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Math 211 Exam 3 May, 3, 1999

(1) (20 pts) Using Laplace transform to solve

$$y'' + y = \sin(3t)$$

subject to initial conditions $y(0) = 1$ and $y'(0) = 0$.

$$\mathcal{L}[y'' + y] = \mathcal{L}[\sin(3t)]$$

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + \mathcal{L}[y] = \frac{3}{s^2+9}.$$

$$(s^2+1)\mathcal{L}[y] - s = \frac{3}{s^2+9}$$

$$\mathcal{L}[y] = \frac{s}{s^2+1} + \frac{3}{(s^2+1)(s^2+9)} = \mathcal{L}[\cos t] + \frac{3}{(s^2+1)(s^2+9)} \quad ①$$

Now, perform partial fraction,

$$\frac{3}{(s^2+1)(s^2+9)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9}$$ with real constants A, B, C, D.

$$3 = (As+B)(s^2+9) + (Cs+D)(s^2+1).$$

set $s = i$, $3 = 8(Ai+B)$, $\therefore A = 0$ & $B = \frac{3}{8}$.

set $s = 3i$, $3 = (3iC+D)(-8)$, $\therefore C = 0$, $D = \frac{-3}{8}$.

So, $$\frac{3}{(s^2+1)(s^2+9)} = \frac{3}{8}\cdot\frac{1}{s^2+1} - \frac{1}{8}\cdot\frac{3}{s^2+9}.$$

$$\mathcal{L}^{-1}\left[\frac{3}{(s^2+1)(s^2+9)}\right] = \frac{3}{8}\sin t - \frac{1}{8}\sin 3t \quad ②$$

By ① & ②, $y = \cos t + \frac{3}{8}\sin t - \frac{1}{8}\sin 3t.$

(2) (20 pts) Find the general solution of problem (1). Then find the solution corresponding to the given initial conditions there. [check that both answers are the same.]

$$y = y_c + y_p \leftarrow \text{particular soln.}$$

complementary soln

where: $y_c'' + y_c = \cancel{\sin(3t)}\ 0$. Let $y_c = e^{\alpha t}$. $\therefore \alpha^2 + 1 = 0$

$\alpha = \pm i$.

$\therefore y_c = e^{it}$ is a soln

$= \cos t + i \sin t$

$\therefore$ general soln to homogeneous eqn, $y_c = c_1 \cos t + c_2 \sin t$.

The forcing term $\sin(3t)$ does not match the complementary soln.

$\therefore$ ~~let $y_p = A$~~ let $y_p = A \sin 3t + B \cos 3t$.

$\therefore$ Put into the eqn,

$$A\{(-3)^2 + 1\} \sin(3t) + B\{(-3)^2 + 1\} \cos 3t = \sin 3t.$$

$$-8A \sin(3t) - 8B \cos 3t = \sin 3t.$$

$\therefore B = 0$ & $A = -\frac{1}{8}$. $\qquad \therefore y_p = -\frac{1}{8} \sin 3t$.

So, general soln to non-homogeneous eqn is:

$$y = c_1 \cos t + c_2 \sin t - \frac{1}{8} \sin(3t).$$

$$y' = -c_1 \sin t + c_2 \cos t - \frac{3}{8} \cos(3t)$$

$y(0) = 1 = c_1$.

$y'(0) = 0 = c_2 - \frac{3}{8}$, $\qquad \therefore c_2 = \frac{3}{8}$.

soln is:

$\therefore y = \cos t + \frac{3}{8} \sin t - \frac{1}{8} \sin 3t$

which checks.

(3) (25 pts) Find the general solution of

$$y'' - 3y' + 2y = e^t + 1$$

$$y = y_c + y_p$$

where $y_c'' - 3y_c' + 2y_c = 0$. $\therefore y_c = e^{\alpha t}$ if $\alpha^2 - 3\alpha + 2 = 0$

$(\alpha-1)(\alpha-2)=0$

$\therefore \alpha = 1$ or 2.

$y_c = c_1 e^t + c_2 e^{2t}$.

For forcing term $f(t) = e^t$, since it matches with y_c, $\therefore$ let $y_{p_1} = Ate^t$.

For forcing term $f(t) = 1$, let $y_{p_2} = B$.

ie let $y_p = y_{p_1} + y_{p_2} = Ate^t + B$.

Put into the eqn,

$$A\left\{(te^t)'' - 3(te^t)' + 2te^t\right\} + 2B = e^t + 1$$

$$-Ae^t + 2B = e^t + 1$$

$$\therefore A = -1 \ \& \ B = \frac{1}{2}.$$

so, $y_p = -te^t + \frac{1}{2}$.

general soln :

$$y = y_c + y_p = c_1 e^t + c_2 e^{2t} - te^t + \frac{1}{2}.$$

(4) (20 pts) Using Laplace transform, find the solution of

$$y' + y = u_2(t)e^{(t-2)}$$

with $y(0) = 0$. Here $u_2(t)$ is the usual step function, i.e., $u_2(t) = 0$ when $t \le 2$, and $u_2(t) = 1$ when $t > 2$.

Recall : $\begin{cases} \mathcal{L}[u_a(t) f(t-a)] = e^{-as} \mathcal{L}[f(t)]. \\ \mathcal{L}[e^{at}] = \frac{1}{s-a} \end{cases}$

$$\therefore \quad \mathcal{L}[y'+y] = \mathcal{L}[u_2(t) e^{(t-2)}]$$

$$s\mathcal{L}[y] - y(0) + \mathcal{L}[y] = e^{-2s} \mathcal{L}[e^t]$$

$$(s+1)\mathcal{L}[y] = \frac{e^{-2s}}{s-1}$$

$$\mathcal{L}[y] = \frac{1}{(s+1)(s-1)} e^{-2s}. \qquad (1)$$

Perform partial fraction on :

$$\frac{1}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}.$$

$\therefore$ $A(s-1) + B(s+1) = 1$ for all s.

set $s=1$, $\therefore$ $2B = 1$ so $B = \frac{1}{2}$.

set $s=-1$, $\therefore$ $-2A = 1$, so, $A = -\frac{1}{2}$

$$\therefore \frac{1}{(s+1)(s-1)} = -\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-1}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s-1)}\right] = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t}. \equiv g(t)$$

Hence, $\mathcal{L}^{-1}\left[\frac{e^{-2s}}{(s+1)(s-1)}\right] = u_2(t) \cdot g(t-2) = u_2(t)\left\{-\frac{1}{2}e^{-(t-2)} + \frac{1}{2}e^{(t-2)}\right\}$ (2)

From (1) & (2), $y = u_2(t)\left\{-\frac{1}{2}e^{-(t-2)} + \frac{1}{2}e^{(t-2)}\right\}.$

(5) (4 pts) (a) Show that

$$\frac{s^2-\omega^2}{(s^2+\omega^2)^2} = \frac{1}{(s^2+\omega^2)} - 2\omega^2\frac{1}{(s^2+\omega^2)^2}$$

Not needed in Exam II

(11 pts) (b) Using results in part (a), and the result listed in tables of Laplace transform, find the inverse Laplace transform of $\frac{1}{(s^2+\omega^2)^2}$.

Not needed in Exam II.