

**Symmetric Groups:**

1.  $S_X = \{f : X \rightarrow X \mid f \text{ is a bijective map}\}$   
 $S_n = S_{\{1,2,\dots,n\}}$  for  $n \in \mathbb{N}$
2. Cycle notation,  $r$ -cycles, transpositions
3. Disjoint cycles (they commute), factorization of any permutation into a product of disjoint cycles.
4. The order of an  $r$ -cycle, the order of a permutation  
 If  $\sigma = \sigma_1 \cdot \sigma_2 \cdots \sigma_n$  is a product of disjoint cycles,  
 then  $o(\sigma) = \text{lcm}(o(\sigma_1), o(\sigma_2), \dots, o(\sigma_n))$
5. Even and odd permutations
6. The subgroups  $A_n$  and  $D_n$  in  $S_n$
7. Generating sets for  $S_n$  and  $A_n$ 
  - $S_n$  is generated by all transpositions.
  - $S_n$  is generated by  $\{(i, i+1) \mid i = 1, \dots, n-1\}$
  - $S_n$  is generated by  $\{(1, i) \mid i = 2, \dots, n\}$
  - $A_n, n \geq 3$ , is generated by the set of all 3-cycles.
8. In general,  $D_n$  is generated by  $f$  and  $g$  where  
 $f^n = i, g^2 = i, f^k g = g f^{n-k}$   
 and can be described as the set of all rigid motions of a regular  $n$ -gon.
9.  $Z(D_n) = \begin{cases} \{i\} & \text{if } n \text{ is odd} \\ \{i, f^{\frac{n}{2}}\} & \text{if } n \text{ is even} \end{cases} = \begin{cases} \langle i \rangle & \text{if } n \text{ is odd} \\ \langle f^{\frac{n}{2}} \rangle & \text{if } n \text{ is even} \end{cases}$
10. For  $x \in G$ ,  $Z(x) =$  the centralizer of  $x$  in  $G = \{y \in G \mid xy = yx\}$

**Partitions, Equivalence Relations, Cosets & Conjugacy Classes:**

1. Equivalence relations: reflexive, symmetric, transitive
2. Equivalence classes and partitions
3. Two important equivalence relations:

**Cosets**

\* if  $G$  is a group,  $H \leq G$ , then  $a \sim b \iff ab^{-1} \in H$

\* equivalence classes are (right) cosets:

$$\bar{a} = Ha = \{ha \mid h \in H\}$$

$$\text{Now, } a \sim b \iff ab^{-1} \in H \iff Ha = Hb \iff a \in Hb$$

\* Be able to work with examples of cosets

\* Be careful with multiplicative notation versus additive notation

\* One-to-one correspondence between the set of right coset and the set of left cosets

\* The number of right cosets = the number of left cosets

\* All cosets of  $H$  in  $G$  have the same cardinality, namely  $|H|$ .

\*  $[G : H] =$  the index of  $H$  in  $G =$  the number of cosets of  $H$  in  $G$ ,

\* Lagrange's Theorem and it's consequences

$$\cdot |G| < \infty \text{ and } H \leq G \implies |H| \mid |G|$$

$$\cdot |G| < \infty \implies o(x) \mid |G| \quad \forall x \in G$$

$$\cdot |G| < \infty \implies x^{|G|} = e \quad \forall x \in G$$

$$\cdot |G| = p, \text{ with } p \text{ a prime} \implies G \cong \mathbb{Z}_p$$

$$\cdot |G| < \infty \text{ and } H \leq G \implies [G : H] = \frac{|G|}{|H|}$$

### Conjugacy Classes

\* if  $G$  is a group and  $a, b \in G$ , then  $a \sim b \iff b = xax^{-1}$  for some  $x \in G$

\* equivalence classes are conjugacy classes

$$\bar{a} = cl(a) = \{xax^{-1} \mid x \in G\}$$

\* be able to work with examples of conjugacy classes

\* One-to-one correspondence between a conjugacy class and the set of cosets of the centralizer

$cl(a) \longrightarrow$  the set of left cosets of  $Z(a)$

$$xax^{-1} \mapsto xZ(a)$$

$$\text{So } |cl(a)| = [G : Z(a)]$$

\* The Class Equation If  $|G| < \infty$  and  $\{a_1, \dots, a_k\}$  consist of representatives of distinct conjugacy classes containing more than one element, then

$$|G| = |Z(G)| + \sum_i [G : Z(a_i)]$$

\* If  $|G| = p^n \implies |Z(G)| > 1$

### Normal Subgroups, Quotient Groups & Homomorphisms:

$$1. H \triangleleft G \iff \forall h \in H \text{ and } \forall g \in G, \quad ghg^{-1} \in H$$

$$\iff gHg^{-1} = H \quad \forall g \in G$$

$$\iff gH = Hg \quad \forall g \in G$$

$$\iff G/H \text{ is a group under coset multiplication (addition)}$$

$$\iff H = \text{kernel of a group homomorphism}$$

2. Be able to show a subgroup is normal in a variety of ways.

3. Example of some normal subgroups

$$- Z(G) \triangleleft G$$

$$- N \triangleleft G \quad \forall N \leq Z(G)$$

$$- G \text{ abelian} \implies H \triangleleft G \quad \forall H \leq G$$

$$- H \leq G \text{ and } [G : H] = 2 \implies H \triangleleft G, \text{ and so, in particular, } A_n \triangleleft S_n$$

4. Be able to work with examples of quotient groups

5. Some theorems and results about quotient groups

$$- G \text{ abelian, } H \leq G \implies G/H \text{ is abelian}$$

$$- G \text{ cyclic, } H \leq G \implies G/H \text{ is cyclic}$$

6.  $\varphi : G \longrightarrow H$  is a homomorphism of groups  $\iff \varphi(xy) = \varphi(x)\varphi(y) \forall x, y \in G$

7. Monomorphisms, epimorphisms, isomorphisms, automorphisms,  $Aut(G)$

8. Facts about compositions of homomorphisms:

If  $\varphi : G \longrightarrow H$  and  $\chi : H \longrightarrow K$  are group homomorphisms,

\*  $\chi \circ \varphi$  is a homomorphism

\*  $\chi$  and  $\varphi$  are  $\left\{ \begin{array}{c} \text{mono} \\ \text{epi} \\ \text{iso} \end{array} \right\}$  morphisms  $\implies \chi \circ \varphi$  is a(n)  $\left\{ \begin{array}{c} \text{mono} \\ \text{epi} \\ \text{iso} \end{array} \right\}$  morphism.

9. The MEGA Theorem: Let  $\varphi : G \longrightarrow G'$  be a homomorphism of groups

-  $\varphi(e) = e$

-  $\varphi(x^n) = (\varphi(x))^n \quad \forall x \in G, \forall n \in \mathbb{Z}$

-  $o(\varphi(x)) \mid o(x) \quad \text{if } o(x) < \infty \text{ and}$

$o(\varphi(x)) = o(x) \quad \text{if } \varphi \text{ is an isomorphism}$

-  $H \leq G \implies \varphi(H) \leq G'$

-  $K \leq G' \implies \varphi^{-1}(K) \leq G$

-  $K \triangleleft G' \implies \varphi^{-1}(K) \triangleleft G$

-  $H \triangleleft G$  and  $\varphi$  is surjective  $\implies \varphi(H) \triangleleft G'$

10. Cayley's Theorem: Every group is isomorphic to a subgroup of a symmetric group.

More precisely:  $\Lambda : G \longrightarrow S_G; g \mapsto \lambda_g$  where  $\lambda_g(x) = gx \quad \forall x \in G$   
is a monomorphism.

### Homomorphisms and Normal subgroups:

• If  $N \triangleleft G$ , then  $\rho : G \longrightarrow G/N; g \mapsto Ng$  is a group homomorphism, called the canonical homomorphism from  $G$  to  $G/N$ . It is an **onto** homomorphism.

• If  $\varphi : G \longrightarrow H$  is a group homomorphism, then  $\ker(\varphi) = \{x \in G \mid \varphi(x) = e\} \triangleleft G$

•  $N = \ker(\rho)$  where  $\rho$  is the canonical homomorphism from  $G$  to  $G/N$

### • The Fundamental Theorem of Group Homomorphisms

Let  $\varphi : G \longrightarrow H$  be an epimorphism of groups.

Then

1.  $H \cong G/\ker(\varphi)$

2.  $\bar{\varphi} : G/\ker(\varphi) \longrightarrow H; x \cdot \ker(\varphi) \mapsto \varphi(x)$

is a group homomorphism and moreover  $\bar{\varphi} \circ \rho = \varphi$

• Any homomorphism of groups can be factored into a composition of epimorphism followed by an monomorphism.

• If  $\varphi : G \longrightarrow G'$  is an epimorphism. Then there is a 1-1 correspondence between the subgroups of  $G'$  and the subgroups of  $G$  containing  $\ker(\varphi)$ .