1. Let $f(x) = x^2$, $g(x) = \sqrt{x^2+1}$. Find
   a) $f(g(x))$; b) $g(f(x))$; c) $f(f(x))$; d) $g(g(0))$.

a) $f(g(x)) = x^2+1$

b) $g(f(x)) = \sqrt{x^4+1}$

c) $f(f(x)) = x^4$

d) $g(g(0)) = g(1) = \sqrt{2}$

2. Simplify the following expressions:
   a) $e^{\ln(x+1)} + 2$; $= x+1+2 = x+3$

   b) $\log_2(9x) - \log_4(x^2)$; $= \log_2(9x) - \log_2 x = \log_2 9 = 2\log_2 3$

   c) $\sin(\cos^{-1}(x^2))$; $= \sqrt{1-\cos^2(\cos^{-1}(x^2))} = \sqrt{1-(x^2)^2} = \sqrt{1-x^4}$

   d) $\frac{x^4-1}{x^2+1}$. $= x^2-1$

3. Calculate the following limits.
   a) $\lim_{x\to 1} \frac{x^3-2x^2+1}{x+1}$; $= \frac{1-2+1}{2} = \frac{0}{2} = 0$

   b) $\lim_{x\to 1} \frac{x^3-2x^2+1}{x-1}$; $= \lim_{x\to 1} \frac{x^3-x^2-\cancel{(x^2-2x)}-(x^2-1)}{x-1} = \lim_{x\to 1} (x^2-x-1) = -1$

   c) $\lim_{x\to\infty} \frac{x^3-2x^2+1}{x^2+1}$; $= \lim_{x\to\infty} \frac{x-2+\frac{1}{x^2}}{1+\frac{1}{x^2}} = \frac{\infty}{1} = \infty$

   d) $\lim_{x\to\infty} \frac{x^3-2x^2+1}{x^3+1}$; $= \lim_{x\to\infty} \frac{1-\frac{2}{x}+\frac{1}{x^3}}{1+\frac{1}{x^3}} = 1$

4. Use the Squeeze Theorem to compute

$$\lim_{x\to 1^+} \sqrt{x-1}\sin\frac{\pi}{x^2-1}.$$

Proof. $-1 \le \sin\frac{\pi}{x^2-1} \le 1$

$-\sqrt{x-1} \le \sqrt{x-1}\,\sin\frac{\pi}{x^2-1} \le \sqrt{x-1}$

$\lim_{x\to 1^+} \sqrt{x-1} = 0 = \lim_{x\to 1^+} -\sqrt{x-1}$,

So $\lim_{x\to 1^+} \sqrt{x-1}\,\sin\frac{\pi}{x^2-1} = 0$

5. Use the $\epsilon, \delta$ definition to prove

$$\lim_{x\to 1}(4x+1) = 5.$$

Proof. $\forall \varepsilon > 0$, we need find $\delta > 0$ so that

$|4x+1-5| < \varepsilon$ whenever

$|x-1| < \delta$.

solve $|4x+1-5| < \varepsilon$,

we have $|4x-4| < \varepsilon$

$|x-1| < \frac{\varepsilon}{4}$

Then $\delta = \frac{\varepsilon}{4}$ is what we choose.

This proves $\lim_{x\to 1}(4x+1) = 5$ by definition.

6. Find the horizontal and vertical asymptotes of the following function and sketch the graph.

$$f(x) = \frac{x^2 + x + 1}{x^2 - 1}.$$

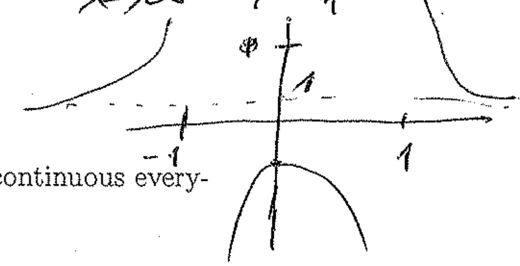
solution: vertical asymptotes $x = \pm 1$ since

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x^2+x+1}{x^2-1} = \infty$$

$$\lim_{x \to 1^-} f(x) = -\infty, \quad \lim_{x \to -1^+} f(x) = -\infty, \quad \lim_{x \to -1^-} f(x) = \infty$$

Horizontal asymptote $y = 1$ since

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2+x+1}{x^2-1} = \lim_{x \to \infty} \frac{1 + \frac{1}{x} + \frac{1}{x^2}}{1 - \frac{1}{x^2}} = 1$$




7. Find the value of $a$ and $b$ that make the following function $f$ continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x > 3 \\ ax + b & \text{if } -3 < x \leq 3 \\ x^2 + 2bx + 3 & \text{if } x \leq 3 \end{cases}$$

solution:

$$\lim_{x \to 3^+} f(x) = 6 = f(3) = 3a + b = \lim_{x \to 3^+} f(x)$$

$$\lim_{x \to 3^-} f(x) = -3a + b = f(-3) = \lim_{x \to -3^-} f(x) = 9 - 6b + 3$$

so
$$\begin{cases} 3a + b = 6 \\ -3a + b = 9 - 6b + 3 \end{cases}$$

solve it for $a$ and $b$

$$\begin{cases} \\ 8b = 18 \end{cases} \qquad \begin{cases} a = \frac{5}{4} \\ b = \frac{9}{4} \end{cases}$$