4.1 Vector Spaces & Subspaces

Many concepts concerning vectors in $\mathbb{R}^n$ can be extended to other mathematical systems. We can think of a vector space in general, as a collection of objects that behave as vectors do in $\mathbb{R}^n$. The objects of such a set are called vectors.

A vector space is a nonempty set $V$ of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms below. The axioms must hold for all $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ in $V$ and for all scalars $c$ and $d$.

1. $\mathbf{u} + \mathbf{v}$ is in $V$.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in $V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in $V$, there is vector $-\mathbf{u}$ in $V$ satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in $V$.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

Vector Space Examples

**EXAMPLE:** Let $M_{2\times2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \text{ are real} \right\}$

In this context, note that the $\mathbf{0}$ vector is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. 
EXAMPLE: Let \( n \geq 0 \) be an integer and let

\[
P_n = \text{the set of all polynomials of degree at most } n \geq 0.
\]

Members of \( P_n \) have the form

\[
p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n
\]

where \( a_0, a_1, \ldots, a_n \) are real numbers and \( t \) is a real variable. The set \( P_n \) is a vector space.

We will just verify 3 out of the 10 axioms here.

Let \( p(t) = a_0 + a_1 t + \cdots + a_n t^n \) and \( q(t) = b_0 + b_1 t + \cdots + b_n t^n \). Let \( c \) be a scalar.

Axiom 1:
The polynomial \( p + q \) is defined as follows: \( (p + q)(t) = p(t) + q(t) \). Therefore,

\[
(p + q)(t) = p(t) + q(t) = (\quad ) + (\quad )t + \cdots + (\quad )t^n
\]

which is also a \( \quad \) of degree at most \( \quad \). So \( p + q \) is in \( P_n \).

Axiom 4:

\[
0 = 0 + 0t + \cdots + 0t^n
\]

(zero vector in \( P_n \))

\[
(p + 0)(t) = p(t) + 0 = (a_0 + 0) + (a_1 + 0)t + \cdots + (a_n + 0)t^n = a_0 + a_1 t + \cdots + a_n t^n = p(t)
\]

and so \( p + 0 = p \)

Axiom 6:

\[
(c p)(t) = c p(t) = (\quad ) + (\quad )t + \cdots + (\quad )t^n
\]

which is in \( P_n \).

The other 7 axioms also hold, so \( P_n \) is a vector space.
Subspaces

Vector spaces may be formed from subsets of other vector spaces. These are called *subspaces*.

A **subspace** of a vector space $V$ is a subset $H$ of $V$ that has three properties:

a. The zero vector of $V$ is in $H$.

b. For each $u$ and $v$ are in $H$, $u + v$ is in $H$. (In this case we say $H$ is closed under vector addition.)

c. For each $u$ in $H$ and each scalar $c$, $cu$ is in $H$. (In this case we say $H$ is closed under scalar multiplication.)

*If the subset $H$ satisfies these three properties, then $H$ itself is a vector space.*

**EXAMPLE:** Let $H =$ \[
\begin{bmatrix}
  a \\
  0 \\
  b
\end{bmatrix}
\] : $a$ and $b$ are real. Show that $H$ is a subspace of $\mathbb{R}^3$.

**Solution:** Verify properties a, b and c of the definition of a subspace.

a. The zero vector of $\mathbb{R}^3$ is in $H$ (let $a = \underline{\phantom{a}}$ and $b = \underline{\phantom{b}}$).

b. Adding two vectors in $H$ always produces another vector whose second entry is $\underline{\phantom{0}}$ and therefore the sum of two vectors in $H$ is also in $H$. ($H$ is closed under addition)

c. Multiplying a vector in $H$ by a scalar produces another vector in $H$ ($H$ is closed under scalar multiplication).

Since properties a, b, and c hold, $V$ is a subspace of $\mathbb{R}^3$. **Note:** Vectors $(a,0,b)$ in $H$ look and act like the points $(a,b)$ in $\mathbb{R}^2$. 
Example: Is \( H = \left\{ \begin{bmatrix} x \\ x + 1 \end{bmatrix} : x \text{ is real} \right\} \) a subspace of _______?

I.e., does \( H \) satisfy properties a, b and c?

\[ \begin{array}{c}
\text{Graphical Depiction of } H \\
\end{array} \]

Solution:

All three properties must hold in order for \( H \) to be a subspace of \( \mathbb{R}^2 \).

Property (a) is not true because

\[ \text{______________________________} \]

Therefore \( H \) is not a subspace of \( \mathbb{R}^2 \).

Another way to show that \( H \) is not a subspace of \( \mathbb{R}^2 \):

Let 

\[ u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{, then } u + v = \begin{bmatrix} \text{________} \\ \text{________} \end{bmatrix} \]

and so 

\[ u + v = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{, which is } \text{____ in } H. \text{ So property (b) fails and so } H \text{ is not a subspace of } \mathbb{R}^2. \]
A Shortcut for Determining Subspaces

THEOREM 1

If \( \mathbf{v}_1, \ldots, \mathbf{v}_p \) are in a vector space \( V \), then \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) is a subspace of \( V \).

Proof: In order to verify this, check properties a, b and c of definition of a subspace.

a. \( \mathbf{0} \) is in \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) since

\[
\mathbf{0} = \ldots + \mathbf{v}_p
\]

b. To show that \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) closed under vector addition, we choose two arbitrary vectors in \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \):

\[
\mathbf{u} = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_p \mathbf{v}_p
\]

and

\[
\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p.
\]

Then

\[
\mathbf{u} + \mathbf{v} = (a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \cdots + a_p \mathbf{v}_p) + (b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p)
\]

\[
= (\ldots + \mathbf{v}_1) + (\ldots + \mathbf{v}_2) + \cdots + (\ldots + \mathbf{v}_p)
\]

\[
= (a_1 + b_1) \mathbf{v}_1 + (a_2 + b_2) \mathbf{v}_2 + \cdots + (a_p + b_p) \mathbf{v}_p.
\]

So \( \mathbf{u} + \mathbf{v} \) is in \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \).

c. To show that \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) closed under scalar multiplication, choose an arbitrary number \( c \) and an arbitrary vector in \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \):

\[
\mathbf{v} = b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p.
\]

Then

\[
c \mathbf{v} = c (b_1 \mathbf{v}_1 + b_2 \mathbf{v}_2 + \cdots + b_p \mathbf{v}_p)
\]

\[
= \ldots + \mathbf{v}_p
\]

So \( c \mathbf{v} \) is in \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \).

Since properties a, b and c hold, \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_p \} \) is a subspace of \( V \).
Recap

1. To show that \( H \) is a subspace of a vector space, use Theorem 1.
2. To show that a set is not a subspace of a vector space, provide a specific example showing that at least one of the axioms a, b or c (from the definition of a subspace) is violated.

**EXAMPLE:** Is \( V = \{(a + 2b, 2a - 3b) : a \text{ and } b \text{ are real}\} \) a subspace of \( \mathbb{R}^2 \)? Why or why not?

*Solution:* Write vectors in \( V \) in column form:

\[
\begin{bmatrix}
a + 2b \\
2a - 3b
\end{bmatrix} = \begin{bmatrix} a \\
2a
\end{bmatrix} + \begin{bmatrix} 2b \\
-3b
\end{bmatrix} = \begin{bmatrix} 1 \\
2
\end{bmatrix} + \begin{bmatrix} 2 \\
-3
\end{bmatrix}
\]

So \( V = \text{Span}\{v_1, v_2\} \) and therefore \( V \) is a subspace of _____ by Theorem 1.

**EXAMPLE:** Is \( H = \left\{ \begin{bmatrix} a + 2b \\ a + 1 \\ a \end{bmatrix} : a \text{ and } b \text{ are real} \right\} \) a subspace of \( \mathbb{R}^3 \)? Why or why not?

*Solution:* \( 0 \) is not in \( H \) since \( a = b = 0 \) or any other combination of values for \( a \) and \( b \) does not produce the zero vector. So property _____ fails to hold and therefore \( H \) is not a subspace of \( \mathbb{R}^3 \).

**EXAMPLE:** Is the set \( H \) of all matrices of the form \( \begin{bmatrix} 2a & b \\ 3a + b & 3b \end{bmatrix} \) a subspace of \( M_{2\times2} \)? Explain.

*Solution:* Since

\[
\begin{bmatrix}
2a & b \\
3a + b & 3b
\end{bmatrix} = \begin{bmatrix} 2a \\
3a
\end{bmatrix} + \begin{bmatrix} 0 \\
b
\end{bmatrix}
\]

Therefore \( H = \text{Span}\left\{\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}\right\} \) and so \( H \) is a subspace of \( M_{2\times2} \).