Math 210Q-02 Spring 2006

Final Exam

You must show your work to receive full credit. Partial credit will be assigned based on the work show and based only on work that counts towards the correct method(s). Please simplify all answers as much ε possible. Use exact answers and do not use decimal approximations unless asked to do so. You probabl expected me to insert another quote in here didn't you. Sucker. Please circle your final answers clearly t avoid any ambiguity over your final answer.

Name: Das Direktor

Score:

(10 points) Evaluate the line integral $\int_C x + y\ dx$ where C is the line that extends from the point $(0, 2)$ to $(3, 0)$.

$$r(t) = (1-t)\langle 0,2\rangle + t\langle 3,0\rangle = \langle 3t, 2-2t\rangle$$

for $t \in [0,1]$

$$\int_0^1 [(3t) + (2-2t)](3)\ dt = 3\int_0^1 t + 2\ dt = 3\left[\frac{t^2}{2} + 2t\Big|_0^1\right]$$

$$= 3 \cdot \frac{5}{2} = \frac{15}{2}$$ ∎

(10 points) Find the curvature formula of the space curve $r(t) = < t, t, 1 + t^2 >$ and also find the curvature at the point $(1, 1, 2)$.

$$\begin{vmatrix} i & -j & k \\ 1 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = |<2,-2,0>| = \sqrt{8}$$

$$|r'(t)|^3 = \left(\sqrt{1^2+1^2+(2t)^2}\right)^3 = (2+4t^2)^{3/2}$$

so $$|\kappa(t)| = \frac{\sqrt{8}}{(2+4t^2)^{3/2}}$$

at $(1,1,2)$, $t=1$

so $$|\kappa(1)| = \frac{\sqrt{8}}{6^{3/2}} = \left(\frac{8}{216}\right)^{1/2} = \frac{1}{\sqrt{27}}$$ ■

(15 points) Find the volume of the space between the paraboloid $z = 9x^2 + 9y^2$ and the plane $z = 0$ that exists above the disc $x^2 + y^2 = 3$.

$$\iint_D 9x^2 + 9y^2 \, dA = \int_0^{\sqrt{3}} \int_0^{2\pi} 9r^3 \, d\theta \, dr$$

$$= 18\pi \int_0^{\sqrt{3}} r^3 \, dr = 18\pi \cdot \frac{9}{4} = \frac{81\pi}{2}$$

4. (15 points) Find the surface integral of the equation $f(x, y, z) = x^2y + z^2$ over the surface S where S is the cylinder $x^2 + y^2 = 9$ between the planes $z = 0$ and $z = 2$.

$x = 3\cos t$ $\quad 0 \le t \le 2\pi$ $\quad r(\theta, t) = (3\cos t)i + (3\sin t)j + (z)k$

$y = 3\sin t$ $\quad r_\theta = \langle -3\sin t, 3\cos t, 0 \rangle$

$z = z$ $\quad r_z = \langle 0, 0, 1 \rangle$

$$|r_\theta \times r_z| = \begin{vmatrix} i & -j & k \\ -3\sin t & 3\cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = |\langle 3\cos t, 3\sin t, 0 \rangle| = 3$$

so

$$\int_0^{2\pi}\int_0^2 ((3\cos t)^2 3\sin t + z^2)(3)\, dz\, d\theta = 3\int_0^{2\pi}\int_0^2 27\cos^2 t \sin t + z^2\, dz\, d\theta$$

$$3\int_0^{2\pi}\left[\left[27\cos^2 t \sin t\, z + \frac{z^3}{3}\right]_0^2\right] d\theta = 3\int_0^{2\pi} 54\cos^2 t \sin t + \frac{8}{3}\, d\theta$$

$$3\left[-18\cos^3 t + \frac{8}{3}\theta \Big|_0^{2\pi}\right] = 3\left[-18 + \frac{16\pi}{3} + 18 - 0\right] = 3 \cdot \frac{16\pi}{3}$$

16π ■

5. (a) (3 points) Find the symmetric equations of the line that goes through the 2 points $(-1, 0, 1)$ and $(2, 4, 6)$.

(b) (7 points) Determine whether the line you found in part (a) is parallel, intersecting, or skew to the line $r(t) = < 2t + 1, 6 - t, 3t + 5 >$.

$V = < 2 - -1, 4 - 0, 6 - 1 > = < 3, 4, 5 >$

so

A $\quad \frac{x+1}{3} = \frac{y}{4} = \frac{z-1}{5}$

B

$3s - 1 = 2t + 1$
$4s = 6 - t$

$3s - 2t = 2$
$4s + t = 6$

$3s - 2t = 2$
$8s + 2t = 12$

$11s = 14$

so $s = \frac{14}{11}$

and

$4 \cdot \frac{14}{11} = 6 - t$

$\frac{56}{11} - \frac{66}{11} = \frac{-10}{11} = -t$

so $t = \frac{10}{11}$

using $s = \frac{14}{11}, t = \frac{10}{11}$

$5s + 1 = \frac{81}{11} \neq 3t + 5 = \frac{85}{11}$

Skew

6. Consider the equation $f(x,y,z) = \ln xz - \ln xy + yz^2$.

(a) (4 points) Find the vector ∇f at the point $(1,1,2)$.

(b) (6 points) Find the derivative of $f(x,y,z)$ in the direction of the vector $< 2,-2,2 >$ at the point $(1,1,2)$.

A

$\nabla f = < 0, -\frac{1}{y} + z^2, \frac{1}{z} + 2zy >$

so $\nabla f(1,1,2) = < 0, 3, \frac{9}{2} >$ ∎

B

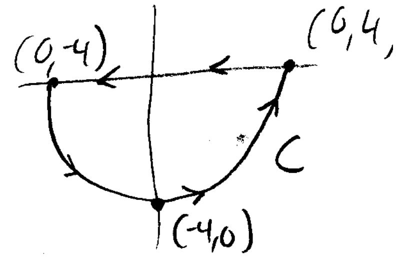
$v = <2,-2,2>$, $|v| = \sqrt{12}$ so $u = < \frac{2}{\sqrt{12}}, \frac{-2}{\sqrt{12}}, \frac{2}{\sqrt{12}} >$

so $D_u f(1,1,2) = <0,3,\frac{9}{2}> \cdot <\frac{2}{\sqrt{12}}, \frac{-2}{\sqrt{12}}, \frac{2}{\sqrt{12}}> = -\frac{6}{\sqrt{12}} + \frac{9}{\sqrt{12}}$

$= \frac{3}{\sqrt{12}} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$ ∎

(15 points) Evaluate the line integral $\int_C -y^3\,dx + x^3\,dy$ where C is the positively oriented, closed curve that forms the boundary of the lower half of the circle of radius 4 centered at the origin.

$$\int_C -y^3\,dx + x^3\,dy = \iint_D 3x^2 + 3y^2\,dA$$



$$\int_{\pi}^{2\pi}\int_0^4 3r^3\,dr\,d\theta = 3\pi\int_0^4 r^3\,dr = \frac{3\pi}{4}\left[r^4\Big|_0^4\right]$$

192π ∎

8. (10 points) Determine whether or not $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exists.

Let $x=0$ then $\frac{xy}{x^2+y^2} \to \frac{0}{y^2}$.

$\lim_{(x,y)\to(0,0)} \frac{0}{y^2} = 0$. So we only need to find a counterexample. Here are two possible ones:

Let $y=x$ then $\frac{xy}{x^2+y^2} \to \frac{x^2}{2x^2} = \frac{1}{2}$

$\neq$ so DNE

Let $y=-x$ then $\frac{xy}{x^2+y^2} \to \frac{-x^2}{2x^2} = -\frac{1}{2}$

9. Consider the function $f(x, y, z) = xyz - \ln xy - \cos z$ where $x(s, t) = 11t^3 - \cos s$, $y(s, t) = \ln s$, and $z(s, t) = t + t^2 - t^4$.

(a) (4 points) Find the derivative of $f(x, y, z)$ with respect to t.

(b) (3 points) Find the implicit derivative of z with respect to y.

(c) (3 points) Find the value of f_{xz} at the point $(1, 2, 3)$.

A

$$f_x = yz - \frac{1}{x} = (\ln s)(t+t^2-t^4) - \frac{1}{11t^3 - \cos s}$$

$$f_y = xz - \frac{1}{y} = (11t^3 - \cos s)(t+t^2-t^4) - \frac{1}{\ln s}$$

$$f_z = xy + \sin z = (11t^3 - \cos s)(\ln s) + \sin(t+t^2-t^4)$$

$$x_t = 33t^2 \qquad z_t = 1+2t-4t^3$$

$$y_t = 0$$

so $f_t = \left[(\ln s)(t+t^2-t^4) - \frac{1}{11t^3 - \cos s}\right](33t^2) + \left[(11t^3-\cos s)(\ln s) + \sin(t+t^2-t^4)\right]$

$(1+2t-4t^3)$ ∎

B

$$\frac{\partial z}{\partial y} = -\frac{xz - \frac{1}{y}}{xy + \sin z}$$ ∎

C

$f_{xz} = y$ so the value is 2 ∎

10. (a) (3 points) Given a vector field F, state the definition of *curl F*.

(b) (6 points) Is the vector field $F(x, y, z) = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + y^2\mathbf{k}$ conservative? Give an explanation for your answer to support it.

(c) (6 points) Use this information in finding the value of $\int_C F \cdot dr$ when F is the vector field from part (b) and C is the arc of the circle of radius 6 about the origin that goes from $\theta = 0$ to $\theta = \frac{3\pi}{2}$.

A

$$\text{Curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ P & Q & R \end{vmatrix} = (R_y - Q_z)i + (P_z - R_x)j + (Q_x - P_y)k$$

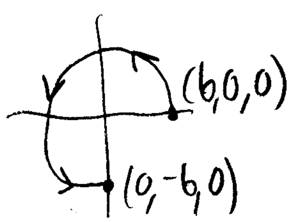
B

$P_y = 2x \quad Q_x = 2x \quad R_x = 0$

$P_z = 0 \quad Q_z = 2y \quad R_y = 2y$

$$\Rightarrow (2y - 2y)i + (0 - 0)j + (2x - 2x)k = \vec{0}$$

C



$\int 2xy\, dx = x^2y + g(y,z)$. $(x^2y + g(y,z))_y$

$= x^2 + g_y(y,z)$ so $g_y(y,z) = 2yz$

$\int g_y(y,z)\, dy = y^2z + h(z)$ so

$(x^2y + y^2z + h(z))_z = y^2 + h'(z)$ so $= 0$

$f = x^2y + y^2z + K.$

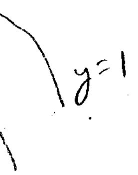
By the Fundamental Theorem of line integrals

$$\int_C F \cdot dr = F(6,0,0) - F(0,-6,0) = 0 - 0 = 0$$

(10 points) Evaluate the triple integral $\iiint_E x^2\, dV$ where E is the space under the parabolic cylin

$z = 1 - y^2$ and trapped by the planes $z = 0$, $x = 1$, and $x = -1$.

$$\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-y^2} x^2\, dz\, dy\, dx = \int_{-1}^{1}\int_{-1}^{1} x^2 - x^2y^2\, dy\, dx$$



$$= \int_{-1}^{1}\left[x^2y - x^2\frac{1}{3}y^3 \Big|_{-1}^{1}\right] dx$$

$$= \int_{-1}^{1}\left[x^2 - \frac{x^2}{3} + x^2 - \frac{x^2}{3}\right] dx = \int_{-1}^{1} \frac{4}{3}x^2\, dx$$

$$= \left[\frac{4}{9}x^3 \Big|_{-1}^{1}\right] = \frac{4}{9} - -\frac{4}{9} = \boxed{\frac{8}{9}}$$

12. (a) (4 points) Find the equation, in standard form, of the plane that includes the 3 points $(1, 2, 3)$ $(-1, 0, 4)$, and $(0, 0, 0)$.

(b) (6 points) Find the angle (to the nearest hundredth) between your plane from part (a) and th plane $3x - 2y + z = 12$.

A

$a = \langle 1-0, 2-0, 3-0 \rangle = \langle 1, 2, 3 \rangle$ $\quad b = \langle -1-0, 0-0, 4-0 \rangle = \langle -1, 0, 4 \rangle$

$a \times b = \begin{vmatrix} i & -j & k \\ 1 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix} = \langle 8, -7, 2 \rangle = n$

so $8(x-0) - 7(y-0) + 2(z-0) = 8x - 7y + 2z = 0$

B

$a = n_1 = \langle 8, -7, 2 \rangle$

$b = n_2 = \langle 3, -2, 1 \rangle$

$\theta = \arccos\left(\frac{a \cdot b}{|a||b|}\right)$

$a \cdot b = 24 + 14 + 2 = 40$

$|a| = \sqrt{117}$

$|b| = \sqrt{14}$

$\theta = \arccos\left(\frac{40}{\sqrt{117} \cdot \sqrt{14}}\right)$

$= \arccos\left(\frac{40}{\sqrt{1638}}\right)$

$= .15$ radians or $8.76°$