Math 2410, Answer to Quiz 7 (4/6/09)

(1) Find the general solution to the equation

$$y'' + 4y' + 20y = 2\cos(2t) \ .$$

Answer: We have $y_h = e^{\alpha t}$ being a homogeneous solution if $\alpha^2 + 4\alpha + 20 = 0$. Thus $\alpha = -2 \pm 4i$. So

$$y_h = c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t) \ .$$

Let $y_p = A\cos(2t) + B\sin(2t)$. Putting into the equations, we have

$$\begin{aligned} 16A + 8B &= 2 \ , \\ -8A + 16B &= 0 \ . \end{aligned}$$

Thus $A = 1/10$ and $B = 1/20$. Hence general solution is

$$y = y_p + y_h = \cos(2t)/10 + \sin(2t)/20 + c_1 e^{-2t}\cos(4t) + c_2 e^{-2t}\sin(4t) \ .$$

(2) Find the solution to the equation

$$y'' + 4y = \sin(2t)$$

with initial condition $y(0) = 0$ and $y'(0) = 0$.

Answer: We have $y_h = e^{\alpha t}$ being a homogeneous solution if $\alpha^2 + 4 = 0$. Thus $\alpha = \pm 2i$. So

$$y_h = c_1\cos(2t) + c_2\sin(2t) \ .$$

Since the forcing term $\sin(2t)$ coincides with the homogeneous solution, we need to add a t to the usual particular solution guess.

Let's consider

$$z'' + 4z = e^{2it}$$

Assume $z_p = Ate^{2it}$. Upon substituting into the equation, we find that $A = 1/(4i) = -i/4$. Therefore $z_p = -ite^{2it}/4$ and

$$y_p = \mathcal{I}m(z_p) = -t\cos(2t)/4 \ .$$

Thus the general solution is

$$y = y_p + y_h = -t\cos(2t)/4 + c_1\cos(2t) + c_2\sin(2t)\ .$$

Putting in the initial conditions, we find $c_1 = 0$ and $c_2 = 1/8$. Hence

$$y = -t\cos(2t)/4 + \sin(2t)/8\ .$$