Answer to Math 2410, Quiz 5 (3/23/09)

(1) Find the solution of $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where

$$A = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} .$$

Answer: We have

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{pmatrix} .$$

To find the eigenvalues of A, we set $det(A - \lambda I) = 0$. This simplifies to $\lambda^2 - 4\lambda - 12 = 0$. Hence the eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -2$.

When $\lambda_1 = 6$, we solve

$$(A - \lambda_1 I) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 5 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and get $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. This is an eigenvector for $\lambda_1 = 6$. Similarly we obtain eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $\lambda_2 = -2$.

Hence the general solution to the given differential equations is

$$\mathbf{x} = c_1 e^{6t} \begin{pmatrix} 3 \\ 5 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

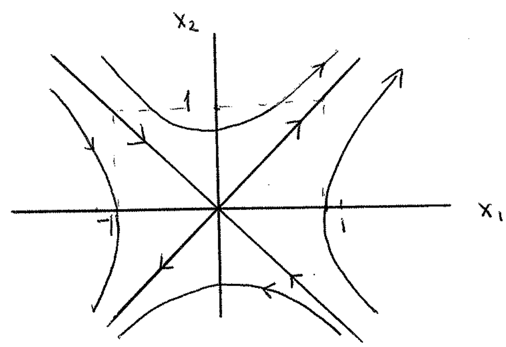
with c_1, c_2 being arbitrary constants. To satisfy the initial condition, we evaluate the above equation at $t = 0$ and obtain

$$\begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

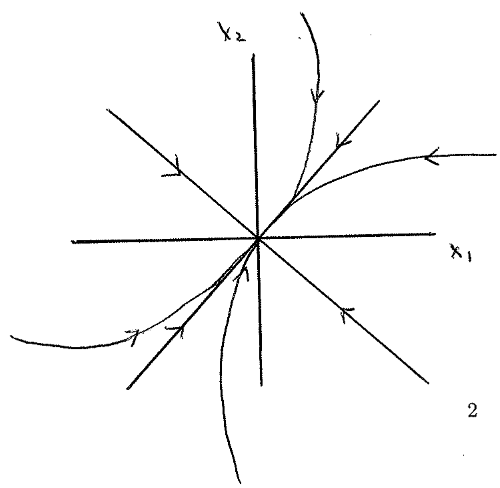
Thus $c_1 = 1/8$ and $c_2 = 5/8$. Therefore the solution is

$$\mathbf{x} = \frac{1}{8} e^{6t} \begin{pmatrix} 3 \\ 5 \end{pmatrix} + \frac{5}{8} e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

(2a) Let a 2×2 matrix A have eigenvalues $\lambda_1 = 2$ with eigenvector $(1, -1)^T$ and $\lambda_2 = -1$ with eigenvector $(1, 1)$. Sketch the phase plane for the equation $d\mathbf{x}/dt = A\mathbf{x}$ with representative solution trajectories.



(2b) Let a 2×2 matrix A have eigenvalues $\lambda_1 = -2$ with eigenvector $(1, -1)^T$ and $\lambda_2 = -1$ with eigenvector $(1, 1)$. Sketch the phase plane for the equation $d\mathbf{x}/dt = A\mathbf{x}$ with representative solution trajectories. (Note that the sign of λ_1 is changed.) Explain why the curved solution trajectories are always tangential to one of the eigenvectors.



The general solution is:

$$\vec{x} = c_1 e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$$

As $t \to \infty$, if c_1 & $c_2 \neq 0$ (this corresponds to curved solution trajectories)

then $\vec{x} \cong c_2 e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$

Hence $\vec{x}$ is parallel & tangential to $\begin{pmatrix}1\\1\end{pmatrix}$.