Math 2410, Answer to Quiz 6 (2/30/09)

(1) Given the matrix

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} .$$

Find the general solution of $d\mathbf{x}/dt = B\mathbf{x}$. Draw the phase plane with representative solution trajectories. In particular explain how you determine whether the trajectories rotate clockwise or anti-clockwise.

Answer: We have

$$B - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} .$$

Hence $det(B - \lambda I) = 0$ leads to $\lambda^2 - 2\lambda + 2 = 0$. Therefore $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$.

For $\lambda_1 = 1 + i$,

$$(B - \lambda_1 I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives the eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Hence $e^{(1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix}$ is a solution.

Since

$$e^{(1+i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^t(\cos t + i \sin t)\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

$$= e^t\{[\cos t \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \end{pmatrix}] + i[\sin t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ 1 \end{pmatrix}]\}$$

By extracting the real and imaginary parts, the general solution is

$$\mathbf{x} = c_1 e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} .$$

The origin $(0, 0)$ is a spiral source since eigenvalues are $1 \pm i$. At the point $(x, y) = (1, 0)$, we find that

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

from the governing differential equation. Since $dy/dt = -1 < 0$, the trajectory moves in the clockwise direction. Therefore, phase plane plot should show a spiral source moving in the clockwise direction.

(2) Find the general solution of $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$, where

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}.$$

Then in the phase plane, draw some representative solution trajectories.

Answer: We have

$$A - \lambda I = \begin{pmatrix} 2-\lambda & 4 \\ 3 & 6-\lambda \end{pmatrix}.$$

Hence $det(A - \lambda I) = 0$ leads to $\lambda^2 - 8\lambda = 0$. Therefore $\lambda_1 = 0$ and $\lambda_2 = 8$.

For $\lambda_1 = 0$,

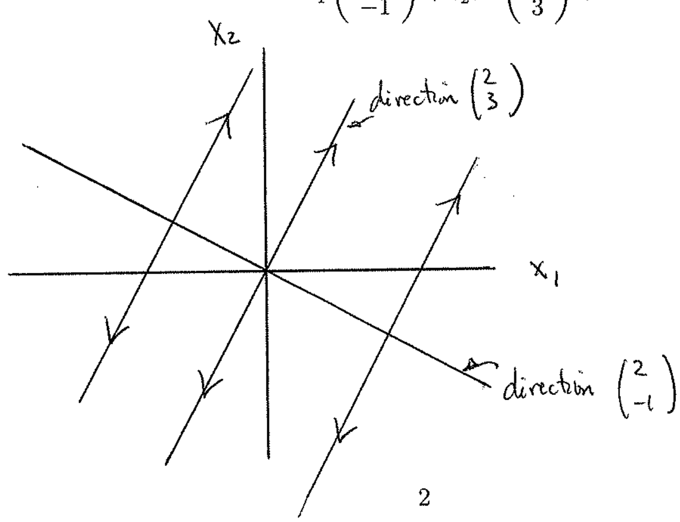
$$(A - \lambda_1 I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives the eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Similarly for $\lambda_2 = 8$, the eigenvector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

The general solution will then be

$$\mathbf{x} = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 e^{8t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$



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