Name: ____________________

Math 211 Exam 2 Nov, 30, 2000

(1) Consider the system of equations:

$$\frac{dx}{dt} = y\,,$$
$$\frac{dy}{dt} = x + t\,,$$

with initial conditions $x(0) = 1$ and $y(0) = 0$.

(a) (16 pts) Use Euler's method with $\Delta t = 0.1$ to find the approximate solution for x and y at $t = 0.2$.

Not needed for

Exam II

(b) (4 pts) If 2 different trajectory starting from $t = 0$ with different initial conditions, can they intersect one another in the x_2 versus x_1 plane (phase plane)? Will that violate the existence and uniqueness theorem? Explanation is absolutely necessary.

Not needed for

Exam II

(c) (5 pts) Consider $u'' - u = t$ with $u(0) = 1$ and $u'(0) = 0$. Explain how you will find the approximate solution u at $t = 0.2$ using $\Delta t = 0.1$. What is this approximate value for $u(0.2)$? (Hint: part (c) is related to part (a)).

Not needed for

Exam II.

(2) (a) (13 pts) Given the matrix

$$B = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of B.

$$B - \lambda I = \begin{pmatrix} 1-\lambda & 3 \\ 5 & 3-\lambda \end{pmatrix}.$$

$\det(B-\lambda I) = 0$ when $(1-\lambda)(3-\lambda) - 15 = 0$

$\lambda^2 - 4\lambda - 12 = 0$, $\therefore (\lambda-6)(\lambda+2) = 0$

so, $\lambda_1 = 6$ & $\lambda_2 = -2$.

When $\lambda_1 = 6$, $(B-\lambda I)\vec{x} = \vec{0}$ gives:

$$\begin{pmatrix} -5 & 3 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$ is an eigenvector for $\lambda_1 = 6$.

When $\lambda_2 = -2$,

$$\begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \therefore \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ is an eigenvector for $\lambda_2 = -2$.

(b) (6 pts) Let A be another 2×2 matrix, with eigenvalues -6 and -1. Let the eigenvector corresponding to eigenvalue -6 be $(2,7)^T$, and the eigenvector corresponding to eigenvalue -1 be $(1,1)^T$.

Consider the first order linear system $d\mathbf{x}/dt = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = (0,5)^T$. Find the solution $\mathbf{x}$.

general soln: $\vec{x} = c_1 e^{-6t}\begin{pmatrix}2\\7\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$

At $t=0$, $\vec{x}(0) = \begin{pmatrix}0\\5\end{pmatrix} = c_1\begin{pmatrix}2\\7\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}$

or $\begin{pmatrix}2 & 1\\7 & 1\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\5\end{pmatrix}$.

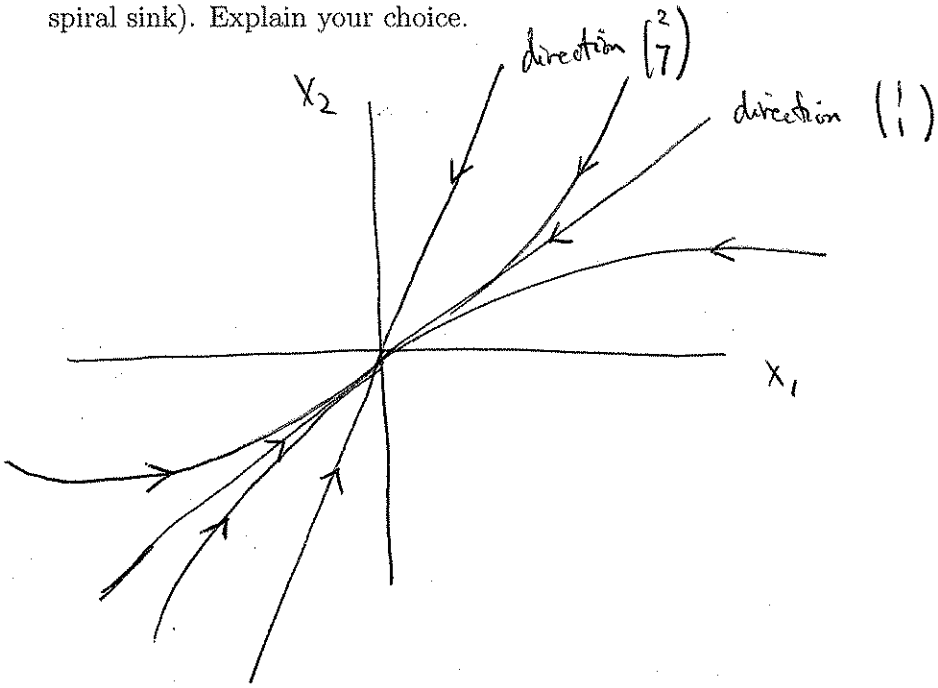
Solve it in any way you prefer, e.g.

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}2 & 1\\7 & 1\end{pmatrix}^{-1}\cdot\begin{pmatrix}0\\5\end{pmatrix} = \frac{1}{\det\begin{pmatrix}2 & 1\\7 & 1\end{pmatrix}}\begin{pmatrix}1 & -1\\-7 & 2\end{pmatrix}\begin{pmatrix}0\\5\end{pmatrix}$$

$$= \frac{1}{-5}\cdot 5\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}.$$

$\therefore$ soln is: $\vec{x} = e^{-6t}\begin{pmatrix}2\\7\end{pmatrix} - 2e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$.

(c) (6 pts) Sketch some solution curves for part (b) in the phase plane (i.e. the x_2 versus x_1 plane) for this system. Indicate the increasing time direction by an arrow. Classify the equilibrium solution at the origin. (i.e. is it a source, a sink, a saddle, a spiral source, or a spiral sink). Explain your choice.



Typical solution curves are tangential to direction $\begin{pmatrix}1\\1\end{pmatrix}$ as $t\to\infty$.

The equilibrium point $(0,0)$ is a sink, as both eigenvalues are negative.

(3) Consider the equation $y'' + 6y' + 10y = 26e^{2t}$.
(a) (14 pts) Find its general solution.

$$y = \underbrace{y_c}_{\text{Complementary soln}} + \underbrace{y_p}_{\text{particular soln.}}$$

where $y_c'' + 6y_c' + 10y_c = 0$. $\therefore y_c = e^{\alpha t}$ if $\alpha^2 + 6\alpha + 10 = 0$

$$\therefore \alpha = \frac{-6 \pm \sqrt{36 - 40}}{2} = -3 \pm i.$$

$\therefore y_c = e^{(-3+i)t} = e^{-3t}(\cos t + i \sin t)$ is a soln.

general soln for the homogeneous eqn is:

$$y_c = c_1 e^{-3t} \cos t + c_2 e^{-3t} \sin t.$$

Since the forcing term $26e^{2t}$ does not match the complementary soln, so, we can assume:

$$y_p = A e^{2t}.$$

Put into the non-homogeneous eqn,

$$A\{2^2 + 6 \cdot 2 + 10\} = 26.$$

$$26A = 26, \quad \therefore A = 1. \quad \& \quad y_p = e^{2t}$$

Hence, $y = \cancel{y_c} + y_p = c_1 e^{-3t} \cos t + c_2 e^{-3t} \sin t + e^{2t}$.

is the general soln to the non-homogeneous eqn.

(b) (7 pts) If $y(0) = 2$, and $y'(0) = -1$, what is the solution for this problem.

$$y = C_1 e^{-3t}\cos t + C_2 e^{-3t}\sin t + e^{2t}$$

$$y' = C_1\{-3e^{-3t}\cos t - e^{-3t}\sin t\} + C_2\{-3e^{-3t}\sin t + e^{-3t}\cos t\} + 2e^{2t}$$

$$\therefore\ y(0) = 2 = C_1 + 1 \quad , \quad \therefore\ C_1 = 1$$

$$y'(0) = -1 = -3C_1 + C_2 + 2 \quad , \quad \therefore\ C_2 = 0$$

Thus, $y = e^{-3t}\cos t + e^{2t}$.

(c) (4 pts) Suppose that $y_1(t)$ is a solution of the equation $y'' + ay' + by = g(t)$ and $y_2(t)$ is a solution of another equation $y'' + ay' + by = h(t)$. Show that $y_1(t) + y_2(t)$ is a solution of the equation $y'' + ay' + by = g(t) + h(t)$.

Define $y = y_1 + y_2$.

$$\therefore\ y'' + ay' + by = (y_1 + y_2)'' + a(y_1 + y_2)' + b(y_1 + y_2)$$

$$= (y_1'' + ay_1' + by_1) + (y_2'' + ay_2' + y_2)$$

$$= g(t) + h(t).$$

Hence $y = y_1 + y_2$ is a soln to $y'' + ay' + by = g(t) + h(t)$

(4) Given the matrix

$$B = \begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix}$$

(a) (13 pts) Find its eigenvalues and eigenvectors.

$$B - \lambda I = \begin{pmatrix} 2-\lambda & 4 \\ -1 & 2-\lambda \end{pmatrix}$$

$\det(B-\lambda I) = 0$ when $(2-\lambda)^2 + 4 = 0$

$\therefore \lambda = 2 \pm 2i$.

When $\lambda = 2+2i$, $(B - \lambda I)\vec{x} = \vec{0}$ gives:

$$\begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

set $x_2 = 1$,

$\therefore x_1 = -2i$

so $\begin{pmatrix} -2i \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda = 2+2i$.

When $\lambda = 2-2i$, its eigenvector is always the conjugate of the above eigenvector.

$\therefore$ the eigenvector is: $\overline{\begin{pmatrix} -2i \\ 1 \end{pmatrix}} = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$.

(b) (8 pts) Let A be another 2×2 matrix with complex eigenvalues $4+i$ and $4-i$. Assume the eigenvector corresponding to eigenvalue $4+i$ is $\begin{pmatrix} 1 \\ 2-i \end{pmatrix}$.

Consider the first order linear system $d\mathrm{x}/dt = A\mathrm{x}$. Find its general solution.

A soln is $\vec{x} = e^{(4+i)t} \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$

$$= e^{4t} e^{it} \begin{pmatrix} 1 \\ 2-i \end{pmatrix}$$

$$= e^{4t} (\cos t + i \sin t) \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}$$

$$= e^{4t} \left\{ \left[\cos t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] + i \left[\sin t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \right\}$$

$$= e^{4t} \left\{ \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix} \right\}$$

$\therefore$ general soln is:

$$\vec{x} = c_1 e^{4t} \begin{pmatrix} \cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} \sin t \\ 2\sin t - \cos t \end{pmatrix}.$$

(c) (4 pts) Continue from part (b), sketch the trajectory starting from $(x_1, x_2) = (1, 0)$ in the phase plane. Indicate the increasing time direction by an arrow. (Let

$$A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}.$$

in part (b), this may help you to determine the increasing time direction.)

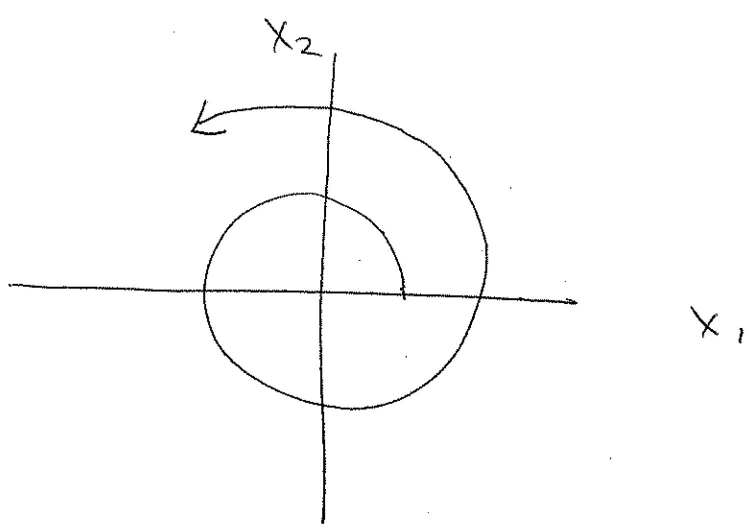
One can determine the spiral direction even without the hint that $A = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}$ & the eqns are $\frac{d\vec{x}}{dt} = A\vec{x}$.

But we'll use the hint. At $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\vec{x} = \begin{pmatrix} 6 & -1 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

$$\therefore \frac{dx_2}{dt} = 5 > 0.$$

Hence it is anti-clockwise spiral.



It is a spiral source because of the term e^{4t}.