

Keep In Mind: Although some of these questions are fairly typical and standard, others are designed to help you prepare for the exam by making you stretch your understanding.

1. Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ are bases of a 3-dimensional vector space V where

$$[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, [\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, [\vec{b}_3]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \text{ Find } P_{\mathcal{C} \leftarrow \mathcal{B}}.$$

2. Let $T : \mathbb{P}_3 \longrightarrow \mathbb{P}_2; \vec{p} \mapsto \vec{p}'$ where \vec{p}' is the derivative of \vec{p} .

- (a) Show that T is a linear transformation. (You can freely use common facts from calculus.)
 (b) List the standard bases of \mathbb{P}_3 and \mathbb{P}_2 respectively.
 (c) Find the corresponding linear transformation $S : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ by using the coordinate vectors relative to the standard bases of \mathbb{P}_3 and \mathbb{P}_2 .

- (d) What does $S(\vec{x})$ equal if $\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 5 \\ 6 \end{bmatrix}$?

3. If $H = \left\{ \begin{bmatrix} a \\ a+b \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$.

- (a) Show that H is a subspace of \mathbb{R}^3 .
 (b) Find a basis for H .
 (c) Find the dimension of H .

4. Let $M = \{\vec{p} \in \mathbb{P}_3 \mid \vec{p}(t) = a + bt^2, \text{ where } a \text{ and } b \text{ are in } \mathbb{R}\}$

- (a) Show that M is a subspace of \mathbb{P}_3
 (b) Find a set of vectors which will span M .
 (c) Find a basis for M .
 (d) Find the dimension of M .
 (e) What is the largest number of linear independent vectors you can have in M ?

5. Let $K = \{\vec{p} \in \mathbb{P}_3 \mid \vec{p}(0) = 0, \vec{p}(1) = 0\}$, so K are all those vectors in \mathbb{P}_3 which vanish as functions at inputs 0 and 1.

- (a) Show that K is a subspace of \mathbb{P}_3 .

- (b) Letting \mathcal{E} be the standard basis of \mathbb{P}_3 , is $\vec{p} \in K$ if $[\vec{p}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$? what about if $[\vec{p}]_{\mathcal{E}} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$?

- (c) Find a set of vectors which will span K .
 (d) Find a basis for K
 (e) Find the dimension of K .

6. Write the standard basis and find the dimension of: $\mathbb{P}_2, \mathbb{P}_3, \mathbb{R}^2, \mathbb{R}^3, M_{2 \times 2}(\mathbb{R}),$.

7. Give examples of nonstandard bases for each of the above vector spaces.
8. Find a basis and dimension of $V = \{A \in M_{2 \times 2}(\mathbb{R}) \mid \text{tr}A = 0\}$, where $\text{tr}A$ is the sum of the diagonal entries.
9. Give examples of two infinite dimensional vector spaces where for one of them, you also give a basis of it.

10. Find all values of h so that $\vec{y} = \begin{bmatrix} 3 \\ 5 \\ h \end{bmatrix}$ will be in the subspace spanned by the three vectors
- $$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}, \text{ and } \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

11. Let $A = \begin{bmatrix} -1 & 3 & 7 & 2 & 0 \\ 1 & -2 & -7 & -1 & 3 \\ 2 & -4 & -9 & -5 & 1 \\ 3 & -6 & -11 & -9 & -1 \end{bmatrix}$.

- (a) Find bases for the $\text{Col}A$, $\text{Nul}A$ and $\text{Row}A$.
- (b) Find $\text{rank}A$ and the dimension of the null space of A .
- (c) If $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4; \vec{x} \mapsto A\vec{x}$, find the dimension of the kernel of T and the range of T .

12. True or False:

- (a) The dimension of a vector space \mathbb{P}_{17} is 17.
- (b) Any line in \mathbb{R}^3 is a one-dimensional subspace of \mathbb{R}^3 .
- (c) If a vector space V has a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_5\}$, then any set in V containing 6 vectors must be linearly dependent.
- (d) If a vector space V has a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_5\}$, then any set in V containing 6 vectors must span all of V .
- (e) In a 6-dimensional vector space, any set of exactly 6 vectors is automatically a basis.
- (f) If the set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_7\}$ span a vector space V , then $\dim V = 7$.
- (g) If H is a subspace of a finite-dimensional vector space V , then $\dim H \leq \dim V$.

13. If the null space of a 7×9 matrix is 3-dimensional, what can you say about $\text{rank}A$? what can you say about $\dim \text{Col}A^T$?

14. If A is a 5×8 matrix, what is the smallest possible dimension of $\text{Nul}A$?