



*University of Connecticut  
Department of Mathematics*

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Math 210Q

Exam IV: Vector Analysis

V0 - FS - 2006

Name:

Section: 002

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**Read This First !**

- Use of a pocket calculator or graphing calculator is only allowed if the calculator cannot perform alpha numeric calculations. Storing of any formulas in memory is forbidden.
- If part of the solution is written outside the space provided for this solution clearly indicate this.
- You might want to use the divergence theorem to calculate the first surface integral. It would be quite lengthy if you didn't.
- There is 10% extra credit in here. Good Luck! ☺

**Grading - For Administrative Use Only**

GRADES	
Page 1	/15
Page 2	/10
Page 3	/15
Page 4	/15
Page 5	/15
Page 6	/15
Page 7	/15
Page 8	/10
Total	/100

1.  $\vec{r}$  and  $r$

(15 points)

Let  $\vec{r} = \langle x, y, z \rangle$  and  $r = \|\vec{r}\|$ . Calculate the following and express your answer in terms of  $\vec{r}$ ,  $r$  and constants *only*.

(a)  $\operatorname{div} \vec{r} =$

(b)  $\operatorname{curl} \vec{r} =$

(c)  $\operatorname{grad} r =$

$\Delta$  &  $\nabla$ 

(10 points)

- (a) Let  $\vec{r} = \langle x, y, z \rangle$  and  $r = \|\vec{r}\|$ . Calculate  $\nabla \cdot (r \vec{r})$  and express this in terms of  $r$  and constants *only*, i.e. no differential operators and no  $x$ 's,  $y$ 's and  $z$ 's either.

- (b) Let  $f$  and  $g$  be functions from  $\mathbb{R}^3$  to  $\mathbb{R}$ . Make a cunning choice of  $g$  in the formula  $\nabla(fg) = f\nabla g + g\nabla f$  to prove

$$\nabla \left( \frac{1}{f} \right) = -\frac{1}{f^2} \nabla f$$

You can also calculate it directly, but that's a lot more work.

**2. Line Integrals**

Calculate these line integrals directly, or by using the Fundamental Theorem of Calculus for Line Integrals, Green's Theorem or Stokes' Theorem if this is appropriate.

- (a) Let  $C$  be the lower half of the circle centered at  $(0, 0)$  with radius 2. *(15 points)*  
traversed counterclockwise.

Calculate  $\oint_C ye^{-x} ds$ .

- (b) Let  $\mathbf{F}(x, y) = \langle x^2 + 3y \cos x, 3 \sin x + y + 1 \rangle$  be a force field. Find a function  $f$  so that  $\mathbf{F}(x, y) = \nabla f(x, y)$ . Now, calculate the line integral

$$\oint_C (x^2 + 3y \cos x) dx + (3 \sin x + y) dy$$

for  $C$  where  $C$  is any curve from  $(0, 0)$  to  $(\pi, \pi)$ .

*(15 points)*

(c) Evaluate the line integral

(15 points)

$$\oint_C (ye^x + xy)dx + (x^2 + y^5)dy$$

where  $C$  is the square with vertices  $(-1, -1)$ ,  $(1, -1)$ ,  $(1, 1)$ , and  $(-1, 1)$ , traversed counterclockwise.

- (d) Find the amount of work done by the force field  $\mathbf{F}$  on a particle moving along the curve  $C$ , where *(15 points)*

$$\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : \langle x, y, z \rangle \mapsto \langle 3y + xz, x + z^2, 2x \rangle$$

and  $C$  is given by

$$C : [0, 1] \rightarrow \mathbb{R}^3 : t \mapsto \langle t, t^2, t^3 \rangle$$

**3. Surface Integrals**

You can calculate these surface integrals by using the Divergence Theorem or Stokes' theorem if this is a valid approach. At least one of them will work in any of these cases.

- (a) Let surface  $S$  be a sphere of radius 3 and let the vector field  $\vec{F}$  be defined by (15 points)  
 $\vec{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 : \langle x, y, z \rangle \mapsto \langle y^2, x^2, z^2 \rangle$ . Calculate the flux of  $\vec{F}$  through  $S$ .

- (b) Assume that  $f$  and  $g$  have continuous second order partial derivatives and let  $S$  be an oriented piecewise smooth surface. Calculate (5 points)

$$\iint_S (\nabla f \times \nabla g) \cdot \vec{dS} =$$

- (c) Let  $E$  be a simple solid region with  $\text{Vol}(E) = 6$  units and let  $S = \partial E$ . (5 points)  
Take  $\vec{F} = \langle x, 2y, 4z \rangle$ . Calculate

$$\iint_S \vec{F} \cdot \vec{dS} =$$

## Basic Math 210Q Formulas

### 1. Vectors

$$\|\lambda \vec{a}\| = |\lambda| \|\vec{a}\|$$

$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\vartheta)$$

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin(\vartheta)$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

### 2. Coordinate Systems

*Polar*

*Cylindrical*

*Spherical*

$$\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \end{cases}$$

$$\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \\ z = z \end{cases}$$

$$\begin{cases} x = \rho \sin(\varphi) \cos(\theta) \\ y = \rho \sin(\varphi) \sin(\theta) \\ z = \rho \cos(\varphi) \end{cases}$$

### 3. Jacobian Determinants

*2 dimensional*

Rectangular:  $J(x, y) = 1$

Polar:  $J(r, \theta) = r$

*3 dimensional*

Rectangular:  $J(x, y, z) = 1$

Cylindrical:  $J(r, \theta, z) = r$

Spherical:  $J(\rho, \theta, \varphi) = \rho^2 \sin(\varphi)$

### 4. Curvature and Tangent Vector

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|} \quad \text{where} \quad \vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$$

# Vector Analysis

## 1. Differential Operators

$$\nabla f = \text{grad } f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\nabla \cdot \vec{F} = \text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\text{curl } \vec{F} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

$$\Delta f = \nabla \cdot (\nabla f) = \text{div}(\text{grad } f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\text{curl}(\text{grad } f) = \vec{0}$$

$$\text{div}(\text{curl } \vec{F}) = 0$$

$$\text{div}(f \vec{F}) = f \text{div } \vec{F} + \vec{F} \cdot \text{grad } f$$

$$\text{curl}(f \vec{F}) = f \text{curl } \vec{F} + (\text{grad } f) \times \vec{F}$$

$$\text{div}(\vec{F} \times \vec{G}) = \vec{G} \cdot \text{curl } \vec{F} - \vec{F} \cdot \text{curl } \vec{G}$$

$$\text{div}(\nabla f \times \nabla g) = 0$$

## 2. Line Integrals

$$\oint_C f ds = \int_a^b f(\vec{r}(t)) \|\vec{r}'(t)\| dt$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

## 3. Surface Integrals

$$\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| dA$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \left( \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) dA$$

## 4. Integral Theorems

*Fundamental Theorem for Line Integrals*

$$\oint_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

*Stokes' Theorem*

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

*Green's Theorem*

$$\iint_D \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dA = \oint_C F_x dx + F_y dy$$

*Divergence Theorem*

$$\iiint_E \text{div } \vec{F} dV = \iint_S \vec{F} \cdot d\vec{S}$$