Math 2210Q-004 **Questions from Chapters 5 & 6** **Spring 2009**

For Review Purposes

Keep In Mind: Although some of these questions are fairly typical and standard, most are designed to help you prepare for the exam by making you stretch your understanding.

1. True or False: If true, give a reason as to why; if false, give a reason or an example. Also, if the statement is false, but some slight modification can be made which will make it true, state the modification.

   (a) If $A\vec{x} = \lambda\vec{x}$ for some scalar λ, then $\vec{x}$ is an eigenvector of A.

   (b) The eigenvalues of a matrix are the values on the diagonal.

   (c) An eigenspace of A is a null space of a certain matrix.

   (d) The multiplicity of a root r of the characteristic equation of a matrix A is equal to the dimension of the eigenspace of A associated with the eigenvalue r.

   (e) If matrix B can be obtained from matrix A by applying elementary row operations to A, then A and B have the same eigenvalues.

   (f) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with standard matrix A, then $[T]_{\mathcal{B}}$ is a diagonal matrix if B is a basis of $\mathbb{R}^n$.

   (g) If $\vec{x}$ is orthogonal to $\vec{y}$ in a subspace W, then $\vec{x} \in W^\perp$.

   (h) The column vectors of A are orthogonal to the vectors in $\operatorname{Nul} A$.

   (i) An orthogonal matrix is invertible

   (j) A basis of $\mathbb{R}^n$ is an orthogonal set of n vectors in $\mathbb{R}^n$.

   (k) If $\vec{y} = \vec{z}_1 + \vec{z}_2$, where $\vec{z}_1 \in W$ and $\vec{z}_2 \in W^\perp$, then $\vec{z}_1$ is the orthogonal projection of $\vec{y}$ onto W.

   (l) If W is a 3-dimensional subspace of $\mathbb{R}^8$ and if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set of vectors, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for W.

   (m) If $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is an ordered basis for a subspace W in $\mathbb{R}^n$, then the Gram-Schmidt process produces an ordered orthonormal basis of W whose first vector is $\vec{x}_1$.

2. Explain why an $n \times n$ matrix can have at most n different eigenvalues.

3. Explain why a matrix A is invertible if and only if A has only nonzero eigenvalues.

4. Find the characteristic equation and eigenvalues for the following matrices:

   (a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 10 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 10 \end{bmatrix}^{-1}$

5. For the matrix $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$, find the characteristic equation, eigenvalues, eigenspaces, a basis for each eigenspace, and determine if A is diagonalizable. If so, give a diagonal matrix D similar to A and exhibit the matrix P so that $A = PDP^{-1}$.

6. For the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3; \vec{x} \mapsto A\vec{x}$, where $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, is there an ordered basis $\mathcal{B}$ of $\mathbb{R}^3$ relative to which $[T]_{\mathcal{B}}$ is diagonal? If so, give the ordered basis and the corresponding matrix $[T]_{\mathcal{B}}$; if not, explain why not.

7. For the linear transformation

$$T : \mathbb{P}_2 \longrightarrow \mathbb{P}_2;\ a + bt + ct^2 \mapsto (a + 3b + 3c) - (3a + 5b + 3c)t + (3a + 3b + c)t^2,$$

is there a basis $\mathcal{C}$ of $\mathbb{P}_2$ relative to which $[T]_{\mathcal{C}}$ is diagonal? If so, give the ordered basis and the corresponding matrix $[T]_{\mathcal{C}}$; if not, explain why not. (*Hint*: consider the relationship of this with question 6 above.)

8. Show that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis of $\mathbb{R}^3$ where

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \ v_3 = \begin{bmatrix} 5 \\ -13 \\ 2 \end{bmatrix}.$$

9. Using the orthogonal basis $\mathcal{B}$ as in question 8 above, find an expression for the vector $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in terms of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

10. Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2\}$, where $\vec{v}_1$ and $\vec{v}_2$ are as in question 8 above, and take $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$. Express $\vec{y} = \hat{y} + \vec{z}$ where $\hat{y}$ is in W and $\vec{z}$ is in $W^\perp$.

11. Consider the basis $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ for a subspace W of $\mathbb{R}^4$ where $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Use the Gram-Schmidt method to find an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ for the W so that $\text{Span}\{\vec{u}_1\} = \text{Span}\{\vec{x}_1\}$ and $\text{Span}\{\vec{u}_1, \vec{u}_2\} = \text{Span}\{\vec{x}_1, \vec{x}_2\}$.