Math 2410 Answer to Exam 2 April 22, 2009

(1) (a) (12 pts) Given the matrix

$$B = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} .$$

Find the eigenvalues and eigenvectors of B.

Answer: We have

$$B - \lambda I = \begin{pmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{pmatrix} .$$

Hence $det(B - \lambda I) = 0$ leads to $\lambda^2 - 5\lambda + 6 = 0$. Therefore $\lambda_1 = 2$ and $\lambda_2 = 3$.

For $\lambda_1 = 2$,

$$(B - \lambda_1 I) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives the eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Similarly for $\lambda_2 = 3$, the eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(b) (7 pts) Let A be another 2×2 matrix, with eigenvalues 1 and 2. Let the eigenvector corresponding to eigenvalue 1 be $(1, 2)^T$, and the eigenvector corresponding to eigenvalue 2 be $(-1, 3)^T$.

Consider the first order linear system $d\mathbf{x}/dt = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = (1, 0)^T$. Find the solution $\mathbf{x}$.

Answer: The general solution is

$$\mathbf{x} = c_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 3 \end{pmatrix} .$$

At $t = 0$,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix} ,$$

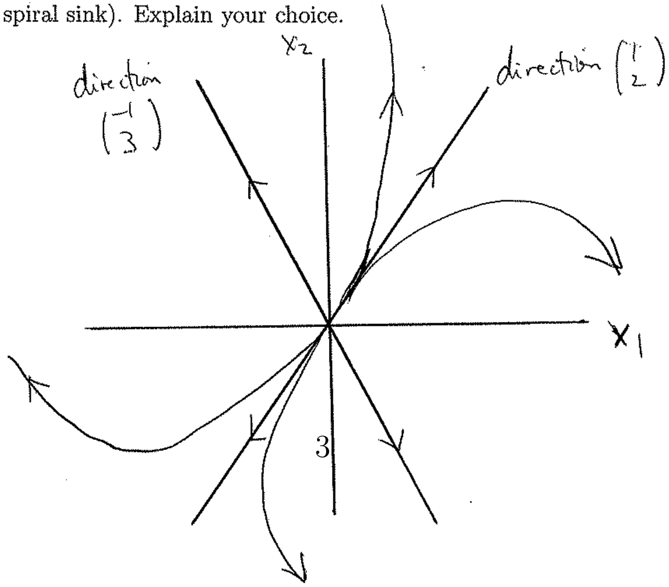
which gives $c_1 = 3/5$ and $c_2 = -2/5$. Thus

$$\mathbf{x} = \frac{3}{5}e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{2}{5}e^{2t} \begin{pmatrix} -1 \\ 3 \end{pmatrix} .$$

(b) (7 pts) Let A be another 2×2 matrix, with eigenvalues 1 and 2. Let the eigenvector corresponding to eigenvalue 1 be $(1,2)^T$, and the eigenvector corresponding to eigenvalue 2 be $(-1,3)^T$.

Consider the first order linear system $d\mathbf{x}/dt = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = (1,0)^T$. Find the solution $\mathbf{x}$.

(c) (6 pts) Sketch some solution curves for part (b) in the phase plane (i.e. the x_2 versus x_1 plane) for this system. Indicate the increasing time direction by an arrow. Classify the equilibrium solution at the origin. (i.e. is it a source, a sink, a saddle, a spiral source, or a spiral sink). Explain your choice.



Since both eigenvalues are positive, the origin is a source.

As $t \to -\infty$, $\vec{x}(t) \to 0$ and is tangential to $\begin{pmatrix}1\\2\end{pmatrix}$ direction.

As $t \to \infty$, $\vec{x}(t) \to \infty$ and is parallel to $\begin{pmatrix}-1\\3\end{pmatrix}$

(2) Given the matrix

$$B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(a) (18 pts) Find the general solution of $d\mathbf{x}/dt = B\mathbf{x}$.

Answer: We have

$$B - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} .$$

Hence $det(B - \lambda I) = 0$ leads to $\lambda^2 - 2\lambda + 2 = 0$. Therefore $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$.

For $\lambda_1 = 1 + i$,

$$(B - \lambda_1 I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which gives the eigenvector $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Hence $e^{(1+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix}$ is a solution.

Since

$$\begin{aligned} e^{(1+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} &= e^t(\cos t + i\sin t)\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\} \\ &= e^t\{[\cos t\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}] + i[\sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix}]\} \end{aligned}$$

By extracting the real and imaginary parts, the general solution is

$$\mathbf{x} = c_1 e^t\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 e^t\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} .$$

(b) (7 pts) Then draw the phase plane with a representative solution curve. Indicate the increasing time direction by an arrow.

Answer: It is a spiral source since eigenvalues are $1 \pm i$. At the point $(x, y) = (1, 0)$, we find that

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

from the governing differential equation. Since $dy/dt = -1 < 0$, the trajectory moves in the clockwise direction. Therefore, phase plane plot should show a spiral source moving in the clockwise direction.

(3a) (13 pts) Using the method of undetermined coefficients, find the solution to the equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = \cos 3t\ .$$

Answer: The homogeneous solution y_h satisfies $y_h'' + 6y_h + 8y_h = 0$. Hence $y_h = e^{\alpha t}$ if $\alpha^2 + 6\alpha + 8 = 0$. Thus $\alpha = -2$ or $\alpha = -4$. In other words,

$$y_h = c_1 e^{-2t} + c_2 e^{-4t}\ .$$

Consider $z'' + 6z' + 8z = e^{3it}$. Let a particular solution $z_p = Ae^{3it}$. Thus

$$A\{(3i)^2 + 18i + 8\} = 1$$

which simplifies to $A = -(1 + 18i)/325$. Thus

$$\begin{aligned} y_p = \mathcal{R}e(z_p) &= -\tfrac{1}{325}(1 + 18i)[\cos 3t + i\sin 3t] \\ &= -\tfrac{1}{325}(\cos 3t - 18\sin 3t) \end{aligned}$$

Thus general solution is

$$y = c_1 e^{-2t} + c_2 e^{-4t} - \frac{1}{325}(\cos 3t - 18\sin 3t)$$

(3b) (4 pts) Write down the form of a particular solution for the equation $y'' + 3y' + 2y = e^{-2t}$. You don't need to find the constant in your guess. Give an explanation on your choice.

Answer: The homogeneous solution is $y_h = c_1 e^{-t} + c_2 e^{-2t}$. Since the forcing function e^{-2t} coincides with the homogeneous solution, we need to assume a particular solution of the form $y_p = Ate^{-2t}$ for some constant A.

(3c) (6 pts) Write down the form of a particular solution for the equation $y'' + 4y' + 4y = 4e^{-2t} - \cos t$. You don't need to find the constants in your guess. Give an explanation on your choice.

Answer: The homogeneous solution is $y_h = c_1 e^{-2t} + c_2 t e^{-2t}$. We can consider separately the forcing functions e^{-2t} and $\cos t$.

The forcing function e^{-2t} coincides with the homogeneous solution, we need to assume a particular solution y_{p1} of the form $y_{p1} = At^2 e^{-2t}$ for some constant A.

For the forcing function $\cos t$, we need to assume a particular solution y_{p2} of the form $y_{p2} = B\cos t + C\sin t$.

Hence the overall form of the particular solution is:

$$y_p = y_{p1} + y_{p2} = At^2 e^{-2t} + B\cos t + C\sin t\ .$$

(4) (25 pts) Using the Laplace transform to solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}$$

with the initial condition $y(0) = 0$ and $y'(0) = 0$.

Answer: Taking the Laplace transform,

$$\mathcal{L}y = \frac{1}{(s^2 + 2s + 5)(s + 1)}$$

Since $s^2 + 2s + 5 = 0$ has complex roots, we can let

$$\frac{1}{(s^2 + 2s + 5)(s + 1)} = \frac{As + B}{s^2 + 2s + 5} + \frac{C}{s + 1}$$

thus $1 = (As + B)(s + 1) + C(s^2 + 2s + 5)$ for all s. Set $s = -1$, we find $C = 1/4$. Comparing coefficient of s^2, $0 = A + C$, so that $A = -1/4$. Comparing coefficient of constant term, $B = -1/4$. Hence

$$\mathcal{L}(y) = -\frac{1}{4}\frac{s + 1}{s^2 + 2s + 5} + \frac{1}{4}\frac{1}{s + 1} = -\frac{1}{4}\frac{s + 1}{(s + 1)^2 + 2^2} + \frac{1}{4}\frac{1}{s + 1}$$

Taking the inverse transform,

$$y = -\frac{1}{4}e^{-t}\cos 2t + \frac{1}{4}e^{-t}$$