Solutions for the Review Questions

1. Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$ are bases of a 3-dimensional vector space V where $[\vec{b}_1]_\mathcal{C} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $[\vec{b}_2]_\mathcal{C} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $[\vec{b}_3]_\mathcal{C} = \begin{bmatrix}0\\1\\1\end{bmatrix}$. Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

   Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the matrix whose columns are the coordinate vectors relative to the basis $\mathcal{C}$ of the basis vectors in $\mathcal{B}$, we have $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix}[\vec{b}_1]_\mathcal{C} & [\vec{b}_2]_\mathcal{C} & [\vec{b}_3]_\mathcal{C}\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix}$.

2. Let $T : \mathbb{P}_3 \longrightarrow \mathbb{P}_2; \vec{p} \mapsto \vec{p}\,'$ where $\vec{p}\,'$ is the derivative of $\vec{p}$.

   (a) Show that T is a linear transformation. (You can freely use common facts from calculus.)

   Since $\frac{d}{dt}(f(t) + g(t)) = f\,'(t) + g\,'(t)$ and $\frac{d}{dt}(r \cdot f(t)) = r \cdot f\,'(t)$ for all differentiable functions f and g and real constants r, it is also true for the collection of polynomial functions (since they are all differentiable functions). Thus, $T(\vec{p} + \vec{q}) = \vec{p}\,' + \vec{q}\,'$ and $T(r \cdot \vec{p}) = r \cdot \vec{p}\,'$. Thus, T is a linear transformation.

   (b) List the standard bases of $\mathbb{P}_3$ and $\mathbb{P}_2$ respectively.

   For $\mathbb{P}_2$, the standard basis is $\{1, t, t^2\}$ and for $\mathbb{P}_3$, the standard basis is $\{1, t, t^2, t^3\}$.

   (c) Find the corresponding linear transformation $S : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ by using the coordinate vectors relative to the standard bases of $\mathbb{P}_3$ and $\mathbb{P}_2$.

   With the bases ordered as above, $S : \mathbb{R}^4 \longrightarrow \mathbb{R}^3; \begin{bmatrix}a\\b\\c\\d\end{bmatrix} \mapsto \begin{bmatrix}b\\2c\\3d\end{bmatrix}$.

   (d) What does $S(\vec{x})$ equal if $\vec{x} = \begin{bmatrix}4\\3\\5\\6\end{bmatrix}$? $S\left(\begin{bmatrix}4\\3\\5\\6\end{bmatrix}\right) = \begin{bmatrix}3\\10\\18\end{bmatrix}$.

3. If $H = \left\{\begin{bmatrix}a\\a+b\\b\end{bmatrix} \,\middle|\, a, b \in \mathbb{R}\right\}$.

   (a) Show that H is a subspace of $\mathbb{R}^3$.

   The easy way: Since $\begin{bmatrix}a\\a+b\\b\end{bmatrix} = a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\1\end{bmatrix}$, $H = \text{Span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ and is therefore a subspace of $\mathbb{R}^3$

(b) Find a basis for H.

Doing it the easy way above, makes this even easier. Since $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent (since the first is nonzero and the second is not a multiple of the first), and it spans H (by part (a)), it is a basis.

(c) Find the dimension of H.

Since there are two vectors in the basis exhibited, $\dim H = 2$.

4. Let $M = \{\vec{p} \in \mathbb{P}_3 \,|\, \vec{p}(t) = a + bt^2,$ where a and b are in $\mathbb{R}\,\}$

(a) Show that M is a subspace of $\mathbb{P}_3$

We'll also do this the easy way: Since any element $\vec{p} = a + bt^2 \in M$ can be written (rather easily as) $a \cdot 1 + b \cdot t^2$, we have $M = \text{Span}\{1, t^2\}$ and is therefore a subspace of $\mathbb{P}_3$.

For instructive purposes, let's also do this NOT the easy way: First, since $\vec{0} = 0 + 0t^2$, we see that $\vec{0} \in M$ and therefore $M \neq \phi$. Now, given two vectors $\vec{p} = a + bt^2$, $\vec{q} = c + dt^2 \in M$, their sum $\vec{p} + \vec{q} = (a + c) + (b + d)t^2$ is in M. Also, given $\vec{p} = a + bt^2 \in M$ and $r \in \mathbb{R}$, we have $r \cdot \vec{p} = ra + rbt^2$ is in M. Thus, M is a subspace of $\mathbb{P}_3$.

(b) Find a set of vectors which will span M.

Clearly, as in the previous question, one spanning set for M (although not the only possible one, by far) is $\{1, t^2\}$.

(c) Find a basis for M.

Since $\{1, t^2\}$ spans and is linearly independent (see the explanation in 3b), it is a basis for M.

(d) Find the dimension of M. Just as above, $\dim M = 2$.

(e) What is the largest number of linear independent vectors you can have in M?

Since $\dim M = 2$, the largest number of linear independent vectors in M is 2. By the way, the smallest number of vectors needed to span M is 2.

5. Let $K = \{\vec{p} \in \mathbb{P}_3 \,|\, \vec{p}(0) = 0, \vec{p}(1) = 0\,\}$, so K are all those vectors in $\mathbb{P}_3$ which vanish as functions at inputs 0 and 1.

(a) Show that K is a subspace of $\mathbb{P}_3$.

To do this the easy way, we have to recall a few facts from college algebra or precalculus: If a polynomial function $f(x)$ vanishes when $x = a$ is plugged in (i.e., $f(a) = 0$, then $x - a$ has to be a factor of $f(x)$. Thus in our situation, for any $\vec{p} \in K$, $\vec{p}(0) = 0$ and $\vec{p}(1) = 0$. Therefore t and $t - 1$ is a factor of every vector $\vec{p} \in K$. But if t and $t - 1$ are both factors, then so is $t(t-1)$. Thus, if $\vec{p}$ is in K, then $\vec{p} = t(t-1)(at+b)$ for some $a, b \in \mathbb{R}$. Multiplying it out, any $\vec{p} = at^3 + (b - a)t^2 - bt$. It is now clear that any $\vec{p} \in K$ can be expressed as $\vec{p} = a(t^3 - t^2) + b(t^2 - t)$. Thus, K =Span$\{t^3 - t^2,\ t^2 - t\}$.

(b) Letting $\mathcal{E}$ be the standard basis of $\mathbb{P}_3$, is $\vec{p} \in K$ if $[\vec{p}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$? what about if $[\vec{p}]_{\mathcal{E}} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$?

If $[\vec{p}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, then $\vec{p} = 1 + 2t + 3t^2 + 4t^3$ and clearly $\vec{p}(0) = 1$, not 0, so $\vec{p}$ is not in K. If $[\vec{p}]_{\mathcal{E}} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$ instead, $\vec{p} = -2t + t^2 + t^3$. So by evaluating$\vec{p}$ at $t = 0$ and $t = 1$, we see that $\vec{p}(0) = 0$ and $\vec{p}(1) = -2 + 1 + 1 = 0$, so $\vec{p}$ is in K.

(c) Find a set of vectors which will span K. We already did this in part (a).

(d) Find a basis for K

Since we found a spanning set, we only need to verify that the spanning set is a linearly independent set and we'll have a basis. Since $t^3 - t^2$ is not zero and $t^2 - t$ is not a scalar multiple of $t^3 - t^2$, $\{t^3 - t^2,\ t^2 - t\}$ is a basis for K.

(e) Find the dimension of K. Since the basis has two vectors, dim $K = 2$.

6. Write the standard bases and find the dimensions of: $\mathbb{P}_2$, $\mathbb{P}_3$, $\mathbb{R}^2$, $\mathbb{R}^3$, $M_{2\times 2}(\mathbb{R})$,.

Respectively, $\{1, t, t^2\}$, $\{1, t, t^2, t^3\}$, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and lastly, $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

7. Give examples of nonstandard bases for each of the above vector spaces.

Respectively, $\{t + t^2,\ 1 + t^2,\ 1 + t\}$, $\{t + t^2 + t^3,\ 1 + t^2 + t^3,\ 1 + t + t^3,\ 1 + t + t^2\}$, $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$, $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$, and finally $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$.

8. Find a basis and dimension of $V = \{ A \in M_{2\times 2}(\mathbb{R}) \mid \text{tr}A = 0 \}$, where tr$A$ is the sum of the diagonal entries.

Any matrix with trace zero is of the form

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} = a \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, one possible basis is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$. Therefore, dim $V = 3$.

9. Give examples of two infinite dimensional vector spaces where for one of them, you also give a basis of it.

One infinite dimensional vector space is the set of all continuous functions defined on $[0,1]$. Another is the set of all polynomials which has an infinite basis $\{1, t, t^2, t^3, \dots\}$.

10. Find all values of h so that $\vec{y} = \begin{bmatrix} 3 \\ 5 \\ h \end{bmatrix}$ will be in the subspace spanned by the three vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$.

$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 2 & 4 & 0 & 5 \\ -4 & -8 & 0 & h \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -2 & 2 & -1 \\ 0 & 4 & -4 & 12+h \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 3 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & 10+h \end{array}\right]$. So this system is consistent so long as $h = -10$.

11. Let $A = \begin{bmatrix} -1 & 3 & 7 & 2 & 0 \\ 1 & -2 & -7 & -1 & 3 \\ 2 & -4 & -9 & -5 & 1 \\ 3 & -6 & -11 & -9 & -1 \end{bmatrix}$.

(a) Find bases for the Col$A$, Nul$A$ and Row$A$.

An echelon form for A is $\begin{bmatrix} 1 & -3 & -7 & -2 & 0 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 5 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore:

a basis for Col$A$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} 7 \\ -7 \\ -9 \\ -11 \end{bmatrix} \right\}$.

The reduced echelon form of A is $\begin{bmatrix} 1 & 0 & 0 & -\frac{16}{5} & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -\frac{3}{5} & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

So a basis for Nul$A$ is $\left\{ \begin{bmatrix} \frac{16}{5} \\ -1 \\ \frac{3}{5} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

A basis for Row$A$ is $\{[-1\ 3\ 7\ 2\ 0],\ [1\ \ -2\ \ -7\ \ -1\ \ 0],\ [2\ \ -4\ \ -9\ \ -5\ \ 1]\}$

(b) Find rank$A$ and the dimension of the null space of A.

The rank of A is 3, and the dim Nul$A$=2.

(c) If $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^4; \vec{x} \mapsto A\vec{x}$, find the dimension of the kernel of T and the range of T.

dim ker$T$ = dim Nul$A$ = 2 and dim range$T$ = dim Col$A$ = 3.

12. True or False:

(a) The dimension of a vector space $\mathbb{P}_{17}$ is 17.

False: Dim $\mathbb{P}_{17} = 18$.

(b) Any line in $\mathbb{R}^3$ is a one-dimensional subspace of $\mathbb{R}^3$.

False: Only those lines that go through the origin in $\mathbb{R}^3$ are 1-dimensional subspaces of $\mathbb{R}^3$.

(c) If a vector space V has a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_5\}$, then any set in V containing 6 vectors must be linearly dependent.

True: The dimension is the largest number of vectors in any linearly independent sets.

(d) If a vector space V has a basis $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_5\}$, then any set in V containing 6 vectors must span all of V.

False: Suppose $\vec{v}$ is any nonzero vector in V, then $\{\vec{v},\ 2\vec{v},\ 3\vec{v},\ 4\vec{v},\ 5\vec{v},\ 6\vec{v}\}$ is)definitely) a set of 6 different vectors that span a 1-dimensional subspace of V and not all of V.

(e) In a 6-dimensional vector space, any set of exactly 6 vectors is automatically a basis.

False: Consider the set above: with $\vec{v}$ any nonzero vector in V, then $\{\vec{v},\ 2\vec{v},\ 3\vec{v},\ 4\vec{v},\ 5\vec{v},\ 6\vec{v}\}$ is not a basis for V.

(f) If the set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_7\}$ span a vector space V, then dim $V = 7$.

False: If a set of 7 vectors span a vector space V, then the most you can say is that dim $V \leq 7$ since the set of 7 vectors might not be linear independent.

(g) If H is a subspace of a finite-dimensional vector space V, then dim $H \leq$ dim V.

True: Since H is inside V, any basis for H is a linearly independent set of vectors in V and can therefore be increased to a basis of V. So the number of vectors in any basis of H is less than or equal to the number of vectors in a basis of V.

13. If the null space of a 7×9 matrix is 3-dimensional, what can you say about rank$A$? what can you say about dim ColA^T?

Rank $A = 6$ and dim Col$A^T =$ dim Row$A =$ rank $A = 6$.

14. If A is a 5×8 matrix, what is the smallest possible dimension of Nul$A$?

The smallest dim Nul$A$ can be is 3. But the way, the largest dim Nul$A$ can possibly be is 8, and that happens if the rank of A is 0, which only happens when the matrix is actually the zero matrix.