Solutions

#1 a) $\begin{cases} xy - 1 = 0 \\ y + 1 = 0 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = -1 \end{cases}$

b) solve $\frac{dy}{dt} = y + 1$, $\Rightarrow y = k_2 e^{t} - 1$

Then $\frac{dx}{dt} = (k_2 e^{t} - 1)x - 1$

$\mu(t) = e^{-\int (k_2 e^{t} - 1)\,dt} = e^{-k_2 e^{t} + t} = e^{-k_2 e^{t}} \cdot e^{t}$

$x(t) = -e^{k_2 e^{t} - t} \int e^{-k_2 e^{t} + t}\,dt = -\frac{1}{k_2} e^{-t} + k_1 e^{k_2 e^{t} - t}$

$y(t) = k_2 e^{t} - 1$

c) $y(0) = 0 \Rightarrow k_2 = 1$

$x(0) = 1 \Rightarrow k_1 = 2e^{-1}$

Then

$\begin{cases} x(t) = -e^{-t} + 2e^{-1} \cdot e^{e^{t} - t}, \\ y(t) = e^{t} - 1. \end{cases}$

[Note: Similar Problem in Exam will be easier in terms of computation of integrals]

#2. skip.

#3 $\det(A - \lambda I) = \lambda^2 - 2\lambda - 7$, $\lambda = \frac{2 \pm \sqrt{32}}{2} = 1 \pm 2\sqrt{2}$

a) $\lambda = 1 + 2\sqrt{2}$, $(A - \lambda I)V_1 = 0 \Rightarrow \begin{pmatrix} -3 - 2\sqrt{2} & -1 \\ 1 & 3 - 2\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 - 2\sqrt{2} \\ -1 \end{pmatrix}$

$Y_1(t) = e^{(1 + 2\sqrt{2})t} \cdot \begin{pmatrix} 3 - 2\sqrt{2} \\ -1 \end{pmatrix}$

$\lambda = 1 - 2\sqrt{2}$, $\begin{pmatrix} -(3 - 2\sqrt{2}) & -1 \\ 1 & 3 + 2\sqrt{2} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$. $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3 + 2\sqrt{2} \\ -1 \end{pmatrix}$

$Y_2 = e^{(1 - 2\sqrt{2})t} \cdot \begin{pmatrix} 3 + 2\sqrt{2} \\ -1 \end{pmatrix}$

b) $Y(t) = k_1 e^{(1+2\sqrt{2})t} \begin{pmatrix} 3-2\sqrt{2} \\ -1 \end{pmatrix} + k_2 e^{(1-2\sqrt{2})t} \begin{pmatrix} 3+2\sqrt{2} \\ -1 \end{pmatrix}$ ②

c) $Y(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = k_1 \begin{pmatrix} 3-2\sqrt{2} \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 3+2\sqrt{2} \\ -1 \end{pmatrix}$

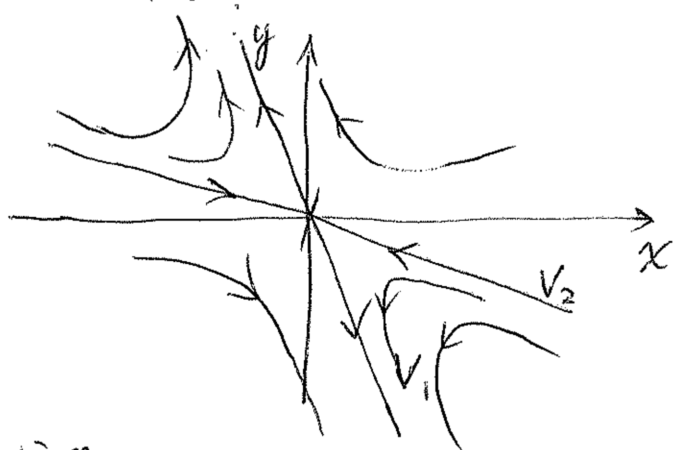
$$\Rightarrow \begin{cases} k_1 \cdot (3-2\sqrt{2}) + k_2 (3+2\sqrt{2}) = 0 \\ -(k_1 + k_2) = -1 \end{cases} \Rightarrow \begin{cases} k_1 = \dfrac{3+2\sqrt{2}}{4\sqrt{2}} \\ k_2 = \dfrac{2\sqrt{2}-3}{4\sqrt{2}} \end{cases}$$

$$Y(t) = \frac{3+2\sqrt{2}}{4\sqrt{2}} e^{(1+2\sqrt{2})t} \begin{pmatrix} 3-2\sqrt{2} \\ -1 \end{pmatrix} + \frac{2\sqrt{2}-3}{4\sqrt{2}} e^{(1-2\sqrt{2})t} \begin{pmatrix} 3+2\sqrt{2} \\ -1 \end{pmatrix}.$$

d) Since $\lambda_1 = 1+2\sqrt{2} > 0$, $\lambda_2 = 1-2\sqrt{2} < 0$

it is a saddle

e)



#4 Correction:
Change $A = \begin{pmatrix} -2 & -1 \\ 1 & 4 \end{pmatrix}$ to $A = \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}$

a) $\det(A - \lambda I) = \lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = 1 \pm 2i$

No straight line solution in x-y plane.

b) $\lambda_1 = 1+2i$, $(A - \lambda I)V = \begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$$V_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2i \\ 1 \end{pmatrix}, \quad Y_1(t) = e^{(1+2i)t} \cdot \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

$$Y_1(t) = e^t \cdot \begin{pmatrix} 2\sin 2t \\ \cos 2t \end{pmatrix} + i e^t \begin{pmatrix} -2\cos 2t \\ \sin 2t \end{pmatrix} = Y_{Re} + i Y_{Im}$$

General solution $Y(t) = k_1 e^t \cdot \begin{pmatrix} 2\sin 2t \\ \cos 2t \end{pmatrix} + k_2 e^t \begin{pmatrix} -2\cos 2t \\ \sin 2t \end{pmatrix}$

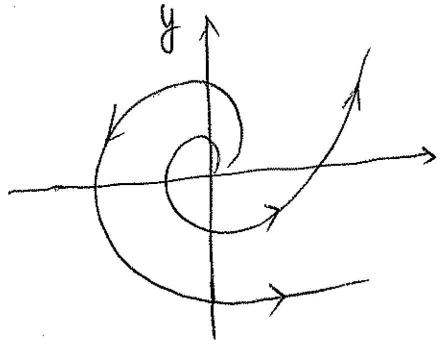
c) $Y(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = k_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} 0 \cdot k_1 - 2k_2 = 0 \\ k_1 = -1 \end{cases} \Rightarrow \begin{cases} k_1 = -1 \\ k_2 = 0 \end{cases}$

$Y(t) = -e^t \begin{pmatrix} 2\sin 2t \\ \cos 2t \end{pmatrix}$

d) $\alpha = 1 > 0$, it is a spiral source.

e)



#5 a) $\frac{dy}{dt} = v$,

$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -5v - 6y$.

Let $Y = \begin{pmatrix} y \\ v \end{pmatrix}$, $\frac{dY}{dt} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} Y$.

b) $\det(A - \lambda I) = \lambda^2 + 5\lambda + 6 = 0$, $\lambda_1 = -2$, $\lambda_2 = -3$

c) $\lambda_1 = -2$, $(A - \lambda I)V = \begin{pmatrix} +2 & 1 \\ -6 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$V_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $Y_1 = e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = -3$ $(A - \lambda I)V = \begin{pmatrix} 3 & 1 \\ -6 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

$V_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $Y_2 = e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$Y(t) = k_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} y \\ v \end{pmatrix}$

Then $y(t) = k_1 e^{-2t} + k_2 e^{-3t}$.