(2) (15 pts) Solve the equation $\frac{dy}{dt} = \frac{\sin t}{1+y}$ subject to $y(0) = 1$ using the method of separation of variable. Express y explicitly as a function of t.

Answer: Separating y from t, we have

$$\int (1+y)\, dy = \int \sin t\, dt \ ,$$

which simplifies to

$$y + \frac{y^2}{2} = -\cos t + C \ .$$

By using the initial condition, $C = 5/2$. Hence

$$\frac{y^2}{2} + y + (\cos t - 5/2) = 0 \ .$$

To express y explicitly as a function of t, we solve this quadratic equation and obtain

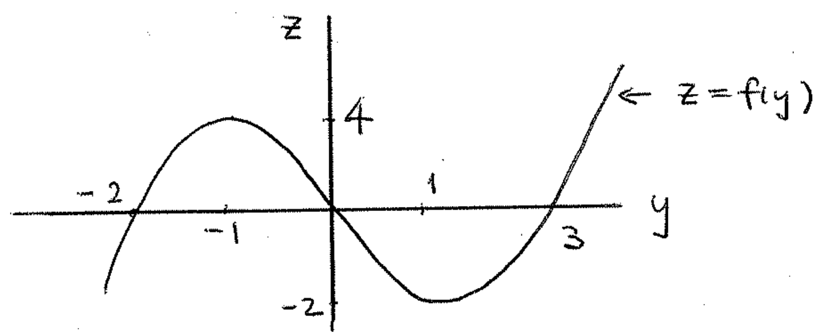
$$y = -1 + \sqrt{6 - 2\cos t} \ .$$

It is necessary to use the positive sign in the quadratic formula because we need $y(0) = 1$.

Name: ____________________

Math 2410 Exam 1 March, 4, 2008

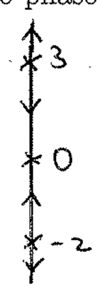
(1) Consider the equation $\frac{dy}{dt} = f(y)$, where the graph $z = f(y)$ is given in the figure below



(a) (5 pts) What are the equilibrium (steady state) solutions for the above equation?

The equilibrium solutions are $x = -2, 0, 3$, which are the roots of f.

(b) (6 pts) Draw the phase lines for the equation. Indicate which equilibrium solution is stable.



$x = -3$ is unstable

$x = 0$ is stable

$x = -2$ is unstable

(c) (4 pts) What happens as $t \to \infty$ if the initial condition is $y(0) = 1/2$?

When $y(0) = \frac{1}{2}$, from the phase line, the solution $y(t) \to 0$ as $t \to \infty$.

(4) (15 pts) Given the system

$$\begin{cases} \dfrac{dx}{dt} & = -2y \ , \\ \\ \dfrac{dy}{dt} & = x^2 + y \ . \end{cases}$$

with initial condition $x(0) = 1$ and $y(0) = -2$. Write down the general formula for the Euler's method. Use it with a step size $\Delta t = 0.5$ to approximate $y(0.5)$. What is this approximate value?

Answer: Let $t_i = i\Delta t$, $i = 0, 1, 2, \ldots$, $\vec{z} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{Z}_i = \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$ be the approximate solution to $\vec{z}(t_i)$. Then Euler's method amounts to

$$\vec{Z}_{i+1} = \vec{Z}_i + \Delta t \begin{pmatrix} -2Y_i \\ X_i^2 + Y_i \end{pmatrix}$$

with $\vec{Z}_0 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Then

$$\vec{Z}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 0.5 \begin{pmatrix} 4 \\ 1 - 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2.5 \end{pmatrix} .$$

Hence $y(0.5) \approx Y_1 = -2.5$.

(3) Consider the equation

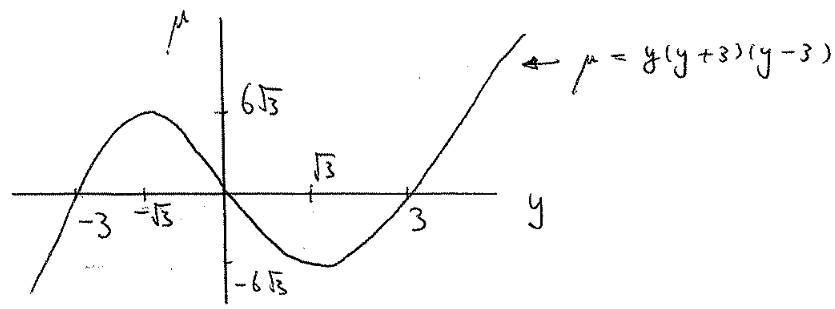
$$\frac{dy}{dt} = y(y^2 - 9) - \mu$$

(a) (9 pts) Draw the bifurcation diagram. (If necessary, use your graphing calculator.)

(b) (5 pts) Put arrow directions in representative phase lines to indicate the stability of the steady state solutions.

(c) (6 pts) At what value of μ will bifurcation occur. Explain

First, we plot $\mu = y(y^2-9) = y(y+3)(y-3)$.

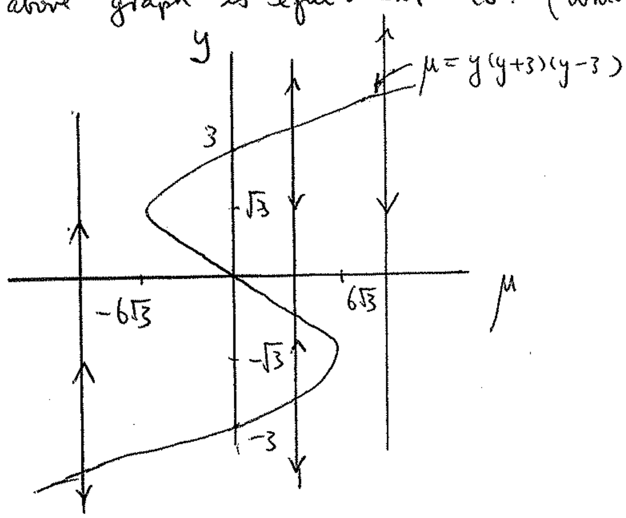


With $\frac{d\mu}{dy} = 3y^2 - 9$, we have $\frac{d\mu}{dy} = 0$ when $y = \pm\sqrt{3}$.

When $y = \sqrt{3}$, $\mu = \sqrt{3}(3-9) = -6\sqrt{3}$.

When $y = -\sqrt{3}$, $\mu = 6\sqrt{3}$.

The above graph is equivalent to: (which is the bifurcation diagram)



At both $\mu = -6\sqrt{3}$ and $\mu = 6\sqrt{3}$, bifurcation occurs because the number of equilibrium solutions change at such values of μ.

(5) (20 pts) Find the general solution to the equation

$$\frac{dy}{dt} = 3y + e^{2t} .$$

Use either the method of undetermined coefficients or the method of integrating factor.

Answer: We will use the method of undetermined coefficients. First consider the homogeneous equation $y_h' - 3y_h = 0$, which has the general solution $y_h = Ce^{3t}$ for any constant C.

Next assume a particular solution to the non-homogeneous equation be of the form $y_p = Ae^{2t}$. By putting it into the equation, we have

$$2Ae^{2t} = 3Ae^{2t} + e^{2t} .$$

This simplifies to $A = -1$ so that $y_p = -e^{2t}$.

General solution of the non-homogeneous equation is then

$$y = y_h + y_p = Ce^{3t} - e^{2t} .$$

(6) (15 pts) Find the general solution of

$$y'' - 2y' + 5y = 0 .$$

Answer: It is known that $y = e^{\alpha t}$ is a solution provided $\alpha^2 - 2\alpha + 5 = 0$. This quadratic equation has roots $\alpha = 1 \pm 2i$. In other words, $e^{(1+2i)t} = e^t\{\cos 2t + i \sin 2t\}$ is a solution. The general solution will be

$$y = C_1 e^t \cos 2t + C_2 e^t \sin 2t .$$