The Concise Handbook of Algebra, Kluwer Publ. (2002), 505-508

# H.6 Homological Characterization of Rings: The Commutative Case

Sarah Glaz
glaz@uconnvm.uconn.edu

A large number of finiteness properties of commutative rings have homological characterizations. For example, it is well known that for a ring to be Noetherian – a condition most commonly described by the finite generation of the ideals of the ring, it is necessary and sufficient that arbitrary direct sums of injective modules be injective modules. One might speculate that this is the reason why homological algebra approaches in Noetherian settings yield such deep and beautiful results.

The same phenomena can be observed in another large class of rings, the class of coherent rings. Chase (1960) attempted to answer the homological question: for what rings arbitrary direct products of flat modules are flat modules. The answer is that this holds true precisely when the ring is coherent. Chase provides no less then seven equivalent characterizations of this homological condition. The most well known are the two equivalent finiteness conditions below:

A ring $R$ is called *a coherent ring* if every finitely generated ideal of $R$ is finitely presented.

Equivalently, $R$ is a coherent ring if and only if for every element $a$ of $R$ and any two finitely generated ideals $I$ and $J$ of $R$, the ideals $I \cap J$ and $(0 : a) = \{r \in R/\ ra = 0\}$ are finitely generated.

Examples of coherent rings include all Noetherian rings, as well as many non-Noetherian rings, among them $k[x_1, x_2, ...]$ – the polynomial ring in infinitely many variables over a field $k$.

A number of other rings characterized by homological conditions were introduced or further developed in an attempt to answer the question: over which coherent rings are the polynomial rings in finitely many variables coherent rings. In contrast to the Noetherian situation, the coherent condition is not always inherited by the polynomial rings. We will follow here the thread of this investigation. A more detailed account of many of the results described below and other results involving coherent and related rings can be found in the books: Vasconcelos (1976), Glaz (1989), Chapman and Glaz (2000), and in the article Glaz (1992).

A ring $R$ is called *a regular ring* if every finitely generated ideal of $R$ has finite projective dimension.

This definition coincides with the usual definition of regularity when the ring is Noetherian. Coherent rings of finite weak global dimensions are regular coherent rings, although, contrary to the Noetherian case, the converse is not necessarily true for quasi-local rings. An example of a quasi-local regular coherent ring of infinite weak global dimension is $k[[x_1, x_2, ...]]$ – the power series ring in infinitely many variables over a field $k$. The class of coherent rings of finite weak global dimension includes many of the classical rings defined by homological conditions, such as Von Neumann regular rings, semihereditary rings, and hereditary rings.

A ring $R$ is called *a Von Neumann regular ring* if for every element $a \in R$ there is an element $b \in R$ such that $a^2b = a$. Von Neumann regular rings have a simple homological description. $R$ is a Von Neumann regular ring if and only if the weak global dimension of $R$ is zero; equivalently, every $R$ module is flat. Von Neumann regular rings are coherent. The class of Von Neumann regular rings includes the very applicable class of the Boolean rings. A Von Neumann integral domain is a field.

A ring $R$ is called *a semihereditary ring* if every finitely generated ideal of $R$ is projective. Semihereditary rings are precisely those rings that are coherent and of weak global dimension one. A semihereditary integral domain is a Prüfer domain. The class of semihereditary rings includes valuation domains and Bezout domains.

A ring $R$ is called *a hereditary ring* if every ideal of $R$ is projective, that is, $R$ is a ring of global dimension one. Hereditary rings are coherent. A hereditary integral domain is a Dedekind domain.

Any coherent ring of finite weak global dimension described above, as well as any coherent ring of global dimension two, satisfies that the polynomial rings in finitely many variables over it is coherent. But there are examples of coherent integral domains $R$ of weak global dimension two over which the polynomial ring in one variable is not a coherent ring. An example of such a ring $R$, constructed by Soublin and Alfonsi, is a localization at a prime ideal of the ring $S = \prod Q[[t, u]]$ – the direct product of countably many power series rings in two variables $t$ and $u$ over the rationals $Q$.

At this point we are compelled to ask to what extent the coherence or regular coherence of the base ring is reflected in coherent-like or regular-like conditions of the polynomial ring over it. The investigation in this direction is of much more recent vintage. The answers involve the definition and development of more rings characterized by homological conditions.

A ring $R$ is called *a PP ring (Principal Projective ring)* if every principal ideal of $R$ is projective.

This homological condition is actually a zero-divisor controlling condition on a ring $R$. In particular, a PP ring $R$ is locally an integral domain, it satisfies that $MinR$ – the space of minimal prime ideals of $R$, is compact in the Zariski topology, and $Q(R)$ – the total ring of fractions of $R$, is a Von Neumann regular ring.

Every regular coherent ring is a PP ring. A regular coherent ring is also locally a GCD (Greatest Common Divisor) domain, that is an integral domain in which every two elements possess a greatest common divisor. A class of rings that generalizes GCD domains is the class of G-GCD rings defined in Glaz (2001). It has the following homological characterization:

A ring $R$ is called *a G-GCD ring (Generalized GCD ring)* if $R$ is a PP ring and the intersection of any two finitely generated flat ideals of $R$ is a finitely generated flat ideal of $R$.

The class of G-GCD rings includes UFDs (Unique Factorization Domains), GCD domains and G-GCD domains – that is integral domains where finite intersections of principal ideals are invertible ideals. Regular coherent rings are G-GCD rings as well. But there are examples of coherent integral domains which are not G-GCD rings, such as $k[x^2, x^3, y, xy]$, where $k$ is a field and $x$ and $y$ are variables over $k$. And there are examples of G-GCD rings which are not coherent, such as the polynomial ring in one variable over the Soublin-Alfonsi ring $R$ mentioned above.

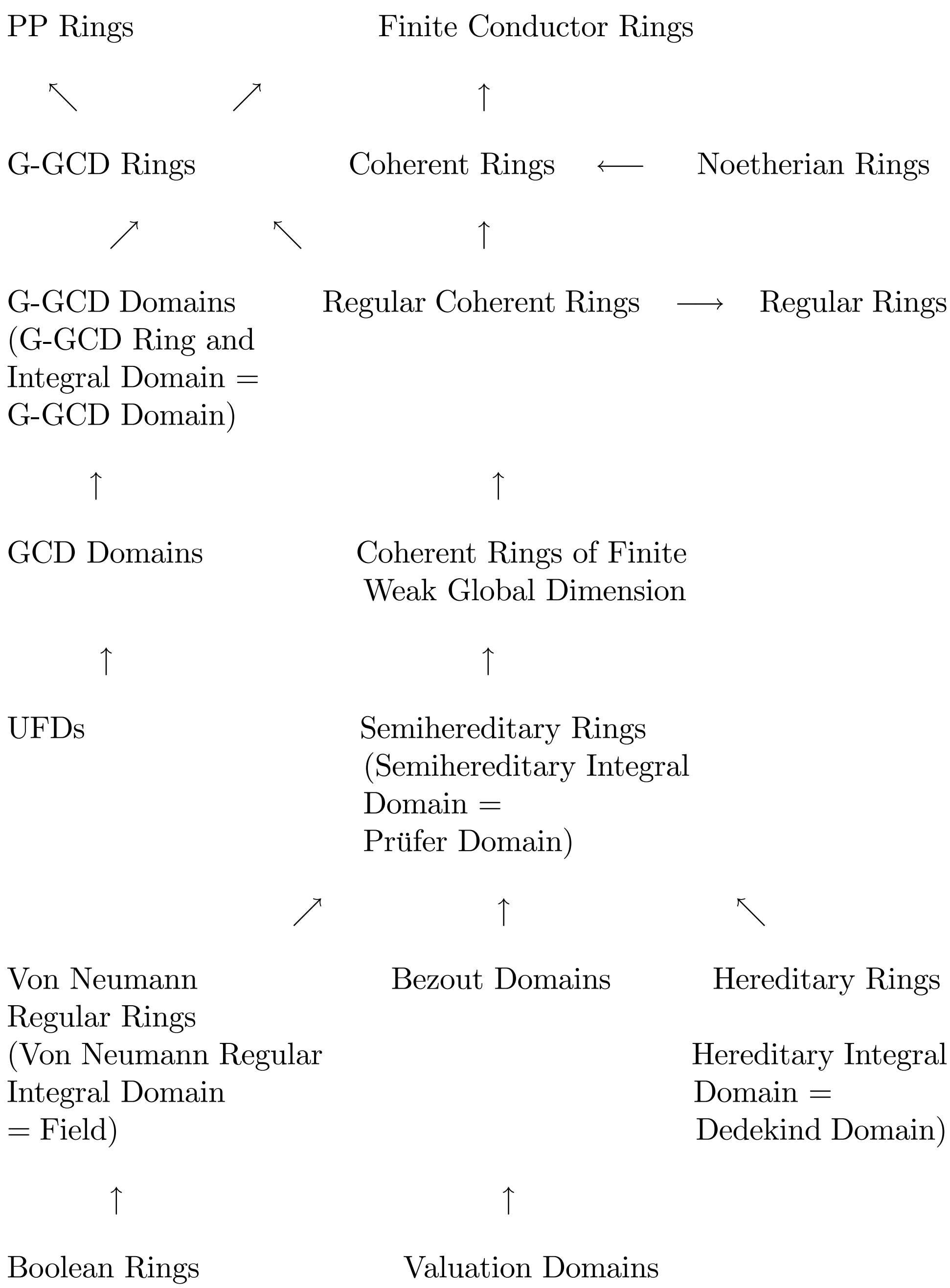
A ring $R$ is called *a finite conductor ring* if for any two elements $a$ and $b$ of $R$ the ideals $(a : b) = \{r \in R/rb \in aR\}$ are finitely generated.

Thus, the class of G-GCD rings and the class of coherent rings include the class of regular coherent rings and are included in the class of finite conductor rings. Moreover, a regular coherent ring and a G-GCD ring have in common the properties of being locally GCD domains and of possessing Von Neumann regular total rings of fractions.

Glaz (2001) has shown that if $R$ is a regular coherent ring, then $R[x]$ – the polynomial ring in one variable over $R$, is a G-GCD ring.

If $R$ is a G-GCD ring and an integral domain then the polynomial rings in finitely many variables over $R$ are G-GCD rings. It is not yet known if this result holds true for G-GCD rings which are neither regular coherent rings nor integral domains.

The diagram below provides an overview of the various containment relationships between the rings mentioned in this section. An arrow signifies containment of the class of rings at the start of the arrow into the class of rings to which the arrow points.



## References


1. S.T. Chapman and S. Glaz (Editors), *Non-Noetherian Commutative Ring Theory*, Math. and Its Appl. Vol. 520, Kluwer Academic Publishers (2000).

2. S.U. Chase, Direct Product of Modules, Trans. AMS 97 (1960), 457-473.

3. S. Glaz, *Commutative Coherent Rings*, Lecture Notes in Math. Vol. 1371, Springer-Verlag (1989).

4. S. Glaz, Commutative Coherent Rings: Historical Perspective and Current Developments, Nieuw Archief. voor Wiskunde 10 (1992), 37-56.

5. S. Glaz, Finite Conductor Rings, to appear in Proc. AMS (2001).

6. W.V. Vasconcelos, *The Rings of Dimension Two*, Lecture Notes in Pure and Appl. Math. Vol. 22, Marcel Dekker (1976)



Sarah Glaz, Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269, USA