Solutions for the Review Questions for Chapters 5 & 6

1. True or False: If true, give a reason as to why; if false, give a reason or an example. Also, if the statement is false, but some slight modification can be made which will make it true, state the modification.

   (a) If $A\vec{x} = \lambda\vec{x}$ for some scalar λ, then $\vec{x}$ is an eigenvector of A.

   False: if vx is a nonzero vector for which $A\vec{x} = \lambda\vec{x}$ for some scalar λ, then $\vec{x}$ would be an eigenvector

   (b) The eigenvalues of a matrix are the values on the diagonal.

   False: Only if the matrix is in triangular form will the eigenvalues be the diagonal entries.

   (c) An eigenspace of A is a null space of a certain matrix.

   True: the eigenspace of A corresponding to eigenvalue λ is $V_\lambda = \text{Nul}(A - \lambda I)$.

   (d) The multiplicity of a root r of the characteristic equation of a matrix A is equal to the dimension of the eigenspace of A associated with the eigenvalue r.

   False: The dim V_λ cannot exceed the algebraic multiplicity of the root λ in the characteristic equation, but it can certainly be less. Eg: the characteristic equation of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is $(1-\lambda)^2 = 0$. The only eigenvalue is $\lambda = 1$, of multiplicity 2. However, $V_1 = \text{Nul}(A - 1 \cdot I) = \text{Nul}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ has dimension 1.

   (e) If matrix B can be obtained from matrix A by applying elementary row operations to A, then A and B have the same eigenvalues.

   False: Elementary row operations alter the eigenvalues (and eigenvectors also). You can see this by considering an invertible matrix A. Then A is certainly row equivalent to the identity which only has one eigenvalue, namely $\lambda = 1$, but clearly not all invertible matrices have only 1 as their eigenvalue. Eg: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. A has eigenvalues 2 and 3 and A is invertible, hence row reducible to the identity.

   (f) If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation with standard matrix A, then $[T]_\mathcal{B}$ is a diagonal matrix if B is a basis of $\mathbb{R}^n$.

   False: $[T]_\mathcal{B}$ is diagonal if and only if $\mathcal{B}$ is a basis of eigenvectors. If A does not have n linearly independent eigenvectors then A is not diagonalizable. So $[T]_\mathcal{B}$ would not be diagonal for any basis $\mathcal{B}$.

   (g) If $\vec{x}$ is orthogonal to $\vec{y}$ in a subspace W, then $\vec{x} \in W^\perp$.

   False: This would be true if W were 1-dimensional. However, if $\dim W \geq 2$ with $\vec{y}, \vec{y}'$ linearly independent in W, then $\vec{x}$ might be orthogonal to $\vec{y}$, but not necessarily orthogonal to $\vec{y}'$. Eg: $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $W = \mathbb{R}^2$. Then vx is orthogonal to $\vec{y}$, but $\vec{x}$ is not in $W^\perp$, which in this example equals $\{\vec{0}\}$.

(h) The column vectors of A are orthogonal to the vectors in $\mathrm{Nul}\, A$.

False: $(\mathrm{Row}\, A)^{\perp} = \mathrm{Nul}\, A$, but $(\mathrm{Col}\, A)^{\perp} = \mathrm{Nul}(A^T)$.

(i) An orthogonal matrix is invertible

True: An orthogonal matrix A is a square matrix such that $A \cdot A^T = I$. Now by the MEGA Theorem way back when, A is invertible.

(j) A basis of $\mathbb{R}^n$ is an orthogonal set of n vectors in $\mathbb{R}^n$.

False: Only certain bases are orthogonal, most aren't. Eg: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ form a basis of $\mathbb{R}^2$ since the second vector is not a multiple of the first and the dimension of $\mathbb{R}^2$ is 2. But they are not orthogonal since their dot product is $4 \neq 0$.

(k) If $\vec{y} = \vec{z}_1 + \vec{z}_2$, where $\vec{z}_1 \in W$ and $\vec{z}_2 \in W^{\perp}$, then $\vec{z}_1$ is the orthogonal projection of $\vec{y}$ onto W.

True: By Theorem 8, page 395, the expression of $\vec{y}$ as a sum of vectors one in W and another in $W^{\perp}$ is unique and the one in W is called "the orthogonal projection of $\vec{y}$ onto W".

(l) If W is a 3-dimensional subspace of $\mathbb{R}^8$ and if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set of vectors, then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for W.

False: If the three vectors are in W, then it would be a basis of W. The orthogonality of the three vectors means that they are linearly independent and any three linearly independent vectors in a 3-dimensional vector space would have to be a basis. But there was nothing that said that the vectors had anything to do with W.

(m) If $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is an ordered basis for a subspace W in $\mathbb{R}^n$, then the Gram-Schmidt process produces an ordered orthonormal basis of W whose first vector is $\vec{x}_1$.

False: The Gram-Schmidt method produces an orthogonal basis whose first vector is $\vec{x}_1$. But unless $\vec{x}_1$ were a unit vector, it would not be part of an orthonormal basis.

2. Explain why an $n \times n$ matrix can have at most n different eigenvalues.

Eigenvectors corresponding to different eigenvalues are linearly independent. If there were more than n different eigenvalues, then there would be more than n linearly independent vectors in $\mathbb{R}^n$, which of course can't happen.

3. Explain why a matrix A is invertible if and only if A has only nonzero eigenvalues.

If A is invertible, then $\mathrm{Nul}\, A = \{\vec{0}\}$, so there are no nonzero solutions to $A\vec{x} = \vec{0}$, i.e., no nonzero vectors $\vec{x}$ where $A\vec{x} = 0\vec{x}$. So 0 is not an eigenvalue. On the other hand, if 0 is not an eigenvalue, then there can be no nonzero solutions to $A\vec{x} = \vec{0}$ and hence by the MEGA Theorem, A is invertible.

4. Find the characteristic equation and eigenvalues for the following matrices:

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 10 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 10 \end{bmatrix}^{-1}$

For (a), the characteristic equation is $(1-\lambda)(4-lambda)(6-\lambda) = 0$ and the eigenvalues are 1, 4 and 6. For (b), the characteristic equation is $\lambda^3 = 0$ and hence the only eigenvalue is 0. For (c), letting A be that matrix, then $\det(A - \lambda I) = (1-\lambda) \cdot \det \begin{bmatrix} -\lambda & 3 \\ 1 & -\lambda \end{bmatrix} = (1-\lambda)(\lambda^2 - 3)$. The characteristic equation is therefore $(1-\lambda)(\lambda^2-3) = 0$ and so the eigenvalues are 1, $\sqrt{3}$ and $-\sqrt{3}$. For (d), the product matrix is similar to the diagonal matrix in the middle and therefore has the same characteristic polynomial. So the characteristic equation is $(2-\lambda)^2(3-\lambda) = 0$ and therefore the eigenvalues are 2 and 3.

5. For the matrix $A = \begin{bmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{bmatrix}$, find the characteristic equation, eigenvalues, eigenspaces, a basis for each eigenspace, and determine if A is diagonalizable. If so, give a diagonal matrix D similar to A and exhibit the matrix P so that $A = PDP^{-1}$.

OK, here we go - we have to do everything for this one:

$$\det(A - \lambda I) = det \begin{bmatrix} -\lambda & -4 & -6 \\ -1-\lambda & -3 & \\ 1 & 2 & 5-\lambda \end{bmatrix} = \cdots = -\lambda^3 + 5\lambda^2 - 8\lambda + 4.$$

So the characteristic equation is: $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$.

Solving this we factor the characteristic polynomial:

$$\begin{aligned} \lambda^3 - 5\lambda^2 + 8\lambda - 4 &= (\lambda^3 - 5\lambda^2 + 4\lambda) + (4\lambda - 4) \\ &= \lambda(\lambda-4)(\lambda-1) + 4(\lambda-1) \\ &= (\lambda-1)\cdot(\lambda^2 - 4\lambda + 4) \\ &= (\lambda-1)(\lambda-2)^2 \end{aligned}$$

Thus, the eigenvalues are 1 and 2.

Remark: You also could have tried values for λ to see if you can get a root of the characteristic equation. That would have gotten you $\lambda = 1$ fairly quickly and then divide $\lambda^3 - 5\lambda^2 + 8\lambda - 4$ by $\lambda - 1$. That would have left you with a quadratic in λ which you could have solved with the quadratic formula in the worst of cases.

$$V_1 = \text{Nul} \begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix} = \cdots = \text{Nul} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$V_2 = \text{Nul} \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix} = \cdots = \text{Nul} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since A has three linearly independent eigenvectors, $\mathbb{R}^3$ has a basis of eigenvectors and therefore A is diagonalizable.

We get that A is similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and moreover if $P = \begin{bmatrix} -2 & -2 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, then $A = PDP^{-1}$.

6. For the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3; \vec{x} \mapsto A\vec{x}$, where $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$, is there an ordered basis $\mathcal{B}$ of $\mathbb{R}^3$ relative to which $[T]_{\mathcal{B}}$ is diagonal? If so, give the ordered basis and the corresponding matrix $[T]_{\mathcal{B}}$; if not, explain why not.

Here we do the almost the same in question 5 above only we don't go as far. $\det(()A - \lambda I) = \cdots = -\lambda^3 - 3\lambda^2 + 4 = -[\lambda^3 + 2\lambda^2 + \lambda^2 - 4] = -[\lambda^2(\lambda+2) + (\lambda-2)(\lambda+2)] = -(\lambda+2)(\lambda^2+\lambda-2) = -(\lambda+2)^2(\lambda-1)$.

The eigenvalues are therefore -2 and 1.

$V_1 = \text{Nul}\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} = \cdots = \text{Nul}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\}$ which is 1-dimensional.

$V_{-2} = \text{Nul}\begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

Thus, there is a basis $\mathcal{B}$ relative to which $[T]_{\mathcal{B}}$ is diagonal, namely $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

The corresponding matrix representation for T is $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

7. For the linear transformation

$$T : \mathbb{P}_2 \longrightarrow \mathbb{P}_2;\ a + bt + ct^2 \mapsto (a + 3b + 3c) - (3a + 5b + 3c)t + (3a + 3b + c)t^2,$$

is there a basis $\mathcal{C}$ of $\mathbb{P}_2$ relative to which $[T]_{\mathcal{C}}$ is diagonal? If so, give the ordered basis and the corresponding matrix $[T]_{\mathcal{C}}$; if not, explain why not. (*Hint*: consider the relationship of this with question 6 above.)

This is the same question as the previous one. Only we have to translate the linear transformation to one from $\mathbb{R}^3$ to $\mathbb{R}^3$ first, then get the standard matrix A, the basis of eigenvectors for it, giving us the $\mathcal{B}$ basis above. Then translate that matrix back to the language of $\mathbb{P}_2$. Thus with

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\},$$

use the following basis $\mathcal{C}$ of $\mathbb{P}_2$:

$$\mathcal{C} = \left\{1 - t + t^2,\ -1 + t,\ -1 + t^2\right\},$$

in which case, $[T]_{\mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

8. Show that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis of $\mathbb{R}^3$ where

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \; \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \; v_3 = \begin{bmatrix} 5 \\ -13 \\ 2 \end{bmatrix}.$$

Just dot all the pairs of vectors in the basis, $\vec{v}_i \cdot \vec{v}_j = 0$ if $i \neq j$.

9. Using the orthogonal basis $\mathcal{B}$ as in question 8 above, find an expression for the vector $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in terms of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

Using Theorem 5, page 385, $\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ where $c_1 = \frac{\vec{y}.\vec{v}_1}{\vec{v}_1.\vec{v}_1}$, $c_2 = \frac{\vec{y}.\vec{v}_2}{\vec{v}_2.\vec{v}_2}$, and $c_3 = \frac{\vec{y}.\vec{v}_3}{\vec{v}_3.\vec{v}_3}$.
This gives us $y = \frac{2}{11}\vec{v}_1 + \frac{15}{18}\vec{v}_2 - \frac{15}{198}\vec{v}_3$.

10. Let $W = \text{Span}\{\vec{v}_1, \vec{v}_2\}$, where $\vec{v}_1$ and $\vec{v}_2$ are as in question 8 above, and take $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$.

Express $\vec{y} = \hat{y} + \vec{z}$ where $\hat{y}$ is in W and $\vec{z}$ is in $W^{\perp}$.

We use Theorem 8, page 395. $\vec{y} = \hat{y} + \vec{z}$ where

$$\begin{aligned} \hat{y} &= \frac{\vec{y}.\vec{v}_1}{\vec{v}_1.\vec{v}_1} + \frac{\vec{y}.\vec{v}_2}{\vec{v}_2.\vec{v}_2} \\ &= \tfrac{2}{11}\vec{v}_1 + \tfrac{15}{18}\vec{v}_2 \\ &= \tfrac{2}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + \tfrac{15}{18}\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{19}{66} \\ -\frac{43}{66} \\ -\frac{116}{33} \end{bmatrix} \end{aligned}$$

Then $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} \frac{91}{66} \\ \frac{67}{66} \\ \frac{104}{33} \end{bmatrix} = \begin{bmatrix} \frac{85}{66} \\ \frac{175}{66} \\ \frac{215}{66} \end{bmatrix}$

You can verify that $\begin{bmatrix} -\frac{19}{66} \\ -\frac{43}{66} \\ -\frac{116}{33} \end{bmatrix} + \begin{bmatrix} \frac{85}{66} \\ \frac{175}{66} \\ \frac{215}{66} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

11. Consider the basis $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ for a subspace W of $\mathbb{R}^4$ where $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Use the Gram-Schmidt method to find an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ for the W so that $\text{Span}\{\vec{u}_1\} = \text{Span}\{\vec{x}_1\}$ and $\text{Span}\{\vec{u}_1, \vec{u}_2\} = \text{Span}\{\vec{x}_1, \vec{x}_2\}$.

We'll first find an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ by the Gram-Schmidt method and then divide each by their lengths to get the orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ requested.

$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ Now, $\vec{x}_2 \cdot \vec{v}_1 = 1$ and $||\vec{v}_1||^2 = 2$,

So, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \\ -\frac{1}{2} \end{bmatrix}$ Then, $||\vec{v}_2||^2 = \frac{5}{2}$, and $\vec{x}_3 \cdot \vec{v}_1 = 2$ and $\vec{x}_3 \cdot \vec{v}_2 = 2$,

So, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{\frac{5}{2}}\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \\ \frac{1}{2} \end{bmatrix}$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \\ \frac{4}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ -\frac{2}{5} \end{bmatrix}$$

Since $||\vec{v}_1|| = \sqrt{2}$, $||\vec{v}_2|| = \sqrt{\frac{5}{2}}$, and $||\vec{v}_3|| = \sqrt{\frac{10}{25}} = \sqrt{\frac{2}{5}} = \frac{2}{\sqrt{10}}$,

We have $\vec{u}_1 = \frac{1}{||\vec{v}_1||}\vec{v}_1 = \frac{1}{\sqrt{2}}\vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$\vec{u}_2 = \sqrt{\frac{2}{5}}\begin{bmatrix} \frac{1}{2} \\ 1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{bmatrix}$ and $\vec{u}_3 = \frac{\sqrt{10}}{2}\begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{5} \\ -\frac{2}{5} \end{bmatrix} = \begin{bmatrix} -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \end{bmatrix}$