}$ there exists a subsequence, which we still denote by $\{s_n\}$, such that $u(y_1 + k(s_n), y_2 + s_n)$ converges in $C_{loc}^2(\mathbb{R}^2)$ to a solution $u_\infty(y)$ of (3.1) in \mathbb{R}^2 . Furthermore, we have

$$(4.18) \quad \begin{cases} u_\infty(0, y_2) = \gamma, \quad \forall y_2 \in \mathbb{R} \\ u_\infty(y_1, y_2) > \gamma, \quad \text{if } y_1 > 0, \quad \forall y_2 \in \mathbb{R} \\ u_\infty(y_1, y_2) < \gamma, \quad \text{if } y_1 < 0, \quad \forall y_2 \in \mathbb{R}. \end{cases}$$

By symmetry results in half plane (See [6]), we know that $u_\infty(y) = g(y_1 + t_0)$, $y \in \mathbb{R}^2$. Since $\{s_n\}$ is arbitrary, we obtain

$$\|u(y_1 + k(s), y_2 + s) - g(y_1 + t_0)\|_{C_{loc}^2(\mathbb{R}^2)} \rightarrow 0, \quad \text{as } s \rightarrow \infty.$$

Then,

$$(4.19) \quad \|u(y_1, y_2) - g(y_1 - k(y_2) + t_0)\|_{C^2([-y_2 \tan \alpha, y_2 \tan \alpha])} \rightarrow 0, \quad \text{as } y_2 \rightarrow \infty.$$

Define

$$\rho_1(y_2) = \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} [F(u(y_1, y_2)) + \frac{1}{2}|u_{y_1}|^2 - \frac{1}{2}|u_{y_2}|^2] dy_1.$$

It follows easily from (4.19) and (4.16) that

$$(4.20) \quad \lim_{y_2 \rightarrow \infty} \rho_1(y_2) = \mathbf{e}_F$$

Combining (4.16), (4.17) and straightforward computations as in (4.6), we obtain

$$(4.21) \quad |\rho_1'(y_2)| \leq C_2 e^{-\kappa_2 y_2}, \quad y_2 \geq M$$

for some positive constants C_2, κ_2 . Hence we conclude that

$$(4.22) \quad |\rho_1(y_2) - \mathbf{e}_F| \leq \frac{C_2}{\kappa_2} e^{-\kappa_2 y_2}, \quad y_2 \geq M.$$

Step 2

We define

$$W(y_2, r) = \|u(\cdot, y_2) - g(\cdot + t_0 - r)\|_{L^2([-y_2 \tan \alpha, y_2 \tan \alpha])}^2.$$

By (4.16) and (4.19), we know that

$$W(y_2, k(y_2)) \rightarrow 0, \quad \text{as } y_2 \rightarrow \infty$$

and

$$(4.23) \quad \frac{\partial^2}{\partial r^2} W(y_2, k(y_2)) = 2 \int [-(u - g)g'' + |g'|^2] dy_1 > 0.$$

when y_2 is sufficiently large.

By the Implicit Function Theorem, for y_2 large there exists a unique $r(y_2)$ such that

$$W(y_2, r(y_2)) = \min_{r \in \mathbb{R}} W(y_2, r).$$

Let

$$v(y_1, y_2) = u(y_1, y_2) - g(y_1 + t_0 - r(y_2)), \quad y_1 \in [-y_2 \tan \alpha, y_2 \tan \alpha].$$

It follows immediately from (4.19) that

$$(4.24) \quad \lim_{y_2 \rightarrow \infty} \|v(\cdot, y_2)\| = 0.$$

Moreover, the function $r(y_2)$ is differentiable and

$$(4.25) \quad \lim_{y_2 \rightarrow 0} r'(y_2) = 0, \quad \lim_{y_2 \rightarrow 0} r(y_2) - k(y_2) = 0.$$

It is also easy to see that

$$(4.26) \quad \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} [(u(y_1, y_2) - g(y_1 + t_0 - r(y_2)))g'(y_1 + t_0 - r(y_2))] dy_1 = 0.$$

Differentiating (4.26) with respect to y_2 leads to

$$(4.27) \quad \left(\int |g'|^2 dy_1 - \int (u - g)g'' dy_1 \right) \cdot r'(y_2) + \int u_{y_2} g' dy_1 = O(e^{-\kappa_2 y_2}).$$

Now we estimate the energy $\rho_1(y_2)$ in terms of $\|v\|$ as follows

$$(4.28) \quad \begin{aligned} \rho_1(y_2) &= \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} [F(g(y_1 + t_0 - r(y_2))) + \frac{1}{2}|g'(y_1 + t_0 - r(y_2))|^2] dy_1 \\ &= \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} [F(u) - F(g) + \frac{1}{2}(|u_{y_1}|^2 - |g'|^2) - \frac{1}{2}|u_{y_2}|^2] dy_1 \\ &= \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} [F(u) - F(g) - \frac{1}{2}(F'(g) + F'(u))(u - g)] dy_1 \\ &\quad + \frac{1}{2} \int u_{y_2 y_2} (u - g) dy_1 - \frac{1}{2} \int u_{y_2}^2 dy_1 + O(e^{-\kappa_2 y_2}) \\ &= \frac{1}{2} \int u_{y_2 y_2} (u - g) dy_1 - \frac{1}{2} \int u_{y_2}^2 dy_1 + O(e^{-\kappa_2 y_2}) + o(\|v\|^2). \end{aligned}$$

In the above estimate, we have used the following estimate

$$\begin{aligned}
 & \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} (|u_{y_1}|^2 - |g'|^2) dy_1 \\
 (4.29) \quad &= - \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} (u_{y_1 y_1} + g'')(u - g) dy_1 + O(e^{-\kappa_2 y_2}) \\
 &= \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} [-(F'(u) + F'(g))(u - g) + u_{y_2 y_2} (u - g)] dy_1 + O(e^{-\kappa_2 y_2}).
 \end{aligned}$$

Hence, in view of (4.22) we obtain

$$(4.30) \quad \int u_{y_2 y_2} (u - g) dy_1 - \int u_{y_2}^2 dy_1 = O(e^{-\kappa_2 y_2}) + o(\|v\|^2).$$

Furthermore, by the spectrum theory we have

$$(4.31) \quad \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} [|\phi'|^2 + F''(g)\phi^2] dy_1 \geq \lambda \|\phi\|^2$$

for some positive constant $\lambda > 0$ when ϕ satisfies

$$(4.32) \quad \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} \phi g' dy_1 = 0, \quad \phi \in H_0^1([-y_2 \tan \alpha, y_2 \tan \alpha]).$$

Choosing $\phi(\cdot) = v(\cdot, y_2)$ in (4.31) for any fixed y_2 , we obtain

$$(4.33) \quad \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} [-(u - g)_{y_1 y_1} + F''(g)(u - g)](u - g) dy_1 \geq \lambda \|v\|^2 + O(e^{-\kappa_2 y_2}).$$

Differentiating $\|v\|^2$ twice leads to

$$\begin{aligned}
 (4.34) \quad & \frac{d^2}{dy_2^2} \|v\|^2 = 2 \int |u_{y_2} + g' r'(y_2)|^2 dy_1 \\
 & + 2 \int [u_{y_2 y_2} - g'' |r'(y_2)|^2 + g' r''(y_2)](u - g) dy_1 + O(e^{-\kappa_2 y_2}) \\
 & = 2 \int |u_{y_2} + g' r'(y_2)|^2 dy_1 + 2 \int u_{y_2 y_2} (u - g) dy_1 \\
 & \quad - 2(r'(y_2))^2 \int g''(u - g) dy_1 + 2r''(y_2) \int g'(u - g) dy_1 + O(e^{-\kappa_2 y_2}) \\
 & \geq \int |u_{y_2} + g' r'(y_2)|^2 dy_1 + \int u_{y_2}^2 dy_1 + \int u_{y_2 y_2} (u - g) dy_1 \quad (\text{by (4.30), (4.26)}) \\
 & \quad + (r'(y_2))^2 O(\|v\|) + o(\|v\|^2) + O(e^{-\kappa_2 y_2}) \\
 & \geq (\|g'\|^2 + O(\|v\|))(r'(y_2))^2 + (\lambda + o(1))\|v\|^2 + O(e^{-\kappa_2 y_2}).
 \end{aligned}$$

Here it is essential to split the term $2 \int u_{y_2 y_2} (u - g) dy_1$ to two terms: one is replaced by $\int u_{y_2}^2 dy_1$ using (4.30) and the other is replaced by $\lambda \|v\|^2$ using the following

estimate

$$\begin{aligned}
& \int u_{y_2 y_2} (u - g) dy_1 = \int_{-y_2 \tan \alpha}^{y_2 \tan \alpha} [(F'(u) - u_{y_1 y_1})(u - g)] dy_1 \\
(4.35) \quad & = \int [(F'(u) - F'(g) - F''(g)(u - g))(u - g)] dy_1 \\
& \quad + \int [(F''(g)(u - g) - (u - g)_{y_1 y_1})(u - g)] dy_1 \quad (\text{by (4.33)}) \\
& \geq o(\|v\|^2) + \lambda \|v\|^2 + O(e^{-\kappa_2 y_2}).
\end{aligned}$$

Therefore we derive a differential inequality

$$(4.36) \quad \frac{d^2}{dy_2^2} \|v\|^2 \geq \frac{\lambda}{2} \|v\|^2 + O(e^{-\kappa_2 y_2}), \quad y_2 \geq M_1$$

where M_1 is a sufficiently large positive constant. By choosing a comparison function of the form $Ce^{-\kappa y_2}$, it is easy to see that

$$(4.37) \quad \|v\| \leq Ce^{-\kappa y_2}, \quad y_2 \geq M_1$$

for appropriately chosen constants C and $\kappa < \min\{\kappa_2, \sqrt{\lambda/2}\}$.

Step 3. From (4.37) and (4.30), we derive

$$(4.38) \quad \int |u_{y_2}|^2 dy_1 \leq Ce^{-\kappa y_2}, \quad y_2 \geq M_1.$$

Then by (4.27), we obtain

$$(4.39) \quad |r'(y_2)| \leq Ce^{-\kappa y_2/2}, \quad y_2 \geq M_1.$$

Therefore

$$(4.40) \quad \lim_{y_2 \rightarrow \infty} r(y_2) = A$$

for some finite number A . The lemma follows immediately from (4.25). \square

Combining Lemma 4.1-4.3, we can prove the following

Theorem 4.4. *Assume that u is a solution to the Allen-Cahn equation (3.1) where F is a double well potential satisfying (3.6). Assume further that u possesses even symmetry (4.1) and monotonicity (4.2). Then every level set of u approaches asymptotically a slant straight line with the same finite positive slope in the first quadrant as y goes to infinity.*

Proof. We just note that after rotating the coordinates clockwise by an angle $\pi/2 - \beta$, then u satisfies the condition of Lemma 4.3 using Lemma 4.2. Hence we can apply Lemma 4.3 to conclude that the γ -level set of u approaches a straight line of slope $\tan \beta$ in the original coordinate. In view of (4.13), the other level set curves of u are essentially parallel to γ -level curve of u asymptotically, the theorem then follows immediately. \square

Remark 4.5. The result in Theorem 4.4 can be generalized to solutions of Allen-Cahn equations in a domain which is a cone at infinity, provided that the level set is a smooth curve contained in a strictly smaller cone near infinity. More details will be provided in a forthcoming paper.

Remark 4.6. The condition that F has only one critical point in $(-1, 1)$ stated in (3.6) can be dropped in most of the discussion. In the case when F has more critical points in $(-1, 1)$, the one dimensional heteroclinic solution of (3.3) may not be unique up to translation. In the case that F is even, the saddle solution satisfying (3.7) and (3.8) may not be unique either. However, we can state the following:

1. For each heteroclinic solution g_i of (3.3) there exists a pair of critical points $[a_i, b_i]$, which are the limits of g_i at plus and minus infinity respectively, such that

$$(4.41) \quad F(u) > F(a_i) = F(b_i), \quad \forall u \in (a_i, b_i).$$

If we assume that $F''(a_i) > 0, F''(b_i) > 0$ at these points, then there are at most countable many of such pairs.

2. Each saddle solution satisfying (4.1) and (4.2) corresponds to a heteroclinic solution g_i of (3.3) and hence a pair of a_i, b_i .

3. For each pair a_i, b_i , the heteroclinic solution g_i connecting a_i and b_i is indeed unique up to translation. Therefore, the discussion in this section as well as in Section 3 can be carried out with $-1, 1$ replaced by a_i, b_i and g replaced by g_i , except for the uniqueness assertions.

4. In the case F is even and $a_i = -b_i$, there exists a saddle solution as in Proposition 3.1 associated with a_i, b_i .

The proofs of the above statements are either easy or can be modified from the arguments in this paper, we leave them to the reader.

Next we study u more carefully at each side of the γ -level curve. For this purpose, we define

$$(4.42) \quad \mathbf{e}_\gamma^+ := \int_\gamma^1 \sqrt{2F(u)} du, \quad \mathbf{e}_\gamma^- := \int_{-1}^\gamma \sqrt{2F(u)} du.$$

We also define

$$\rho_2(y_2) = \int_{k(y_2)}^\infty [F(u(y_1, y_2)) + \frac{1}{2}|u_{y_1}|^2 - \frac{1}{2}|u_{y_2}|^2] dy_1.$$

Since $u(k(y_2), y_2) = \gamma$ for $y_2 \geq 0$, then

$$u_{y_1}(k(y_2), y_2) \cdot k'(y_2) + u_{y_2}(k(y_2), y_2) = 0, \quad \forall y_2 > 0.$$

Hence, by straight forward computations and the Allen-Cahn equation (3.1) we obtain

$$\rho_2'(y_2) = -[F(\gamma) - \frac{1}{2}|\nabla u|^2(k(y_2), y_2)]k'(y_2).$$

Hence

$$(4.43) \quad \begin{aligned} \rho_2(y_2) &= - \int_0^{y_2} [F(\gamma) - \frac{1}{2}|\nabla u|^2(k(s), s)]k'(s)ds + \rho_2(0) \\ &= - \int_0^{k(y_2)} [F(\gamma) - \frac{1}{2}|\nabla u|^2(y_1, h(y_1))]dy_1 + \rho_2(0). \end{aligned}$$

Using (1.6) and computations as in (2.12), we can obtain

$$(4.44) \quad \rho_2(0) = \mathbf{e} \sin \beta, \quad \lim_{y_2 \rightarrow \infty} \rho_2(y_2) = \mathbf{e}_\gamma^+ \sin \beta.$$

Therefore

$$(4.45) \quad \int_0^\infty [F(\gamma) - \frac{1}{2}|\nabla u|^2(y_1, h(y_1))]dy_1 = \mathbf{e}_\gamma^- \sin \beta.$$

Similarly, we can define

$$\rho_3(y_1) = \int_{h(y_1)}^{\infty} [F(u(y_1, y_2)) + \frac{1}{2}|u_{y_2}|^2 - \frac{1}{2}|u_{y_1}|^2] dy_2.$$

and obtain

$$\rho_3'(y_1) = -[F(\gamma) - \frac{1}{2}|\nabla u|^2(y_1, h(y_1))]h'(y_1).$$

Hence

$$\begin{aligned} \rho_3(y_1) &= -\int_0^{y_1} [F(\gamma) - \frac{1}{2}|\nabla u|^2(s, h(s))]h'(s)ds + \rho_3(0) \\ (4.46) \quad &= -\int_0^{h(y_1)} [F(\gamma) - \frac{1}{2}|\nabla u|^2(k(y_2), y_2)]dy_2 + \rho_3(0). \end{aligned}$$

Using (1.6) and computations as in (2.12), we can also obtain

$$(4.47) \quad \rho_3(0) = \mathbf{e} \cos \beta, \quad \lim_{y_1 \rightarrow \infty} \rho_3(y_1) = \mathbf{e}_\gamma^+ \cos \beta.$$

and therefore

$$(4.48) \quad \int_0^{\infty} [F(\gamma) - \frac{1}{2}|\nabla u|^2(k(y_2), y_2)]dy_2 = \mathbf{e}_\gamma^+ \cos \beta.$$

Combining (4.48) and (4.45), we can conclude the following result.

Theorem 4.7. *Under the assumptions of Theorem 4.4, if we assume further $u(0) = \gamma$ and the γ -level curve $y_2 = h(y_1)$ is close to a straight line in C^1 norm globally in the first quadrant of \mathbb{R}^2 , i.e., for some $\beta \in (0, \pi/2)$ and small positive constant ϵ*

$$\|h(y_1) - y_1 \tan \beta\|_{C^1([0, \infty))} \leq \epsilon,$$

then we have

$$(4.49) \quad (\tan \beta - \epsilon) \tan \beta \leq \frac{\mathbf{e}_\gamma^-}{\mathbf{e}_\gamma^+} \leq (\tan \beta + \epsilon) \tan \beta.$$

In particular, if γ -level curve is a straight line, i.e., $\epsilon = 0$, then

$$(4.50) \quad \tan^2 \beta = \frac{\mathbf{e}_\gamma^-}{\mathbf{e}_\gamma^+}.$$

Remark 4.8. Theorem 3.4 and Theorem 3.10 are special cases of the above theorem.

In general, we have the following estimate of the contact angle $\theta = 2\beta$ of the γ -level curves.

Theorem 4.9. *Under the assumption of Theorem 4.7, the angle β of the γ -level curve with y_1 -axis at infinity satisfies*

$$(4.51) \quad \frac{\mathbf{e}_\gamma^+}{\mathbf{e}} \leq \sin \beta \leq \frac{\mathbf{e}_\gamma^+}{\mathbf{e}} \sqrt{1 + 2\frac{\mathbf{e}_\gamma^-}{\mathbf{e}_\gamma^+}}.$$

In particular,

$$(4.52) \quad \lim_{\gamma \rightarrow -1} \beta = \frac{\pi}{2}, \quad \lim_{\gamma \rightarrow 1} \beta = 0.$$

Proof. We just note that $\rho_2(0) \geq \mathbf{e}_\gamma^+$ and $\rho_3(0) \geq \mathbf{e}_{-\gamma}^-$. Then (4.44) and (4.47) lead to (4.51) immediately. The limits of the angle in terms of γ follows from the following fact

$$(4.53) \quad \lim_{\gamma \rightarrow -1} \mathbf{e}_\gamma^+ = \mathbf{e}, \quad \lim_{\gamma \rightarrow 1} \mathbf{e}_\gamma^+ = 0.$$

□

Remark 4.10. Theorem 3.6 is a special case of the above theorem.

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CHANGFENG GUI, SCHOOL OF MATHEMATICS, HUNAN UNIVERSITY, CHANGSHA, 410082, PRC,
AND DEPARTMENT OF MATHEMATICS, U-9, UNIVERSITY OF CONNECTICUT, STORRS, CT 06269,
USA

E-mail address: `gui@math.uconn.edu`