1. Let $\{\mathcal{F}(t)\}$ be the filtration generated by a Brownian motion $W(t)$ and assume the existence of a risk-neutral measure $\tilde{\mathbb{P}}$ and a risk-free discount process $D(t)$ and interest rate process $R(t)$ with $dD(t) = -R(t)D(t)dt$. Let $V$ be an $\mathcal{F}(T)$-measurable random variable with $V > 0$ almost surely. Show that there is a generalized geometric Brownian motion $V(t)$ adapted to $\{\mathcal{F}(t)\}$ with $V(T) = V$. In other words, all strictly positive payoffs come from generalized geometric Brownian motions.

**Solution 1**  
By the definition of a risk-neutral measure, $\tilde{\mathbb{E}}[D(T)V \mid \mathcal{F}(t)]$ is a $\tilde{\mathbb{P}}$ martingale. Define $V(t)$ by $V(t) = \tilde{\mathbb{E}} \left[ -\int_t^T R(u)du \mid \mathcal{F}(t) \right] V$. Notice that according to this definition $V(t) > 0$ almost surely and $V(T)$ is defined by $V(T) = \tilde{\mathbb{E}}[V \mid \mathcal{F}(T)] = V\tilde{\mathbb{E}}[1 \mid \mathcal{F}(T)] = V$ (taking out what’s known.) $D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T) \mid \mathcal{F}(t)]$ is another expression for the $\tilde{\mathbb{P}}$ martingale. By the martingale representation theorem there is an adapted process $\tilde{\Gamma}(t)$ where $d(D(t)V(t)) = \tilde{\Gamma}(t) d\tilde{W}(t)$ for $\tilde{W}(t)$ Brownian under $\tilde{\mathbb{P}}$. But Itô’s product formula says that $d(D(t)V(t)) = dD(t)V(t) + D(t)dV(t) + dD(t)dV(t)$. Equating the two expressions for $d(D(t)V(t))$
gives

\[ dD(t)V(t) + D(t)dV(t) + dD(t)dV(t) = \tilde{\Gamma}(t)d\tilde{W}(t) \]

\[ dV(t) = -\frac{dD(t)}{D(t)} (V(t) + dV(t)) + \frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t) \]

\[ = \frac{dD(t)}{D(t)} (V(t) + dV(t)) dt + \frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t) \]

a solution for which is

\[ dV(t) = \frac{dD(t)}{D(t)} (V(t) + dV(t)) dt + \frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t) \]

because \( dt dt = 0 \) and \( dtd\tilde{W}(t) = 0 \)

This can be rewritten as \( dV(t) = \frac{dD(t)}{D(t)} (V(t) + dV(t)) dt + \frac{\tilde{\Gamma}(t)}{D(t)}d\tilde{W}(t) \)

because \( V(t) > 0 \) almost surely.

Therefore, \( V(t) \) is a generalized \( \tilde{P} \) Brownian motion with drift process \( R(t) \) and volatility process \( \frac{\tilde{\Gamma}(t)}{D(t)V(t)} \). From the Girsanov Theorem, \( d\tilde{W}(t) = \Theta(t)dt + dW(t) \) for a market price of risk process \( \Theta(t) \) so \( dV(t) = (R(t) + \Theta(t))V(t)dt + \frac{\tilde{\Gamma}(t)}{D(t)V(t)}V(t)dW(t) \), which shows that \( V(t) \) is a generalized \( P \) Brownian motion with drift process \( R(t) + \Theta(t) \) and volatility process \( \frac{\tilde{\Gamma}(t)}{D(t)V(t)} \).

2. In the situation of question 1, assume that the risk-neutral measure \( \tilde{P} \) is equal to \( \tilde{P}_{S_1} \), the risk-neutral measure derived from some generalized geometric Brownian motion \( S_1(t) \) such that \( D(t)S_1(t) \) is a \( \tilde{P}_{S_1} \)-martingale. Let \( \tilde{P}_{S_2} \) be the risk-neutral measure derived from some other generalized geometric Brownian motion \( S_2(t) \) so that \( D(t)S_2(t) \) is a \( \tilde{P}_{S_2} \)-martingale. Show that \( \tilde{P}_{S_1}(A) = \tilde{P}_{S_2}(A) \) for all sets \( A \in \mathcal{F} \). In other words, the risk-neutral measure is unique.

**Solution 2** Let \( dS_1(t) = \alpha_1(t)S_1(t)dt + \sigma_1(t)S_1(t)dW(t) \) and \( dS_2(t) = \alpha_2(t)S_2(t)dt + \sigma_2(t)S_2(t)dW(t) \) be the two generalized geometric Brownian motions. Since \( \mathcal{F} = \bigcup_t \mathcal{F}(t) \), for any \( A \in \mathcal{F} \), there is a \( t \) such that
3. If $S(t) = S(0)e^{\int_0^t \nu(u)du + \int_0^t \sigma(u)dW(u)}$ is a martingale, what is the relationship between $\nu(t)$ and $\sigma(t)$? Prove it.
Solution 3  Let $X(t) = \int_0^t \nu(u)du + \int_0^t \sigma(u)dW(u)$, so $S(t) = S(0)e^{X(t)}$.

By the Itô Lemma

$$dS(t) = S(0) e^{X(t)} dX(t) + \frac{1}{2} S(0) e^{X(t)} dX(t) dX(t)$$

$$= S(t) \left\{ \nu(t) dt + \sigma(t) dW(t) + \frac{1}{2} (\nu(t) dt + \sigma(t) dW(t)) (\nu(t) dt + \sigma(t) dW(t)) \right\}$$

$$= S(t) \left\{ (\nu(t) + \frac{1}{2} \sigma^2(t)) dt + \sigma(t) dW(t) \right\}$$

So $S(t)$ is a martingale only if $\nu(t) = -\frac{1}{2} \sigma^2(t)$, making the $dt$ term equal to 0.

4. If $M(t) = \int_0^t h(u)dN(u)$ where $N(t)$ is the semi-martingale with $dN(t) = \alpha(t)dt + \beta(t)dW(t)$, $g(t)$ is an adapted process, and $[M, M](t)$ is the quadratic variation of $M(t)$, what is the simplest expression for $\int_0^t g(u)d[M, M](u)$ in terms of $g(t)$, $h(t)$, $\alpha(t)$, $\beta(t)$, and $W(t)$?

Solution 4

$$\int_0^t g(u)d[M, M](u) = \int_0^t g(u)dM(u)dM(u)$$

$$= \int_0^t g(u)h(u)dN(u)h(u)dN(u)$$

$$= \int_0^t g(u)h^2(u) (\alpha(u)du + \beta(u)dW(u)) (\alpha(u)du + \beta(u)dW(u))$$

$$= \int_0^t g(u)h^2(u) \beta^2(u)dW(u)W(u)$$

$$= \int_0^t g(u)h^2(u) \beta^2(u)du$$

5. Use stochastic calculus to derive a formula for $E[W^6(t)]$ assuming only that you know that $dW(t)dW(t) = dt$ and the usual rules of stochastic calculus.
Solution 5

\[ dW^2(t) = 2W(t)dW(t) + dW(t)dW(t) \]
\[ W^2(t) = 2 \int_0^t W(u)dW(u) + \int_0^t du \]
\[ \mathbb{E}[W^2(t)] = t \text{ because the first integral is a martingale.} \]
\[ dW^4(t) = 4W^3(t)dW(t) + 6W^2(t)dW(t)dW(t) \]
\[ W^4(t) = 4 \int_0^t W^3(u)dW(u) + 6 \int_0^t W^2(u)du \]
\[ \mathbb{E}[W^4(t)] = 6 \int_0^t \mathbb{E}[W^2(u)] du \text{ because the first integral is a martingale.} \]
\[ = 6 \int_0^t udu = 3t^2 \]
\[ dW^6(t) = 6W^5(t)dW(t) + 15W^4(t)dW(t)dW(t) \]
\[ W^6(t) = 6 \int_0^t W^5(u)dW(u) + 15 \int_0^t W^4(u)du \]
\[ \mathbb{E}[W^6(t)] = 15 \int_0^t \mathbb{E}[W^4(u)] du \text{ because the first integral is a martingale.} \]
\[ = 15 \int_0^t 3u^2 du = 15t^3 \]

6. If \( B_1(t) \) and \( B_2(t) \) are Brownian motions with \( dB_1(t)dB_2(t) = \rho(t)dt \) for an adapted process \(-1 < \rho(t) < 1\) find an adapted process \( W(t) \) that is a Brownian motion independent of \( B_1(t) \).

Solution 6 Define \( W(t) \) by

\[ W(0) = 0 \]
\[ dW(t) = \frac{dB_2(t) - \rho(t)dB_1(t)}{\sqrt{1 - \rho^2(t)}}. \text{ Then} \]
\[ dW(t)dW(t) = \frac{(dB_2(t) - \rho(t)dB_1(t))(dB_2(t) - \rho(t)dB_1(t))}{1 - \rho^2(t)} \]
\[ = \frac{dB_2(t)dB_2(t) - 2\rho(t)dB_1(t)dB_2(t) + \rho^2(t)dB_1(t)dB_1(t)}{1 - \rho^2(t)} \]
\[ = \frac{dt - 2\rho(t)\rho(t)dt + \rho^2(t)dt}{1 - \rho^2(t)} \]
\[ = \frac{1 - \rho^2(t)}{1 - \rho^2(t)} dt = dt \]
and by Lévy’s Theorem $W(t)$ is Brownian.

\[

dW(t)dB_1(t) = \frac{dB_2(t) - \rho(t)dB_1(t)}{\sqrt{1 - \rho^2(t)}}dB_1(t)
\]

\[
= \frac{dB_2(t)dB_1(t) - \rho(t)dB_1(t)dB_1(t)}{\sqrt{1 - \rho^2(t)}}
\]

\[
= \frac{\rho(t)dt - \rho(t)dt}{\sqrt{1 - \rho^2(t)}} = 0
\]

so $W(t)$ is independent of $B_1(t)$.