RIEMANNIAN GEOMETRY OF $\mathrm{Diff}(S^1)/S^1$ REVISITED

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Abstract. A further study of Riemannian geometry $\mathrm{Diff}(\mathrm{S}^1)/\mathrm{S}^1$ is presented. We describe Hermitian and Riemannian metrics on the complexification of the homogeneous space, as well as the complexified symplectic form. It is based on the ideas from [12], where instead of using the Kähler structure symmetries to compute the Ricci curvature, the authors rely on classical finite-dimensional results of Nomizu et al on Riemannian geometry of homogeneous spaces.

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1. Introduction

Let $\mathrm{Diff}(\mathrm{S}^1)$ be the Virasoro group of orientation-preserving diffeomorphisms of the unit circle. Then the quotient space $\mathrm{Diff}(\mathrm{S}^1)/\mathrm{S}^1$ describes those diffeomorphisms that fix a point on the circle. The geometry of this infinite-dimensional space has been of interest to physicists (e.g. [8], [7], [19]).

We follow the approach taken in [8, 7, 19, 14] in that we describe the space $\mathrm{Diff}(S^1)/S^1$ as an infinite dimensional complex manifold with a Kähler metric. Theorem 3.3 describes properties of the Hermitian and Riemannian metrics, as well as of the complexified symplectic form. Then we introduce the covariant derivative ∇ which is consistent with the Kähler structure. We use the expression for the derivative found in [12], where the classical finite-dimensional results of K.Nomizu in [16] for homogeneous spaces were used in this infinite-dimensional setting. The goal of the present article is to clarify certain parts of [12], in particular, Theorem 4.5. This theorem stated that the covariant derivative in question is Levi-Civita, but the details were omitted. In the present paper we explicitly define the Riemannian metric g for which ∇ is the Levi-Civita covariant derivative. This is proven in part (3) of Theorem 3.5 of the present paper. To complete the exposition we present the computation of the Riemannian curvature tensor and the Ricci curvature for the covariant derivative ∇. This proof follows the one in [12].

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Our interest to the geometry of this infinite-dimensional manifold comes from attempts to develop stochastic analysis on infinite-dimensional manifolds. Relevant references include works by H. Airault, V. Bogachev, P. Malliavin, A. Thalmaier ([2, 6, 3, 4, 5, 10]). A group Brownian motion in $\mathrm{Diff}(\mathrm{S}^1)$ has been constructed by P.Malliavin in [15]. From the finite-dimensional case we know that the lower bound of the Ricci curvature controls the growth of the Brownian motion, therefore a better understanding of the geometry of $\mathrm{Diff}(S^1)/S^1$ might help in studying a Brownian motion on this homogeneous space. For further references to the works exploring the connections between stochastic analysis and Riemannian geometry in infinite dimensions, mostly in loop groups and their extensions such as current groups, path spaces and complex Wiener spaces see [9], [11], [17], [18].

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2. Virasoro algebra

Notation 2.1. *Let* $\mathrm{Diff}(\mathrm{S}^1)$ *be the group of orientation preserving* C^∞*-diffeomorphisms of the unit circle, and* $\mathrm{diff}(\mathrm{S}^1)$ *its Lie algebra. The elements of* $\mathrm{diff}(\mathrm{S}^1)$ *will be identified with the* C^∞ *left-invariant vector fields* $f(t)\frac{d}{dt}$*, with the Lie bracket given by*

$$[f,g]=fg'-f'g, f,g\in \mathrm{diff}(\mathrm{S}^1).$$

Definition 2.2. Suppose c,h are positive constants. Then the **Virasoro algebra** $\mathcal{V}_{c,h}$ is the vector space $\mathbb{R}\oplus\mathrm{diff}(\mathrm{S}^1)$ with the Lie bracket given by

$$[a\kappa+f,b\kappa+g]_{\mathcal{V}_{c,h}}=\omega_{c,h}(f,g)\kappa+[f,g], \tag{2.1}$$

where $\kappa\in\mathbb{R}$ is the central element, and ω is the bilinear symmetric form

$$\omega_{c,h}(f,g)=\int_0^{2\pi}\left((2h-\frac{c}{12})f^{'}(t)-\frac{c}{12}f^{(3)}(t)\right)g(t)\frac{dt}{2\pi}.$$

Remark 2.3. If $h=0$, $c=6$, then $\omega_{c,h}$ is the fundamental cocycle ω (see [3])

$$\omega(f,g)=-\int_0^{2\pi}\left(f'+f^{(3)}\right)g\frac{dt}{4\pi}.$$

Remark 2.4. A simple verification shows that $\mathcal{V}_{c,h}$ with $\omega_{c,h}$ satisfies the Jacobi identity, and therefore $\mathcal{V}_{c,h}$ with this bracket is indeed a Lie algebra. In addition, by the integration by parts formula $\omega_{c,h}$ satisfies

$$\omega_{c,h}\left(f',g\right)=-\omega_{c,h}\left(f,g'\right). \tag{2.2}$$

Moreover, $\omega_{c,h}$ is anti-symmetric

$$\omega_{c,h}\left(f,g\right)=-\omega_{c,h}\left(g,f\right). \tag{2.3}$$

Notation 2.5. *Throughout this work we use* $k,m,n...\in\mathbb{N}$*, and* $\alpha,\beta,\gamma...\in\mathbb{Z}$*.*

Below we introduce an inner product on the Lie algebra $\mathrm{diff}(\mathrm{S}^1)$ which has a natural basis

$$f_k = \cos kt, g_m = \sin mt, \ k = 0, 1, 2..., m = 1, 2.... \tag{2.4}$$

The Lie bracket in this basis satisfies the following identities

$$\begin{aligned}
[f_m, f_n] &= \frac{1}{2}\left((m{-}n)g_{m+n}{+}(m{+}n)\frac{m{-}n}{|m{-}n|}g_{|m-n|}\right), \ m \neq n, \\
[g_m, g_n] &= \frac{1}{2}\left((n{-}m)g_{m+n}{+}(m{+}n)\frac{m{-}n}{|m{-}n|}g_{|m-n|}\right), \ m \neq n, \\
[f_m, g_n] &= \frac{1}{2}\left((n{-}m)f_{m+n}{+}(m{+}n)f_{|m-n|}\right).
\end{aligned} \tag{2.5}$$

Notation 2.6. *By* $\mathrm{diff}_0(\mathrm{S}^1)$ *we denote the space of functions having mean* 0*. This space can be identified with* $\mathrm{diff}(\mathrm{S}^1)/\mathrm{S}^1$*, where* S^1 *is being viewed as constant vector fields corresponding to rotations of* S^1*.*

Then any element of $f \in \mathrm{diff}_0(\mathrm{S}^1)$ can be written

$$f(t) = \sum_{k=1}^{\infty}(a_k f_k{+}b_k g_k),$$

with $\{a_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty} \in \ell^2$ since f is smooth. We will also need the following endomorphism J of $\mathrm{diff}_0(\mathrm{S}^1)$

$$J(f)(t) = \sum_{k=1}^{\infty}(b_k f_k{-}a_k g_k). \tag{2.6}$$

It satisfies $J^2 = -I$.

Notation 2.7. *For any* $k \in \mathbb{Z}$ *we denote* $\theta_k = 2hk{+}\frac{c}{12}(k^3{-}k)$.

Remark 2.8. Note that $\theta_{-k} = -\theta_k$, for any $k \in \mathbb{Z}$.

The form $\omega_{c,h}$ and the endomorphism J induce an inner product on $\mathrm{diff}_0(\mathrm{S}^1)$ by

$$\langle f, g\rangle = \omega_{c,h}(f, Jg) = \omega_{c,h}(g, Jf).$$

The last identity follows from Equation (2.3).

Proposition 2.9. $\langle f, g\rangle$ *is an inner product on* $\mathrm{diff}_0(\mathrm{S}^1)$*.*

Proof. Let $b_0 = 0$, then

$$\omega_{c,h}(f, Jf) = \int_0^{2\pi}\left((2h{-}\frac{c}{12})f^{'}(t){-}\frac{c}{12}f^{(3)}(t)\right)(Jf)(t)\frac{dt}{2\pi} =$$
$$\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\theta_k(b_k f_k{-}a_k g_k)\right)\left(\sum_{m=1}^{\infty}(b_m f_m{-}a_m g_m)\right)\frac{dt}{2\pi} = \frac{1}{2}\sum_{k=1}^{\infty}\theta_k(a_k^2{+}b_k^2).$$

Then for any $f \in \mathrm{diff}_0(\mathrm{S}^1)$

$$\langle f, f \rangle = \frac{1}{2} \sum_{k=1}^{\infty} \theta_k \left(a_k(f)^2 + b_k(f)^2 \right).$$

□

Notation 2.10. *Let $\lambda_{m,n} = \frac{(2n+m)\theta_m}{2\theta_{m+n}}$ for any $n, m \in \mathbb{Z}$. Then it is easy to check that*

$$\lambda_{m,n} = \lambda_{n,m} + \frac{m-n}{2}. \tag{2.7}$$

3. $\mathrm{Diff}(S^1)/S^1$ as a Kähler manifold

Denote $\mathfrak{g} = \mathrm{diff}(\mathrm{S}^1)$, $\mathfrak{m} = \mathrm{diff}_0(\mathrm{S}^1)$, $\mathfrak{h} = f_0\mathbb{R}$, so that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Then $\mathfrak{g}$ is an infinite-dimensional Lie algebra equipped with an inner product $\langle \cdot, \cdot \rangle$. Note that for any $n \in \mathbb{N}$

$$[f_0, f_n] = -ng_n \in \mathfrak{m}, \quad [g_0, g_n] = nf_n \in \mathfrak{m},$$

and therefore $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. In addition, $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, but $\mathfrak{m}$ is not a Lie subalgebra of $\mathfrak{g}$ since $[f_m, g_m] = mf_0$.

Let $G = \mathrm{Diff}(\mathrm{S}^1)$ with the associated Lie algebra $\mathrm{diff}(\mathrm{S}^1)$, the subgroup $H = S^1$ with the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, then $\mathfrak{m}$ is a tangent space naturally associated with the quotient $\mathrm{Diff}(\mathrm{S}^1)/\mathrm{S}^1$. For any $g \in \mathfrak{g}$ we denote by $g_{\mathfrak{m}}$ (respectively $g_{\mathfrak{h}}$) its $\mathfrak{m}$-(respectively $\mathfrak{h}$-)component, that is, $g = g_{\mathfrak{m}} + g_{\mathfrak{h}}$, $g_{\mathfrak{m}} \in \mathfrak{m}$, $g_{\mathfrak{h}} \in \mathfrak{h}$. The fact $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ implies that for any $h \in \mathfrak{h}$ the adjoint representation $ad(h) = [h, \cdot] : \mathfrak{g} \to \mathfrak{g}$ maps $\mathfrak{m}$ into $\mathfrak{m}$. We will abuse notation by using $ad(h)$ for the corresponding endomorphism of $\mathfrak{m}$.

Recall that $J : \mathrm{diff}_0(\mathrm{S}^1) \to \mathrm{diff}_0(\mathrm{S}^1)$ is an endomorphism defined by (2.6), or equivalently, in the basis $\{f_m, g_n\}$, $m, n = 1, ...$ by

$$Jf_m = -g_m, \; Jg_n = f_n.$$

This is an almost complex structure on $\mathrm{diff}_0(\mathrm{S}^1)$, and as was shown in [12] it is actually a complex structure for an appropriately chosen connection.

Let $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{m}_{\mathbb{C}}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{m}$ respectively. Now we would like to introduce Hermitian metric, Riemannian metric and the complexified symplectic form $\omega_{\mathbb{C}}$.

Notation 3.1. *For any $f + ig, u + iv \in \mathfrak{g}_{\mathbb{C}}$, where $f, g, u, v \in \mathfrak{g}$ and $i^2 = -1$ we denote*

$$h\,(f + ig, u + iv) = \langle f, u \rangle + \langle g, v \rangle + i\,(\langle g, u \rangle - \langle f, v \rangle)\,;$$
$$g\,(f + ig, u + iv) = \langle f, u \rangle + \langle g, v \rangle = \mathrm{Re}\,(h\,(f + ig, u + iv))\,;$$
$$\omega_{\mathbb{C}}\,(f + ig, u + iv) = \langle g, u \rangle - \langle f, v \rangle = \mathrm{Im}\,(h\,(f + ig, u + iv))\,.$$

We will call h a Hermitian metric, g a Riemannian metric, and $\omega_{\mathbb{C}}$ a symplectic form.

The endomorphism J can be naturally extended to $\mathfrak{g}_{\mathbb{C}}$ by $J\,(f + ig) = Jf + iJg$ for any $f, g \in \mathfrak{g}$. We will abuse notation and use the same J for this extended endomorphism. It is easy to check that J is complex-linear.

Notation 3.2. *Let $m \in \mathbb{N}$, then denote*

$$L_m = f_m + i g_m,\ L_{-m} = f_m - i g_m.$$

One can see that on elements $\{L_\alpha\}_{\alpha\in\mathbb{Z}}$ the endomorphism J acts in the following way

$$J(L_\alpha) = i\,\mathrm{sgn}(\alpha) L_\alpha,\ \alpha \in \mathbb{Z}.$$

Using Equation(2.5) we can easily check that for any $\alpha, \beta \in \mathbb{Z}$

$$[L_\alpha, L_\beta] = i(\beta - \alpha) L_{\alpha+\beta}. \tag{3.1}$$

In addition to these properties, we can easily see that $\{L_\alpha\}_{\alpha\in\mathbb{Z}}$ form an orthogonal system in the Riemannian metric g. Indeed, for any $m, n \in \mathbb{N}$

$$\begin{aligned}
&h(L_m, L_n) = h(f_m + i g_m, f_n + i g_n) = \\
&\langle f_m, f_n\rangle + \langle g_m, g_n\rangle = \theta_m \delta_{m,n} = g(L_m, L_n);\\
&h(L_m, L_{-n}) = h(L_{-m}, L_n) = 0.
\end{aligned}$$

In particular,

$$\omega_{\mathbb{C}}(L_\alpha, L_\beta) = 0$$

for any $\alpha, \beta \in \mathbb{Z}$.

Theorem 3.3. *For any $F, G \in \mathfrak{g}_{\mathbb{C}}$, $a, b \in \mathbb{C}$ we have*

(1)

$$h(F,G) = \overline{h(G,F)},\ g(F,G) = g(G,F),\ \omega_{\mathbb{C}}(F,G) = -\omega_{\mathbb{C}}(G,F);$$

(2)

$$\begin{aligned}
h(aF, bG) &= a\bar{b} h(F,G),\\
g(aF, bG) &= \mathrm{Re}\left(a\bar{b}\right) g(F,G) - \mathrm{Im}\left(a\bar{b}\right)\omega_{\mathbb{C}}(F,G),\\
\omega_{\mathbb{C}}(aF, bG) &= \mathrm{Re}\left(a\bar{b}\right)\omega_{\mathbb{C}}(F,G) + \mathrm{Im}\left(a\bar{b}\right) g(F,G);
\end{aligned}$$

(3) Both the Hermitian and Riemannian inner products, as well as the form $\omega_{\mathbb{C}}$ are invariant under J;

(4) the form $\omega_{\mathbb{C}}$ is closed.

Proof. Parts (1) and (2) follow from the definition of the metrics h and g and form $\omega_{\mathbb{C}}$ by a straightforward computation. Let us check the invariance of h, g and $\omega_{\mathbb{C}}$ under J and closedness of $\omega_{\mathbb{C}}$. It is enough to check that $h(JF, JF) = h(F,F)$ on the complex basis $\{f_m, if_m, g_n, g_n\}$. This is indeed so for $\{f_m, g_n\}$, and for $F = if_m$ by the second part of this Proposition

$$h(J(if_m), J(if_m)) = h(f_m, f_m) = h(if_m, if_m).$$

The rest of the identities can be checked similarly, and invariance of the Hermitian metric under J obviously implies invariance of g and $\omega_{\mathbb{C}}$.

To check closedness of $\omega_{\mathbb{C}}$ we can first note that (2.2) holds on $\mathfrak{g}_{\mathbb{C}}$ as well, and so

$$d\omega_{\mathbb{C}}(F,G) = d\omega_{\mathbb{C}}(F',G) + d\omega_{\mathbb{C}}(F,G') = 0.$$

□

Notation 3.4. *Define a complex-linear connection on* $\mathfrak{m}_{\mathbb{C}}$ *by*

$$\begin{aligned} &\nabla_{L_m} L_n = -2i\lambda_{m,n} L_{m+n}; \\ &\nabla_{L_{-m}} L_{-n} = 2i\lambda_{m,n} L_{-m-n}; \\ &\nabla_{L_{-m}} L_n = i(m+n) L_{n-m},\ n > m; \\ &\nabla_{L_m} L_{-n} = -i(m+n) L_{m-n},\ n > m; \\ &\nabla_{L_{-m}} L_n = \nabla_{L_m} L_{-n} = 0,\ m \geqslant n, \end{aligned} \tag{3.2}$$

for any $m, n \in \mathbb{N}$.

As we can see from (3.2), for any $\alpha, \beta \in \mathbb{Z}$

$$\nabla_{L_\alpha} L_\beta = \Gamma_{\alpha,\beta} L_{\alpha+\beta}$$

for some $\Gamma_{\alpha,\beta} \in \mathbb{C}$. In addition, it is easy to check that the definition of the covariant derivative ∇ in (3.2) implies that

$$\Gamma_{-\alpha,-\beta} = -\Gamma_{\alpha,\beta}, \tag{3.3}$$

and (3.2) can be described as

$$\begin{aligned} &\Gamma_{m,n} = -2i\lambda_{m,n},\ \Gamma_{-m,n} = i(m+n), n > m,\ \Gamma_{-m,n} = 0, n \leqslant m, \\ &\Gamma_{-\alpha,-\beta} = -\Gamma_{\alpha,\beta},\ m, n \in \mathbb{N},\ \alpha, \beta \in \mathbb{Z}. \end{aligned}$$

The first part of the following theorem is an analogue of the Newlander-Nirenberg theorem in our setting.

Theorem 3.5. *The covariant derivative* ∇ *and the almost complex structure* J *satisfy the following properties.*

(1) *The Nijenhuis tensor* N *(the torsion of the almost complex structure* J*) defined by*

$$N_J(X,Y) = 2\left([JX,JY]_{\mathfrak{m}_{\mathbb{C}}} - [X,Y]_{\mathfrak{m}_{\mathbb{C}}} - J[X,JY]_{\mathfrak{m}_{\mathbb{C}}} - J[JX,Y]_{\mathfrak{m}_{\mathbb{C}}}\right)$$

vanishes on $\mathfrak{m}_{\mathbb{C}} = \operatorname{diff}_0(\mathrm{S}^1)_{\mathbb{C}}$*;*

(2) J *is a complex structure on* $\mathfrak{m}_{\mathbb{C}} = \operatorname{diff}_0(\mathrm{S}^1)_{\mathbb{C}}$ *with the covariant derivative* ∇ *defined in Notation 3.2. In other words,* J *is integrable;*

(3) *the covariant derivative* ∇ *is the Levi-Civita connection, that is, it is torsion-free and compatible with the Riemannian metric* g.

Proof. (1) It is enough to check the statement on the real basis elements.

$$\begin{aligned} &N_J(L_\alpha, L_\beta) = \\ &2\left([JL_\alpha, JL_\beta]_{\mathfrak{m}_{\mathbb{C}}} - [L_\alpha, L_\beta]_{\mathfrak{m}_{\mathbb{C}}} - J\left([L_\alpha, JL_\beta]_{\mathfrak{m}_{\mathbb{C}}}\right) - J\left([JL_\alpha, L_\beta]_{\mathfrak{m}_{\mathbb{C}}}\right)\right) = \\ &-2\left((1+\operatorname{sgn}(\alpha)\operatorname{sgn}(\beta))[L_\alpha, L_\beta]_{\mathfrak{m}_{\mathbb{C}}} + i\left(\operatorname{sgn}(\alpha)+\operatorname{sgn}(\beta)\right)J\left([L_\alpha, L_\beta]_{\mathfrak{m}_{\mathbb{C}}}\right)\right) = 0. \end{aligned}$$

Here we used Equation (3.1) for the Lie bracket $[L_\alpha, L_\beta]_{\mathfrak{m}_{\mathbb{C}}}$, and then considered the cases when α and β are of the same or opposite signs.

(2) We will use the fact that

$$(\nabla_X J)(Y) = \nabla_X(JY) - J(\nabla_X Y).$$

One can check using Equation (3.2) that $J(\nabla_{L_\alpha} L_\beta) = i\,\mathrm{sgn}(\beta)\nabla_{L_\alpha} L_\beta$. Then again on the real basis elements

$$(\nabla_{L_\alpha} J)(L_\beta) = \nabla_{L_\alpha}(JL_\beta) - J(\nabla_{L_\alpha} L_\beta) = i\,\mathrm{sgn}(\beta)\nabla_{L_\alpha} L_\beta - J(\nabla_{L_\alpha} L_\beta) = 0$$

since the covariant derivative $\nabla_{L_\alpha} L_\beta$ is again a basis element.
(3) The torsion of the covariant derivative ∇ is defined by

$$T_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]_{\mathfrak{m}_\mathbb{C}}.$$

Using Equation (2.7) and the definition of the covariant derivative in Notation 3.2 we see that

$$\begin{aligned}
&\nabla_{L_m} L_n - \nabla_{L_n} L_m = 2i\left(\lambda_{n,m} - \lambda_{m,n}\right) L_{m+n} = i(n-m)L_{m+n};\\
&\nabla_{L_{-m}} L_{-n} - \nabla_{L_{-n}} L_{-m} = 2i\left(\lambda_{m,n} - \lambda_{n,m}\right) L_{-m-n} = i(m-n)L_{-m-n};\\
&\nabla_{L_{-m}} L_n - \nabla_{L_n} L_{-m} = i(m+n)L_{n-m},\ n > m;\\
&\nabla_{L_{-m}} L_n - \nabla_{L_n} L_{-m} = i(m+n)L_{n-m},\ n < m;\\
&\nabla_{L_{-n}} L_n - \nabla_{L_n} L_{-n} = \nabla_{L_n} L_{-n} - \nabla_{L_{-n}} L_n = 0;\\
&\nabla_{L_m} L_{-n} - \nabla_{L_{-n}} L_m = -i(m+n)L_{m-n},\ n > m;\\
&\nabla_{L_m} L_{-n} - \nabla_{L_{-n}} L_m = -i(m+n)L_{m-n},\ n < m,
\end{aligned}$$

which together with the Lie bracket expression for the basis elements $\{L_\alpha\}$ in Equation(3.1) gives the result.

To check that ∇ is compatible with the Riemannian metric g, it is enough to check that

$$g\left(\nabla_{L_\alpha} L_\beta, L_\gamma\right) + g\left(L_\beta, \nabla_{L_\alpha} L_\gamma\right) = 0$$

for any $L_\alpha, L_\beta, L_\gamma \in \mathbb{Z}$. Note that all $\Gamma_{\delta,\varepsilon}$ are either 0 or purely imaginary, so

$$\begin{aligned}
&g\left(\nabla_{L_\alpha} L_\beta, L_\gamma\right) + g\left(L_\beta, \nabla_{L_\alpha} L_\gamma\right) =\\
&g\left(\Gamma_{\alpha,\beta} L_{\alpha+\beta}, L_\gamma\right) + g\left(L_\beta, \Gamma_{\alpha,\gamma} L_{\alpha+\gamma}\right) =\\
&\Gamma_{\alpha,\beta}\omega_\mathbb{C}\left(L_{\alpha+\beta}, L_\gamma\right) - \Gamma_{\alpha,\gamma}\omega_\mathbb{C}\left(L_\beta, L_{\alpha+\gamma}\right) = 0.
\end{aligned}$$

□

Definition 3.6. The **curvature tensor** $R_{xy} : \mathfrak{m}_\mathbb{C} \to \mathfrak{m}_\mathbb{C}$ is defined by

$$R_{xy} = \nabla_x \nabla_y - \nabla_y \nabla_x - \nabla_{[x,y]_{\mathfrak{m}_\mathbb{C}}} - ad([x, y]_{\mathfrak{h}_\mathbb{C}}),\ x, y \in \mathfrak{m}_\mathbb{C};$$

the **Ricci tensor** $\mathrm{Ric}(x, y) : \mathfrak{m}_\mathbb{C} \to \mathfrak{m}_\mathbb{C}$ is defined by

$$\mathrm{Ric}(x, y) = \sum_{n\in\mathbb{N}} \frac{1}{\theta_n}\left(\langle R_{L_n,x} y, L_n\rangle + \langle R_{L_{-n},x} y, L_{-n}\rangle\right).$$

Proposition 3.7. *For any* $\alpha, \beta, \gamma \in \mathbb{Z}$

$$R_{L_\alpha, L_\beta} L_\gamma = R_{\alpha,\beta,\gamma} L_{\alpha+\beta+\gamma},$$

where the coefficients $R_{\alpha,\beta,\gamma}$ *satisfy*

$$R_{-\alpha,-\beta,-\gamma} = R_{\alpha,\beta,\gamma}.$$

Proof.

$$\begin{aligned}&R_{L_\alpha, L_\beta} L_\gamma = \nabla_{L_\alpha} \nabla_{L_\beta} L_\gamma - \nabla_{L_\beta} \nabla_{L_\alpha} L_\gamma - i(\beta - \alpha) \nabla_{L_{\alpha+\beta}} L_\gamma = \\ &\left(\Gamma_{\beta,\gamma}\Gamma_{\alpha,\beta+\gamma} - \Gamma_{\alpha,\gamma}\Gamma_{\beta,\alpha+\gamma} - i(\beta-\alpha)\Gamma_{\alpha+\beta,\gamma}\right) L_{\alpha+\beta+\gamma}.\end{aligned}$$

Then Equation(3.3) implies

$$R_{-\alpha,-\beta,-\gamma} = \Gamma_{-\beta,-\gamma}\Gamma_{-\alpha,-\beta-\gamma} - \Gamma_{-\alpha,-\gamma}\Gamma_{-\beta,-\alpha-\gamma} - i(\alpha - \beta)\Gamma_{-\alpha-\beta,-\gamma} = R_{\alpha,\beta,\gamma}.$$

□

Theorem 3.8. *[Theorem 4.11 in* [12]*] The only non-zero components of the Ricci tensor are*

$$Ric\left(\frac{L_n}{\sqrt{|\theta_n|}}, \frac{L_{-n}}{\sqrt{|\theta_n|}}\right) = -\frac{13n^3 - n}{6\theta_n}, \ n \in \mathbb{Z}.$$

Proof.

$$\begin{aligned}&\mathrm{Ric}\left(\frac{L_\alpha}{\sqrt{|\theta_\alpha|}}, \frac{L_\beta}{\sqrt{|\theta_\beta|}}\right) = \sum_{m \in \mathbb{N}} \frac{\left(\langle R_{L_m, L_\alpha} L_\beta, L_m\rangle + \langle R_{L_{-m}, L_\alpha} L_\beta, L_{-m}\rangle\right)}{\sqrt{|\theta_\alpha|}\sqrt{|\theta_\beta|}\theta_m} = \\ &\delta_{\alpha,-\beta} \sum_{m \in \mathbb{N}} \frac{\left(\langle R_{L_m, L_\alpha} L_{-\alpha}, L_{-m}\rangle + \langle R_{L_{-m}, L_\alpha} L_{-\alpha}, L_m\rangle\right)}{|\theta_\alpha|\theta_m}.\end{aligned}$$

Thus the only non-zero components of the Ricci tensor are the ones when $\alpha + \beta = 0$. For example, let $\alpha = n \in \mathbb{N}$, then according to Equation (3.2) we have that

$$\begin{aligned}R_{L_m, L_n} L_{-n} =& \nabla_{L_m} \nabla_{L_n} L_{-n} - \nabla_{L_n} \nabla_{L_m} L_{-n} - \nabla_{[L_m, L_n]_{\mathfrak{m}}} L_{-n} - ad([L_m, L_n]_{\mathfrak{h}}) L_{-n} = \\ & -\nabla_{L_n} \nabla_{L_m} L_{-n} - i(n-m) \nabla_{L_{m+n}} L_{-n} = -\nabla_{L_n} \nabla_{L_m} L_{-n},\end{aligned}$$

therefore

$$\begin{aligned}R_{L_m, L_n} L_{-n} =& 0, \quad m \geqslant n; \\ R_{L_m, L_n} L_{-n} =& i(m+n) \nabla_{L_n} L_{m-n} = 0, \quad m < n.\end{aligned}$$

Now by Proposition 3.7

$$\begin{aligned}&R_{L_m, L_{-n}} L_n = R_{L_{-m}, L_n} L_{-n} = \\ &\nabla_{L_{-m}} \nabla_{L_n} L_{-n} - \nabla_{L_n} \nabla_{L_{-m}} L_{-n} \\ &-\nabla_{[L_{-m}, L_n]_{\mathfrak{m}}} L_{-n} - \mathrm{ad}([L_{-m}, L_n]_{\mathfrak{h}}) L_{-n} = \\ &-\nabla_{L_n} \nabla_{L_{-m}} L_{-n} - i(m+n) \nabla_{L_{n-m}} L_{-n} = \\ &-2i\lambda_{m,n} \nabla_{L_n} L_{-m-n} - i(m+n) \nabla_{L_{n-m}} L_{-n} = \\ &-2(m+2n)\lambda_{m,n} L_{-m} - i(m+n) \nabla_{L_{n-m}} L_{-n},\end{aligned}$$

thus

$$R_{L_{-m},L_n}L_{-n} = -2(m+2n)\lambda_{m,n}L_{-m} + 2(m+n)\lambda_{m-n,n}L_{-m}, \quad m > n;$$
$$R_{L_{-m},L_n}L_{-n} = -\left(2(m+2n)\lambda_{m,n} + (2n-m)(m+n)\right)L_{-m}, \quad m \leqslant n.$$

Thus

$$\mathrm{Ric}\left(\frac{L_n}{\sqrt{\theta_n}}, \frac{L_{-n}}{\sqrt{\theta_n}}\right) =$$
$$\sum_{m=n+1}^{\infty} \frac{2(m+n)\lambda_{m-n,n} - 2(m+2n)\lambda_{m,n}}{\theta_n} - \sum_{m=1}^{n} \frac{(m+n)(2n-m) + 2(m+2n)\lambda_{m,n}}{\theta_n} =$$
$$\sum_{m=1}^{n} \frac{2(m+2n)\lambda_{m,n}}{\theta_n} - \sum_{m=1}^{n} \frac{(m+n)(2n-m) + 2(m+2n)\lambda_{m,n}}{\theta_n} =$$
$$-\sum_{m=1}^{n} \frac{(m+n)(2n-m)}{\theta_n} = -\frac{13n^3 - n}{6\theta_n}.$$

Now we can use Proposition 3.7 to extend the result to all $n \in \mathbb{Z}$.

□

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