Section 6.2 Orthogonal Sets

A set of vectors \( \{u_1, u_2, \ldots, u_p\} \) in \( \mathbb{R}^n \) is called an orthogonal set if \( u_i \cdot u_j = 0 \) whenever \( i \neq j \).

**EXAMPLE:** Is \( \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \) an orthogonal set?

**Solution:** Label the vectors \( u_1, u_2, \) and \( u_3 \) respectively. Then

\[
\begin{align*}
 u_1 \cdot u_2 &= \\
 u_1 \cdot u_3 &= \\
 u_2 \cdot u_3 &= 
\end{align*}
\]

Therefore, \( \{u_1, u_2, u_3\} \) is an orthogonal set.

**THEOREM 4**

Suppose \( S = \{u_1, u_2, \ldots, u_p\} \) is an orthogonal set of nonzero vectors in \( \mathbb{R}^n \) and \( W = \text{span}\{u_1, u_2, \ldots, u_p\} \). Then \( S \) is a linearly independent set and is therefore a basis for \( W \).

**Partial Proof:** Suppose

\[
\begin{align*}
 c_1 u_1 + c_2 u_2 + \cdots + c_p u_p &= 0 \\
 (c_1 u_1 + c_2 u_2 + \cdots + c_p u_p) \cdot u_1 &= 0 \\
 (c_1 u_1) \cdot u_1 + (c_2 u_2) \cdot u_1 + \cdots + (c_p u_p) \cdot u_1 &= 0 \\
 c_1(u_1 \cdot u_1) + c_2(u_2 \cdot u_1) + \cdots + c_p(u_p \cdot u_1) &= 0 \\
 c_1(u_1 \cdot u_1) &= 0 
\end{align*}
\]

Since \( u_1 \neq 0, u_1 \cdot u_1 > 0 \) which means \( c_1 = \ldots \).

In a similar manner, \( c_2, \ldots, c_p \) can be shown to by all 0. So \( S \) is a linearly independent set. ■

An orthogonal basis for a subspace \( W \) of \( \mathbb{R}^n \) is a basis for \( W \) that is also an orthogonal set.
EXAMPLE: Suppose $S = \{u_1, u_2, \ldots, u_p\}$ is an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$ and suppose $y$ is in $W$. Find $c_1, \ldots, c_p$ so that

$$y = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p.$$ 

Solution:

$$y \cdot = (c_1 u_1 + c_2 u_2 + \cdots + c_p u_p) \cdot$$

$$y \cdot u_1 = (c_1 u_1 + c_2 u_2 + \cdots + c_p u_p) \cdot u_1$$

$$y \cdot u_1 = c_1 (u_1 \cdot u_1) + c_2 (u_2 \cdot u_1) + \cdots + c_p (u_p \cdot u_1)$$

$$y \cdot u_1 = c_1 (u_1 \cdot u_1)$$

$$c_1 = \frac{y \cdot u_1}{u_1 \cdot u_1}$$

Similarly, $c_2 = \ldots$, $c_3 = \ldots$, $c_p = \ldots$

THEOREM 5

Let $\{u_1, u_2, \ldots, u_p\}$ be an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$. Then each $y$ in $W$ has a unique representation as a linear combination of $u_1, u_2, \ldots, u_p$. In fact, if

$$y = c_1 u_1 + c_2 u_2 + \cdots + c_p u_p$$

then

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \quad (j = 1, \ldots, p)$$

EXAMPLE: Express $y = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ as a linear combination of the orthogonal basis

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution:

$$\frac{y \cdot u_1}{u_1 \cdot u_1} = \frac{y \cdot u_2}{u_2 \cdot u_2} = \frac{y \cdot u_3}{u_3 \cdot u_3} = \ldots$$

Hence

$$y = \ldots u_1 + \ldots u_2 + \ldots u_3$$
Orthogonal Projections

For a nonzero vector $\mathbf{u}$ in $\mathbb{R}^n$, suppose we want to write $\mathbf{y}$ in $\mathbb{R}^n$ as the the following

$$\mathbf{y} = (\text{multiple of } \mathbf{u}) + (\text{multiple a vector } \perp \text{ to } \mathbf{u})$$

$$\mathbf{y} \cdot \mathbf{u} - \alpha (\mathbf{u} \cdot \mathbf{u}) = 0 \quad \Rightarrow \quad \alpha = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}$$

and

$$\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \quad \text{(orthogonal projection of } \mathbf{y} \text{ onto } \mathbf{u})$$

$$\mathbf{z} = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \quad \text{(component of } \mathbf{y} \text{ orthogonal to } \mathbf{u})$$
EXAMPLE: Let \( y = \begin{bmatrix} -8 \\ 4 \end{bmatrix} \) and \( u = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \). Compute the distance from \( y \) to the line through \( 0 \) and \( u \).

\[
\hat{y} = \frac{y \cdot u}{u \cdot u} u =
\]

Distance from \( y \) to the line through \( 0 \) and \( u \) = distance from \( \hat{y} \) to \( y \)

\[
= \| \hat{y} - y \| =
\]

Orthonormal Sets

A set of vectors \( \{u_1, u_2, \ldots, u_p\} \) in \( \mathbb{R}^n \) is called an orthonormal set if it is an orthogonal set of unit vectors.

If \( W = \text{span} \{u_1, u_2, \ldots, u_p\} \), then \( \{u_1, u_2, \ldots, u_p\} \) is an orthonormal basis for \( W \).
Recall that \( \mathbf{v} \) is a unit vector if \( \| \mathbf{v} \| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{v}^T \mathbf{v}} = 1 \).

Suppose \( \mathbf{U} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} \) where \( \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \) is an orthonormal set.

Then \( \mathbf{U}^T \mathbf{U} = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \)

It can be shown that \( \mathbf{U} \mathbf{U}^T = \mathbf{I} \) also. So \( \mathbf{U}^{-1} = \mathbf{U}^T \) (such a matrix is called an \textbf{orthogonal matrix}).

**THEOREM 6**  An \( m \times n \) matrix \( \mathbf{U} \) has orthonormal columns if and only if \( \mathbf{U}^T \mathbf{U} = \mathbf{I} \).

**THEOREM 7**  Let \( \mathbf{U} \) be an \( m \times n \) matrix with orthonormal columns, and let \( \mathbf{x} \) and \( \mathbf{y} \) be in \( \mathbb{R}^n \). Then

a. \( \| \mathbf{U} \mathbf{x} \| = \| \mathbf{x} \| \)

b. \( (\mathbf{U} \mathbf{x}) \cdot (\mathbf{U} \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \)

c. \( (\mathbf{U} \mathbf{x}) \cdot (\mathbf{U} \mathbf{y}) = 0 \) if and only if \( \mathbf{x} \cdot \mathbf{y} = 0 \).

*Proof of part b: \( (\mathbf{U} \mathbf{x}) \cdot (\mathbf{U} \mathbf{y}) = \)