Math 2210Q-004

Solutions for the Quiz Questions For Review Purposes

Spring 2009

1. Suppose A and B are 4×4 matrices where the last column of AB is the zero vector, but where no column of B is the zero vector. What can you say about the columns of A and how does that impact on whether or not A is invertible? Explain your reasoning.

   Remark: It might be easier to explain if you write $B = [\vec{b_1}\ \vec{b_2}\ \vec{b_3}\ \vec{b_4}]$.

   Notice that using the remark given, $AB = [A\vec{b_1}\ A\vec{b_2}\ A\vec{b_3}]$. That means that if the third column of AB is zero, namely $A\vec{b_3} = \vec{0}$, and $\vec{b_3} \neq \vec{0}$, then $\vec{b_3}$ is a nontrivial solution to $A\vec{x} = \vec{0}$ and by Theorem 8, p 129, parts (a) and (d), A is not invertible.

2. Express $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ as a product of elementary matrices.

   This is not the only solution, but it is one of various valid solutions

   Since $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \overset{-3R_1+R_2 \to R_2}{\Longrightarrow} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \overset{R-1+R_2 \to R_1}{\Longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \overset{-\frac{1}{2}R_2 \to R_2}{\Longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   We have the following product of elementary matrices multiplying together with the original matrix B to give the identity:

   $$\begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} B = I$$

   So

   $$B = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}^{-1}$$

   $$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

3. Without finding the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 4 & 7 \end{bmatrix}$, find the 2^{nd} column of A^{-1}. Show all your work.

   If $A^{-1} = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$, then $A \cdot A^{-1} = [A\vec{v}_1\ A\vec{v}_2\ A\vec{v}_3]$. But $A \cdot A^{-1} = I = [\vec{e}_1\ \vec{e}_2\ \vec{e}_3]$. That means that $[A\vec{v}_1\ A\vec{v}_2\ A\vec{v}_3] = [\vec{e}_1\ \vec{e}_2\ \vec{e}_3]$. So, to find $\vec{v}_2$, we have to solve $A\vec{x} = \vec{e}_2$ for $\vec{x}$.

   $$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 0 & 4 & 7 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 4 & 7 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 4 & 7 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 4 \end{array}\right]$$

   $$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -4 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -4 \end{array}\right]$$ So the second column of A^{-1} is $\begin{bmatrix} -2 \\ 7 \\ -4 \end{bmatrix}$

4. If the columns of a matrix P are linearly independent, what can you say about the solution(s) of the nonhomogeneous system $P\vec{x} = \vec{b}$? Explain why?

   Since the columns are linearly independent, by Theorem 12 page 89, the linear transformation $\vec{x} \mapsto P\vec{x}$ is one-to-one. The definition (page 87) says that means that each $\vec{b}$ is the image of *at most one* $\vec{x}$ in $\mathbb{R}^n$. This means that the matrix equation $P\vec{x} = \vec{b}$ either has no solutions or one solution. But it cannot have infinitely many solutions.

5. If A is an invertible matrix, must A^2 also be invertible? Explain why or why not.

   If A is invertible, then A^2 is the product of two invertible matrices and by Theorem 6, page 121, is invertible. In fact, that theorem tells you that the inverse of A^2 is $(A^{-1})^2$

6. If A is a square matrix with linearly independent columns, then the rows of A are also linearly independent. Explain why this is so.

   If A is an $n \times n$ matrix with n linearly independent columns, then by theorem 8, A is invertible. But then also by Theorem 8, A^T is invertible, so again, its columns are linearly independent. But the columns of A^T are precisely the rows of A. So therefore, the rows of A are linearly independent.

7. Using your knowledge of invertible matrices, explain without row reduction computations why $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is or is not invertible.

   Since the second column A is not a multiple of the first column of A and the first column is nonzero, the first two columns of A are linearly independent. Now, if the third column is not a linear combination of the first two columns, then all three columns would be linearly independent. (Look at the blue box on page 67, Theorem 7 on page 68 and the following example on that page.)

   So now we need to figure out if the third column is a linear combination of the first two columns: Let the columns of A be denoted by $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$. Suppose for the moment that $\vec{a}_3 = k\vec{a}_1 + l\vec{a}_2$. Since $\vec{a}_3$ has a 2 in the first entry and a 3 in the second entry, then it must be that $\vec{a}_3 = 2\vec{a}_1 + 3\vec{a}_2$. But this can't be true since the third entries wouldn't be right. The third entry of $\vec{a}_3$ would have to be $2 \cdot 4 + 3 \cdot 5$ which equals 23, not 6 which is the 3rd entry of $\vec{a}_3$.

   So $\vec{a}_3$ is not a linear combination of $\vec{a}_1$ and $\vec{a}_2$ and $\vec{a}_2$ is not a multiple of $\vec{a}_1$, and $\vec{a}_1$ is nonzero, so all three are linearly independent.

   Since the matrix is a square matrix and its columns are linearly independent, Theorem 8, section 2.3, says that the matrix is invertible.

8. Suppose that $A^{-1} = \begin{bmatrix} -3 & 0 & 2 \\ -6 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix}$. Solve the system $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Since we know that A is invertible, then there is only one solution to the matrix equation $A\vec{x} = \vec{b}$, namely $\vec{x} = A^{-1}\vec{b}$. So all we have to do is multiply A^{-1} and $\vec{b}$ correctly. So $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 3 \\ 5 \\ -5 \end{bmatrix}$.

9. Determine whether each of the following matrices is invertible. No arithmetic is needed; explain your answers.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

The second matrix is invertible because in question #7 we found that the columns are linearly independent and the matrix is a square matrix. Then Theorem 8, section 2.3 in the text, tells us that the matrix is invertible.

This gives us a way to figure out whether the first matrix is invertible or not. Because of the well placement of zeros in the matrix, you can see that the first column in nonzero, the second column is not a multiple of the first and the third column in not a linear combination (i.e., a sum of multiples) of the first and second column. So all three columns are linearly independent and therefore, as before, the matrix, being square, is invertible.