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Commutative group rings with von Neumann regular total rings of quotients

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ABSTRACT

Let R be a commutative ring and let G be an abelian group. We show that if G is either torsion free or R is uniquely divisible by the order of every element of G, then the von Neumann regularity of the total ring of quotients of R ascends to the total ring of quotients of RG. Examples are given to show that the converse does not hold. These results are applied in the group ring setting to explore a number of zero divisor controlling conditions, such as being a PF or a PP ring as well as a number of Prüfer conditions, such as being an arithmetical, a Gaussian, or a Prüfer ring.

1. Introduction

Let R be a commutative ring, let G be an abelian group, and denote by RG the group ring of G over R. In this article we explore a number of zero divisor and Prüfer conditions in the group ring setting.

In 1940, Higman [14] characterized when RG is a domain. A classical result in [16] addresses the case when RG is reduced. Subsequent efforts to control zero divisors of commutative group rings have shifted towards homological conditions that were recently linked to Prüfer conditions. Two such notions are the PP (Principal Projective) and the PF (Principal Flat) properties of rings. Both PP and

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PF rings are locally domains [9,20]. These conditions were introduced in the 1960s by Hattori [13] and Endo [5] primarily to develop a torsion theory for modules. They were extensively studied in the 1970s in conjunction with another zero divisor controlling condition, namely the property of having von Neumann regular total ring of quotients [6,15,20,23,25]. More recently, the PP, PF, and von Neumann regular total ring of quotients conditions have been used to explore the extensions of the notion of a Prüfer domain to rings with zero divisors [2,10–12,19]. These extensions of the Prüfer domain notion include semihereditary rings, rings of weak global dimension one, arithmetical rings, Gaussian rings, locally Prüfer rings, and Prüfer rings. The PP and PF conditions have also been investigated recently in other contexts, e.g. [1,26,17]. In particular [22,26] began to address these conditions in the group ring setting.

In Section 2 we determine conditions under which the von Neumann regularity of the total ring of quotients of R ascends to the total ring of quotients of RG. This happens when G is either torsion free or R is uniquely divisible by the order of every element of G (Theorem 2.3). We also provide examples (Example 2.4) to show that descent of this property from RG to R does not necessarily hold under either condition.

Section 3 explores the PP and PF conditions, while Section 4 explores the six Prüfer conditions mentioned above in the group ring setting. Many of the results and examples in both of these two sections make use of the ascent of von Neumann regularity of the total ring of quotients results obtained in Section 2.

In Section 3 we show that if G is torsion free, then RG is a PF (respectively, PP) ring if and only if R is a PF (respectively, a PP) ring (Theorem 3.3). In general, if RG is a PF (respectively, PP) ring then either G is torsion free or R is uniquely divisible by the order of every element of G (Proposition 3.5). But, in general, neither the PF nor the PP conditions ascend from R to RG (Example 3.6).

In Section 4 we show that if G is a torsion free or a mixed abelian group, then the six Prüfer conditions are equivalent to each other and also equivalent to R being von Neumann regular and $\operatorname{rank} G = 1$ (Theorem 4.3(i)). If G is torsion and R is uniquely divisible by the order of every element of G, the equivalence of the six Prüfer conditions, and analogous statements on R and G, require the additional assumption of von Neumann regularity of the total ring of quotients of R (Theorem 4.3(ii)). An example is provided (Example 4.4) that shows that this assumption is necessary.

Throughout the paper R will always denote a commutative ring with identity, G will denote an abelian group written multiplicatively, and $Q(R)$ will denote the total ring of quotients of R. Also $\mathbb{Q}$ will denote the rational numbers, and $\mathbb{C}$ will denote the complex numbers.

2. Von Neumann regularity of the total ring of quotients

In this section we explore ascent and descent of the von Neumann regularity condition between the total ring of quotients of R, $Q(R)$, and the total ring of quotients of RG, $Q(RG)$. The following condition, which appears often in investigations involving homological properties of group rings, links R and G via a divisibility property.

Definition 2.1. Let R be a commutative ring, and let G be an abelian group. R is said to be uniquely divisible by the order of every element of G if for every g in G of finite order n, n divides every element $r \in R$, and if for $r \in R$, we have $r = ns = nt$ for some $t, s \in R$, then $s = t$.

This condition is equivalent to asking that every prime number p, which is the order of an element g in G, be a unit in R. For $x = \sum x_g g \in RG$, let $\operatorname{supp} x = \{g \in G\colon\ x_g \neq 0\}$.

Lemma 2.2. *Let G be an abelian group. Write $G = \varinjlim G_i$ where G_i are the finitely generated subgroups of G ordered by inclusion. Then $Q(RG) = \varinjlim Q(RG_i)$ and $\text{w.gl.dim}\, Q(RG) \leqslant \sup\{\text{w.gl.dim}\, Q(RG_i)\}$.*

Proof. Consider $f/g \in Q(RG_i)$. $RG_i \subseteq RG_j$, where $i < j$, is a free extension and g is not a zero divisor in RG_i. It follows that g is not a zero divisor in RG_j. Therefore $Q(RG_i) \subseteq Q(RG_j)$ and $\varinjlim Q(RG_i) = \bigcup Q(RG_i)$ exists. Since $RG_i \subset RG$ is a flat extention, g is also not a zero divisor in RG. This yields $\varinjlim Q(RG_i) \subseteq Q(RG)$. Conversely, let $f/g \in Q(RG)$, and let G_i be the subgroup of G generated by $\operatorname{supp} f$ and $\operatorname{supp} g$. Then clearly $f/g \in Q(RG_i)$ and therefore belongs to $\varinjlim Q(RG_i)$, establishing $\varinjlim Q(RG_i) = Q(RG)$. Now, $\operatorname{Tor}^n_{Q(RG)}(N, M) = \varinjlim \operatorname{Tor}^n_{Q(RG_i)}(N, M)$ for any $Q(RG)$-modules M and N. It follows that $\text{w.gl.dim}\, Q(RG) \leqslant \sup\{\text{w.gl.dim}\, Q(RG_i)\}$. $\square$

Theorem 2.3. *Let R be a ring and let G be an abelian group which is either torsion free or R is uniquely divisible by the order of every element of G. If $Q(R)$ is von Neumann regular, then $Q(RG)$ is von Neumann regular.*

Proof. By Lemma 2.2, we may assume that G is finitely generated. We first consider the case where G is torsion free. If G is generated by n elements we can write $RG \cong R[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ [16, Proposition 2.2.6] and so

$$Q(R)G = Q(R)\bigl[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\bigr] = S^{-1}\bigl(Q(R)[x_1, \ldots, x_n]\bigr)$$

where S is the multiplicatively closed set consisting of all products of powers of the x_i. As $Q(R)$ is von Neumann regular, the polynomial ring $Q(R)[x_1, \ldots, x_n]$ is coherent [9, Theorem 7.3.1], therefore $Q(R)G$, as a localization of a coherent ring, is a coherent ring. By the syzygy theorem, $\text{w.gl.dim}\, Q(R)[x_1, \ldots, x_n] = \text{w.gl.dim}\, Q(R) + n$, hence $\text{w.gl.dim}\, Q(R)[x_1, \ldots, x_n] = n$. It follows that $\text{w.gl.dim}\, Q(R)G \leqslant n$, and therefore $Q(R)G$ is a coherent regular ring. Since coherent regular rings have von Neumann regular total rings of quotients [9, Corollary 6.2.4], we have that $Q(Q(R)G) = Q(RG)$ is von Neumann regular, completing the case where G is torsion free.

In general, write $G = F \times G'$ where F is a torsion free group and G' is the finite torsion part of G. Then $RG = (RF)G'$ [16, Proposition 2.2.16]. Since $Q(RF)$ is von Neumann regular, we may reduce to the case where G is a finite torsion group. As RG is flat over R, $Q(RG) = Q(Q(R)G)$. Since R is uniquely divisible by the order of every element of G and G is torsion, it follows that $Q(R)$ is also uniquely divisible by the order of every element of G. This, combined with the hypothesis that $Q(R)$ is von Neumann regular is equivalent to $Q(R)G$ being von Neumann regular [8]. In particular, $Q(R)G$ is a total ring of quotients hence $Q(R)G = Q(Q(R)G)$ is the von Neumann regular total ring of quotients of RG. $\square$

The converse of Theorem 2.3 does not hold.

Example 2.4 *(A torsion free or mixed abelian group G such that the group ring RG has $Q(RG)$ von Neumann regular, but $Q(R)$ is not von Neumann regular).* Let K be a countable algebraically closed field. Quentel [23,24] (see [9] for details) constructed a K-algebra R containing K that satisfies the following conditions:

1. R is reduced.
2. $R = Q(R)$.
3. $\operatorname{Min} R$ is compact.
4. R is not a von Neumann regular ring.

Let G be an infinite cyclic group. Then $RG \cong R[x, x^{-1}]$, where x is an indeterminate over R. It follows that $Q(RG) = Q(R[x, x^{-1}]) = Q(R[x])$. Since R is reduced, $Q(R[x])$ being von Neumann regular is equivalent to $\operatorname{Min} R$ being compact [23]. Thus $Q(RG)$ is von Neumann regular, although $R = Q(R)$ is not. To obtain an example where the group G is mixed we require K to contain $\mathbb{Q}$, so that every prime number is a unit in R. Now let $G = G_1 \oplus G_2$, where G_1 is an infinite cyclic group and G_2 is a torsion group. Then $RG = (RG_1)G_2$, and $Q(RG_1)$ is von Neumann regular. Since G_2 is torsion and R is uniquely divisible by the order of every element of G we obtain that $Q(RG_1)G_2$ is a von Neumann regular ring [8] and therefore equal to its own total ring of quotients. It follows that $Q(RG) = Q(R(G_1)G_2) = Q(Q(RG_1)G_2) = Q(RG_1)G_2$ is von Neumann regular.

3. The PF and PP conditions

In this section we explore the ascent and descent of the PF and PP conditions between R and RG.

Definition 3.1. A commutative ring R is called a PP ring, or a weak Baer ring, if every principal ideal of R is projective. R is called a PF ring if every principal ideal of R is flat.

The first step is to consider the case where G is a torsion free group. For this purpose we start by proving the following lemma:

Lemma 3.2. *Let $R \to S$ be a ring homomorphism making S a faithfully flat R-module.*

(i) *If S is a PF ring, then R is a PF ring.*
(ii) *If S is a PP ring, then R is a PP ring.*

Proof. (i) Let $a \in R$. We aim to show that aR is a flat ideal of R. Since S is a flat R-module we have $aS \cong aR \otimes_R S$. Since S is a PF ring, aS is a flat S-module and by [21, Exercise 7.1] aR is a flat R-module.

(ii) Let $a \in R$. We aim to show that aR is a projective ideal of R. Since S is a PP ring, by (i), aR is flat. It therefore suffices to show that aR is finitely presented, that is, $(0 :_R a)$ is a finitely generated ideal of R. Since S is a flat R-module we have $(0 :_R a)S \cong (0 :_R a) \otimes S \cong (0 :_S a)$. S is a PP ring and therefore aS is projective, implying that $(0 :_S a)$ is a finitely generated ideal of S. The faithful flatness of S over R implies that $(0 :_R a)$ is a finitely generated ideal of R [21, Exercise 7.3]. □

The next result pertains to the ascent and descent of the PF and PP properties between R and RG in the case where G is torsion free. Theorem 3.3(ii) makes use of Theorem 2.3, but, contrary to the situation for the von Neumann regularity of the total ring of quotients, both the PF and the PP properties ascend as well as descend between R and RG.

Theorem 3.3. *Let R be a commutative ring and let G be a torsion free abelian group. Then*:

(i) *RG is a PF ring if and only if R is a PF ring.*
(ii) *RG is a PP ring if and only if R is a PP ring.*

Proof. (i) Lemma 3.2(i) yields the forward implication since RG is faithfully flat (actually free) over R. Conversely, assume R is a PF ring. Then R is locally a domain [9, Theorem 4.2.2]. Let $p \in \operatorname{Spec} R$. Since R_p is a domain and G is torsion free, it follows that R_pG is a domain [16, Proposition 2.2.21]. Let $P \in \operatorname{Spec} RG$. Then $p = P \cap R \in \operatorname{Spec} R$, and $(RG)_P = (R_pG)_{PR_pG}$. Consequently, $(RG)_P$ is a domain for all $P \in \operatorname{Spec} RG$ and therefore RG is a PF ring.

(ii) The forward implication follows from Lemma 3.2(ii). Conversely, assume that R is a PP ring. Then R is a PF ring and $Q(R)$ is von Neumann regular [9, Theorem 4.2.10]. Theorem 2.3 yields $Q(RG)$ von Neumann regular, while the first part of this theorem implies that RG is a PF ring. This implies that RG is a PP ring [9, Theorem 4.2.10]. □

It is worth noting the following reformulation of Theorem 3.3(i):

Corollary 3.4. *Let R be a commutative ring and let G be a torsion free abelian group. Then RG is locally a domain if and only if R is locally a domain.*

Extending the ascent and descent results for the PF and PP conditions from the torsion free setting to a more general group setting can be partially achieved using the unique divisibility condition.

Proposition 3.5. *Let R be a commutative ring and let G be an abelian group. Assume that RG is a PF ring (respectively, PP ring). Then R is a PF ring (respectively, PP ring), and either G is torsion free or R is uniquely divisible by the order of every element of G.*

Proof. Assume that RG is a PF ring. Then by Lemma 3.2(i), R is a PF ring. To show the second claim, assume that G is not torsion free and let $H = \langle g \rangle$ be a cyclic subgroup of order p of the group G. Then RG is a free extension of RH and we may employ Lemma 3.2(i) to conclude that RH is a PF ring. We may therefore assume that G is a cyclic group generated by g, which has prime order p. Consider the short exact sequence

$$0 \to I(G) \to RG \to R \to 0 \tag{$\star$}$$

where $I(G)$ is the augmentation ideal associated with the augmentation map $\sum x_g g \to \sum x_g$. Then $I(G) = (g-1)RG$. Since RG is a PF ring, $I(G)$ is a flat ideal of RG, and the exact sequence $(\star)$ implies that $\mathrm{w.dim}_{RG} R \leqslant 1$. This means that $\mathrm{Tor}^k_{RG}(R, N) = 0$ for all RG-modules N and for all $k \geqslant 2$. Choosing $N = R$ and setting $k = 2n$ for $n \geqslant 1$, we have $\mathrm{Tor}^{2n}_{RG}(R, R) = 0$. By [9, Theorem 8.2.2], $\mathrm{Tor}^{2n}_{RG}(R, R) \cong R/\mathrm{p}R$. It follows that $R/\mathrm{p}R = 0$ and hence $R = \mathrm{p}R$, which implies that R is uniquely divisible by p.

The proof of the proposition for the PP condition is similar. □

The converse of the previous proposition does not hold for either the PF or the PP condition.

Example 3.6 *(An abelian group G and a ring R which is PP and is uniquely divisible by the order of every element of G, but whose group ring RG is neither a PP ring nor a PF ring).* Zan and Chen [26, Example 2.5] constructed a subring, R, of the complex numbers $\mathbb{C}$ in which 2 is a unit. Being a domain, R is a PP ring and a PF ring. It is shown in [26] that RG, where G is a cyclic group of order 4, is not a PP ring. We show that R is not a PF ring either. Since R is a PP ring, $Q(R)$ is von Neumann regular. It follows from Theorem 2.3 that $Q(RG)$ is also von Neumann regular. If RG is a PF ring, then RG is a PP ring [9, Theorem 4.2.10].

4. Prüfer conditions

In this section we consider the following six extensions of the Prüfer domain notion to rings with zero divisors:

(1) R is a semihereditary ring.
(2) $\mathrm{w.gl.dim}\, R \leqslant 1$.
(3) R is an arithmetical ring.
(4) R is a Gaussian ring.
(5) R is a locally Prüfer ring.
(6) R is a Prüfer ring.

These six Prüfer conditions have been extensively studied for the last 5 to 7 years. For a comprehensive survey and an extensive list of references on the subject see [12]. In particular, the classes of rings described above are contained in each other in the order of their numbering, and the containments are strict [10]. Glaz [8] and Bazzoni and Glaz [2] found conditions that allow for reversals of containments for properties (1)–(4) and (6), while Boynton [3] covered the same ground for property (5). In particular, the three zero divisor controlling conditions described in Sections 2 and 3 allow several reversals of containment, and if $Q(R)$ is a von Neumann regular ring, then conditions (1)–(6) are equivalent for the ring R.

Lemma 4.1. *Let $R \to S$ be a ring homomorphism making S a faithfully flat R-module. If S is a Prüfer ring, then so is R.*

Proof. Let I be a finitely generated regular ideal of R. We aim to show that I is invertible. Since I contains a nonzero divisor, IS is a regular ideal of S. Also, S is Prüfer and therefore IS is invertible and hence a projective ideal of S. It follows that IS_P is flat for all $P \in \mathrm{Spec}\, S$. Let $p \in \mathrm{Spec}\, R$. Since S is faithfully flat over R there is a prime $P \in \mathrm{Spec}\, S$ such that $P \cap R = p$ and S_P is faithfully flat over R_p.

It follows that IR_p is a flat ideal of R_p. Thus $IR_p \neq 0$ is projective [9, Lemma 4.2.1] and therefore free. We conclude that IR_p is principal, and I is invertible, as desired. □

Before exploring the relations among these Prüfer conditions in the group ring setting we record the following characterizations of group rings RG with $\text{w.gl.dim}\, RG = 1$.

Proposition 4.2. *Let R be a commutative ring and let G be an abelian group. Then RG is a semihereditary ring (respectively, $\text{w.gl.dim}\, RG = 1$) if and only if exactly one of the following two conditions hold:*

(i) *R is a semihereditary ring (respectively, $\text{w.gl.dim}\, R = 1$), G is a torsion group, and R is uniquely divisible by the order of every element of G.*
(ii) *R is a von Neumann regular ring, $\text{rank}\, G = 1$, and R is uniquely divisible by the order of every element of G.*

Proof. The semihereditary case is proved in [8]. For the case where $\text{w.gl.dim}\, RG = 1$, we note that in [4] it is proved that $\text{w.gl.dim}\, RG < \infty$ if and only if R is uniquely divisible by the order of every element of G and both $\text{rank}\, G$ and $\text{w.gl.dim}\, R$ are finite. Moreover, when the unique divisibility condition holds, $\text{w.gl.dim}\, RG = \text{w.gl.dim}\, R + \text{rank}\, G$. The conclusion now follows. □

The next theorem characterizes the conditions under which the group ring RG satisfies any of the Prüfer conditions (1)–(6), provided that R is uniquely divisible by the order of every element of G.

Theorem 4.3. *Let R be a commutative ring and let G be an abelian group. If G is not torsion free, assume that R is uniquely divisible by the order of every element of G. Then:*

(i) *If G is either a torsion free or a mixed group, each one of the Prüfer conditions (1)–(6) is equivalent to: R is a von Neumann regular ring and $\text{rank}\, G = 1$.*
(ii) *If G is a torsion group and, in addition, $Q(R)$ is von Neumann regular, each one of the Prüfer conditions (1)–(6) is equivalent to: R is a semihereditary ring.*

Proof. The implications of the Prüfer conditions $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6)$ hold for any ring [10]. To prove (i) we first note that by Proposition 4.2 we only need to show that if RG is a Prüfer ring then R is a von Neumann regular ring and $\text{rank}\, G = 1$. We first note that it suffices to show that if RG is a Prüfer ring, then R is a von Neumann regular ring. For if this is indeed true, Theorem 2.3 will imply that $Q(RG)$ is a von Neumann regular ring. According to [2] properties (1)–(6) are equivalent for a ring with von Neumann regular total ring of quotients. Therefore using Proposition 4.2 we can conclude that $\text{rank}\, G = 1$. Now let $H = \langle g \rangle$ be an infinite cyclic subgroup of G. Then RG is free over $RH \cong R[x, x^{-1}]$, for an indeterminate x, and this ring, in turn, is free over a polynomial ring in one variable over R [8]. By Lemma 4.1 we can therefore reduce to the case where the polynomial ring $R[x]$ is a Prüfer ring. Let $a \neq 0$ be a non unit element of R. Since $R[x]$ is a Prüfer ring, we have equality of the following two ideals: $(a, x)^2 = (a^2, x^2)$ [18]. It follows that $ax = fa^2 + gx^2$ for some f and g in $R[x]$. Therefore $f = xf'$ for some f' in $R[x]$. Since x is a nonzero divisor, $a = f'a^2 + gx$. If $g \neq 0$, then x divides $(1 - f'a)$, and $(a, x) = R[x]$. We conclude that $g = 0$, and therefore $a \in a^2R$. It follows that R is von Neumann regular as desired.

To prove (ii) we note that by Theorem 2.3 the total ring of quotients of RG is von Neumann regular. It follows that Prüfer conditions (1) through (6) are equivalent for RG [2], and therefore, by Proposition 4.2, each one is also equivalent to: R is a semihereditary ring. □

We remark that if G is torsion free, the equivalence of "R is von Neumann regular and $\text{rank}\, G = 1$" to Prüfer conditions (3) and (6) can also be deduced by different methods from the results proved in [7]. If G is a torsion group, Example 4.4 below shows that the result of Theorem 4.3(ii) need not hold without the additional assumption that $Q(R)$ is von Neumann regular.

Example 4.4 *(A ring R which is not semihereditary and a torsion group G such that R is uniquely divisible by the order of every element of G with* w.gl.dim $RG = 1$*).* Let R be any non coherent ring containing $\mathbb{Q}$ which satisfies w.gl.dim $R = 1$. Such a ring is not semihereditary and, equivalently, does not have a von Neumann regular total ring of quotients [9, Corollary 4.2.19]. Any prime p is invertible in R and thus the unique divisibility condition holds regardless of G. By Proposition 4.2, w.gl.dim $RG = 1$. Such a ring R appears in a number of sources. For example, see the ring T on page 51 of [9], where the field K is chosen to be $\mathbb{Q}$ and for all λ the rings $R_\lambda = \mathbb{Q}[x]_{(x)}$, for an indeterminate x over $\mathbb{Q}$.

At this point the exact conditions under which a group ring RG satisfies any of the Prüfer conditions (3) through (6) for a general group G are not clear. There are several results appearing in the literature, for example [7], pointing out cases when RG is arithmetical. Those seem to be ad hoc and do not generalize. Some examples of group rings satisfying some of these conditions but not others also muddy the waters. We conclude this section with one such example, a locally Prüfer group ring which is not Gaussian.

Example 4.5 *(A ring R of characteristic* 2 *and a group G of order* 4 *such that RG is locally Prüfer but not Gaussian).* Let R be a field of characteristic 2, and let G be a non-cyclic group of order 4. Let g and h generate G. Then $RG \cong R[x,y]/(x^2,y^2)$, where x and y are indeterminates over R (via the isomorphism $g \mapsto x-1$, $h \mapsto y-1$). The group ring RG is local with maximal ideal $(x,y) = m$ satisfying $m^3 = 0$. Thus RG is a total ring of quotients, and as such is locally Prüfer. To see that RG is not Gaussian consider $f = xT + y$ and $g = xT - y$ in $RG[T]$. Now, $fg = 0$ but $c(f)c(g) = m^2 \neq 0$.

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