Math 2410, Answer to Quiz 2 (2/9/09)

There are 2 questions. All answers have to be accompanied by supporting calculation or reasoning.

(1) For the initial value problem

$$\begin{cases} \dfrac{dy}{dt} = y^2 + t + 1 \ , \\ \\ y(0) = 1 \ , \end{cases}$$

use the Euler's method with $\Delta t = 1/2$ to find the approximate solution at $t = 1$.

Answer: With $\Delta t = 1/2$, we have $t_k = 0 + k * \Delta t = k/2$. Let Y_k be the approximate solution at time t_k. We have $Y_0 = 1$ and like to find Y_2.

The Euler's method formula is $Y_{k+1} = Y_k + \Delta t \ f(t_k, Y_k)$. Hence

$$Y_1 = Y_0 + (Y_0^2 + t_0 + 1)/2 = 2 \ ,$$

$$Y_2 = Y_1 + (Y_1^2 + t_1 + 1)/2 = 2 + (4 + 1/2 + 1)/2 = 4.75 \ .$$

Hence $y(1) \approx 4.75$.

(2) It can be checked that $y_1(t) = -\frac{1}{1-t}$ and $y_2(t) = -\frac{1}{2-t}$ are solution to $dy/dt = -y^2$. Without solving for the solution explicitly, can you give an argument why the solution $y_3(t)$ to this differential equation with initial condition $y_3(0) = -3/4$ should go to $-\infty$ at a time between $t = 1$ and $t = 2$.

Answer: Since $f(t, y) = -y^2$ is nice everywhere, we can employ the existence and uniqueness theorem. In particular it says that solution trajectories do not cross one another at any point in the (t, y) plane. With $y_1(0) = -1 < y_3(0) = -\frac{3}{4} < y_2(0) = -\frac{1}{2}$, we have $y_1(t) < y_3(t) < y_2(t)$ at any time t. Since y_2 goes to $-\infty$ at $t = 2$ and y_1 goes to $-\infty$ at $t = 1$, we must have $y_3 \to -\infty$ at some t between 1 and 2.