Regime-Switching Interest Rate Models With Randomized Regimes

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University of Connecticut

Actuarial Science Seminar Jan. 29, 2008
Introduction

- Work in progress on cash flow testing interest rate models
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- Empirical Work In Valuation Actuary Practice (1990’s):

  - Unconstrained lognormal models have too much tail
  - Mean-reverting ones have too little shoulder
  - Randomizing the reversion target fixes it
  - Trial and error calibration

2001 Valuation Actuary Symposium Proceedings

Theoretical Work (2006):

  - Closed form calibration for the mean-reverting lognormal
  - A surprising drift formula for the mean-reverting lognormal
  - Couldn’t get closed form calibration with randomized targets

ARCH 2007.1

More Recent Results (2007):

  - Asymptotic closed form calibration with randomized targets
  - Interesting probability results/techniques

ARCH 2008.1

This year? Numerical examples and extensions
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Bridgeman (University of Connecticut) Random Regimes
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- This year? Numerical examples and extensions
Example: 55 Years of the 10-year Treasury Rate
The Distribution of those Interest Rates

FREQUENCY OF 10 YEAR RATES

DATA

LOGNORMAL
Lognormal 4th Moment Is Just Too High (6th too)
MONTHLY LOG-CHANGE IN 10 YEAR RATE

-0.2 -0.15 -0.1 -0.05 0 0.05 0.1 0.15 0.2
What is the Distribution of Those Changes?

FREQUENCY OF MONTHLY LOG-CHANGE IN 10 YEAR RATES

DATA GAUSSIAN
For Rate Changes, Lognormal 4th Moment Too Low
The Fix: Randomize the Reversion Target

50 YEAR SAMPLE PATH (A DANGEROUS ONE)

PATH
TARGET

Bridgeman (University of Connecticut) Random Regimes
Lognormal Models

- Unconstrained:

\[
\begin{align*}
\text{Unconstrained:} \\
\ln(r_t) &= D_t dt + \sigma dW_t
\end{align*}
\]
Lognormal Models

- Unconstrained:
  \[ d \ln (r_t) = D_t \, dt + \sigma \sqrt{dt} \, N_t \]
Lognormal Models

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  \[ d \ln (r_t) = D_t dt + \sigma \sqrt{dt} N_t \]
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Lognormal Models

- Unconstrained:
  - $d \ln(r_t) = D_t dt + \sigma \sqrt{dt} N_t$
  - $d \ln(r_t) = D_t dt + \sigma dW_t$

- Mean-reverting:

actuarial folklore (circa 1970)

Black-Karasinski (1991)

With Randomized Reversion Target

$\sum_{j=0}^{\infty} I_{t_j, t_{j+1}} (t) \ln(T_j) \ln(r_t dt) + \int \sigma dW_t,$

where $I_{t_j, t_{j+1}} (t)$ is the indicator for $t$ to be in a random interval $t_j, t_{j+1}$. 
Lognormal Models

- **Unconstrained:**
  
  \[ d \ln(r_t) = D_t \, dt + \sigma \sqrt{dt} \mathcal{N}_t \]
  
  \[ d \ln(r_t) = D_t \, dt + \sigma dW_t \]

- **Mean-reverting:**
  
  \[ d \ln(r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \ln(T_0) - \ln(r_{t-}dt) \right] + (1 - F)^{dt} D_t \, dt + (1 - F)^{dt} \sigma \sqrt{dt} \mathcal{N}_t \]

  *actuarial folklore (circa 1970)*
Lognormal Models

- **Unconstrained:**
  
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  actuarial folklore (circa 1970)

  \[ d \ln (r_t) = \left\{ - \ln (1 - F) \left[ \ln (T_0) - \ln (r_t) \right] + D_t \right\} \, dt + \sigma \, dW_t \]

  Black-Karasinski (1991)
Lognormal Models

- **Unconstrained:**
  \[
  d \ln (r_t) = D_t \, dt + \sigma \sqrt{d t} \, N_t
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  \[
  d \ln(r_t) = D_t \, dt + \sigma \, dW_t
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  \]
  \[
  + (1 - F)^{d t} D_t \, dt + (1 - F)^{d t} \sigma \sqrt{d t} \, N_t
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  \[
  d \ln(r_t) = \{- \ln(1 - F) \left[ \ln(T_0) - \ln(r_t) \right] + D_t \} \, dt + \sigma \, dW_t
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  Black-Karasinski (1991)

- **With Randomized Reversion Target**
Lognormal Models

- Unconstrained:
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  \[ d \ln(r_t) = \left\{ - \ln (1 - F) \left[ \ln(T_0) - \ln(r_t) \right] + D_t \right\} \, dt + \sigma \, d\mathbf{W}_t \]
  
  Black-Karasinski (1991)

- With Randomized Reversion Target
  \[ d \ln(r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \sum_{j=0}^{\infty} 1_{[j, j+1)}(t) \ln(T_j) - \ln(r_{t-dt}) \right] \]
  \[ + (1 - F)^{dt} \, D_t \, dt + (1 - F)^{dt} \sigma \sqrt{dt} \, \mathbf{N}_t, \text{ where } 1_{[j, j+1)}(t) \text{ is} \]
  \[ \text{the indicator for } t \text{ to be in a random interval } [t_j, t_{j+1}) \]
Lognormal Models

- Mean-reverting:

\[
d\ln(r_t) = \frac{1}{2} \left( \frac{1}{F} \right) dt + \sigma dW_t
\]
Lognormal Models

- Mean-reverting:

\[ d \ln (r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \ln(T_0) - \ln(r_{t-dt}) \right] + (1 - F)^{dt} D_t dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t \]

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Lognormal Models

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- \[ d \ln(r_t) = \{- \ln(1 - F) \left[ \ln(T_0) - \ln(r_t) \right] + D_t \} dt + \sigma dW_t \]
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- With Randomized Reversion Target
Lognormal Models

- Mean-reverting:
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- Black-Karasinski (1991)

  \[
  d \ln(r_t) = \left\{ - \ln(1 - F) \left[ \ln(T_0) - \ln(r_t) \right] + D_t \right\} dt + \sigma dW_t
  \]

- With Randomized Reversion Target

  \[
  d \ln(r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \sum_{j=0}^{\infty} \mathbf{1}_{[j,j+1)}(t) \ln(T_j) - \ln(r_{t-dt}) \right] + (1 - F)^{dt} D_t dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t, \text{ where } \mathbf{1}_{[j,j+1)}(t) \text{ is the indicator for } t \text{ to be in a random interval} \ [t_j, t_{j+1}]
  \]
Lognormal Models

- **Mean-reverting:**
  
  \[ d \ln(r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \ln(T_0) - \ln(r_{t-dt}) \right] \]
  
  + \( (1 - F)^{dt} D_t dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t \)

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  \[ d \ln(r_t) = \left\{ - \ln(1 - F) \left[ \ln(T_0) - \ln(r_t) \right] + D_t \right\} dt + \sigma dW_t \]

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- **With Randomized Reversion Target**
  
  \[ d \ln(r_t) = \left[ 1 - (1 - F)^{dt} \right] \left[ \sum_{j=0}^{\infty} 1_{[j,j+1)}(t) \ln(T_j) - \ln(r_{t-dt}) \right] \]

  + \( (1 - F)^{dt} D_t dt + (1 - F)^{dt} \sigma \sqrt{dt} N_t \), where \( 1_{[j,j+1)}(t) \) is the indicator for \( t \) to be in a random interval \([t_j, t_{j+1})\)

  \[ d \ln(r_t) = - \ln(1 - F) \left[ \sum_{j=0}^{\infty} 1_{[j,j+1)}(t) \ln(T_j) - \ln(r_t) \right] dt \]

  + \( D_t dt + \sigma dW_t \)
It would be intuitive to have:

$$E[r_t] = r_0^{(1-F)^t} T_0^{1-(1-F)^t}$$
Drift Compensation and Calibration: plain mean-reversion

- It would be intuitive to have:
  \[ \mathbb{E}[r_t] = r_0^{(1-F)^t} T_0^{1-(1-F)^t} \]

- To find out what drift \( D_t \) will ensure it, you can integrate \( d\ln(r_t) \):

  \[
  \ln(r_t) = \ln(r_0) (1 - F)^{\frac{t}{dt}} dt + \sigma \sqrt{dt} \sum_{s=1}^{\frac{t}{dt}} N_{t-(s-1)dt} (1 - F)^{sdt}
  \]

  \[
  + \ln(T_0) \left[ 1 - (1 - F)^{\frac{t}{dt}} \right] \sum_{s=1}^{\frac{t}{dt}} (1 - F)^{(s-1)dt} \quad \text{notice geom. series}
  \]

  \[
  + dt \sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt} (1 - F)^{sdt} \quad \text{which simplifies to:}
  \]

  \[
  \ln(r_t) = \ln(r_0) (1 - F)^{t} + \sigma \sqrt{dt} \sum_{s=1}^{\frac{t}{dt}} N_{t-(s-1)dt} (1 - F)^{sdt}
  \]

  \[
  + \ln(T_0) \left[ 1 - (1 - F)^{t} \right] + dt \sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt} (1 - F)^{sdt}, \text{ which is Gaussian.}
  \]
Drift Compensation and Calibration: plain mean-reversion

Since $\ln(r_t)$ is Gaussian, $\mathbb{E}[r_t] = e^{\mu + \frac{1}{2}\sigma^2}$ where the $\mu$ and $\sigma^2$ are some mess determined by the constants in the expression for $\ln(r_t)$. 

There is a similar closed form for the variance of $r_t$ based on $\mathbb{E}[r_t^2] = e^{2\mu + \frac{1}{2}(2\sigma^2)}$ which can help calibrate the model to historical variance $\mathbb{E}[r_t^2] = \frac{1}{n} \sigma^2 \mathbb{E}[\ln(V_{obs})^2]$. 

Practical work with the randomized reversion target model all but requires you to know similar closed forms for drift compensation and variance, but now when you integrate no geometric series appears.
Drift Compensation and Calibration: plain mean-reversion

- Since \( \ln(r_t) \) is Gaussian, \( \mathbb{E}[r_t] = e^{\mu + \frac{1}{2}\sigma^2} \) where the \( \mu \) and \( \sigma^2 \) are some mess determined by the constants in the expression for \( \ln(r_t) \).

- If you work that mess out and set it equal to \( r_0(1-F)^t T_0[1-(1-F)^t] \), and require that it be true for all \( t \), you can arrive at what the drift compensation function \( D_t \) must be to deliver the intuitive \( \mathbb{E}[r_t] \):
Drift Compensation and Calibration: plain mean-reversion

- Since \( \ln(r_t) \) is Gaussian, \( \mathbb{E}[r_t] = e^{\mu + \frac{1}{2}\sigma^2} \) where the \( \mu \) and \( \sigma^2 \) are some mess determined by the constants in the expression for \( \ln(r_t) \).

- If you work that mess out and set it equal to \( r_0 (1-F)^t \int_0^t [1-(1-F)^t] \), and require that it be true for all \( t \), you can arrive at what the drift compensation function \( D_t \) must be to deliver the intuitive \( \mathbb{E}[r_t] \):
  
  \[
  D_t = -\frac{1}{2} \sigma^2 \int_0^t \frac{(1-F)^{dt}}{1+(1-F)^{dt}} \left[ 1 + (1 - F)^{2t-dt} \right],
  \]
  or
Drift Compensation and Calibration: plain mean-reversion

Since \( \ln(r_t) \) is Gaussian, \( \mathbb{E}[r_t] = e^{\mu + \frac{1}{2} \sigma^2} \) where the \( \mu \) and \( \sigma^2 \) are some mess determined by the constants in the expression for \( \ln(r_t) \).

If you work that mess out and set it equal to \( r_0^{(1-F)^t} T_0^{[1-(1-F)^t]} \), and require that it be true for all \( t \), you can arrive at what the drift compensation function \( D_t \) must be to deliver the intuitive \( \mathbb{E}[r_t] \):

- \[ D_t = -\frac{1}{2} \sigma^2 \frac{(1-F)^{dt}}{1+(1-F)^{dt}} \left[ 1 + (1 - F)^{2t} - dt \right] \], or
- \[ D_t = -\frac{1}{4} \sigma^2 \left[ 1 + (1 - F)^{2t} \right] \] in the continuous case.

There is a similar closed form for the variance of \( r_t \) based on \( \mathbb{E}[r_t^2] = e^{2\mu + \frac{1}{2}(2\sigma)^2} \) which can help calibrate the model to historical variance.

Practical work with the randomized reversion target model all but requires you to know similar closed forms for drift compensation and variance, but now when you integrate no geometric series appears.
Drift Compensation and Calibration: plain mean-reversion

- Since $\ln(r_t)$ is Gaussian, $\mathbb{E}[r_t] = e^{\mu + \frac{1}{2} \sigma^2}$ where the $\mu$ and $\sigma^2$ are some mess determined by the constants in the expression for $\ln(r_t)$.

- If you work that mess out and set it equal to $r_0^{(1-F)^t} T_0^{1-(1-F)^t}$, and require that it be true for all $t$, you can arrive at what the drift compensation function $D_t$ must be to deliver the intuitive $\mathbb{E}[r_t]$:
  
  $$D_t = -\frac{1}{2} \sigma^2 \frac{(1-F)^{dt}}{1+(1-F)^{dt}} \left[ 1 + (1 - F)^{2t} dt \right],$$  
  
  or  
  $$D_t = -\frac{1}{4} \sigma^2 \left[ 1 + (1 - F)^{2t} \right] \text{ in the continuous case}$$

- There is a similar closed form for the variance of $r_t$ based on $\mathbb{E}[r_t^2] = e^{2\mu + \frac{1}{2}(2\sigma)^2}$ which can help calibrate the model to historical variance $F = 1 - \left\{ 1 - \frac{\sigma_{\text{obs}}^2 dt}{\ln(V_{\text{obs}} + T^2) - \ln(T^2)} \right\}^{\frac{1}{2dt}}$
Drift Compensation and Calibration: plain mean-reversion

- Since $\ln(r_t)$ is Gaussian, $\mathbb{E}[r_t] = e^{\mu + \frac{1}{2}\sigma^2}$ where the $\mu$ and $\sigma^2$ are some mess determined by the constants in the expression for $\ln(r_t)$.

- If you work that mess out and set it equal to $r_0(1-F)^t T_0[1-(1-F)^t]$, and require that it be true for all $t$, you can arrive at what the drift compensation function $D_t$ must be to deliver the intuitive $\mathbb{E}[r_t]$:

  $$D_t = -\frac{1}{2}\sigma^2 \frac{(1-F)^t dt}{1+(1-F)^t dt} \left[1 + (1 - F)^2 t - dt \right],$$
  or

  $$D_t = -\frac{1}{4}\sigma^2 \left[1 + (1 - F)^2 t \right]$$
  in the continuous case

- There is a similar closed form for the variance of $r_t$ based on $\mathbb{E}[r_t^2] = e^{2\mu + \frac{1}{2}(2\sigma)^2}$ which can help calibrate the model to historical variance $F = 1 - \left\{1 - \frac{\sigma^2_{obs} dt}{\ln(V_{obs} + T^2) - \ln(T^2)}\right\} \frac{1}{2dt}$

- Practical work with the randomized reversion target model all but requires you to know similar closed forms for drift compensation and variance, but now when you integrate no geometric series appears.
\[
\ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{t} N_{t-(s-1)dt} (1 - F)^{sdt} \\
+ \left[ 1 - (1 - F)^{dt} \right] \sum_{s=1}^{t} \sum_{j=0}^{\infty} 1_{[j,j+1)}(sdt) \ln(T_j) (1 - F)^{t-sdt} \leftarrow \text{ugly} \\
+ dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{sdt}
\]
\[ \ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{t \frac{dt}{dt}} N_{t-(s-1)dt} (1 - F)^{sdt} \]

\[ + \left[ 1 - (1 - F)^{dt} \right] \sum_{s=1}^{t \frac{dt}{dt}} \sum_{j=0}^{\infty} \mathbf{1}_{[j,j+1]}(sdt) \ln(T_j) (1 - F)^{t-sdt} \leftarrow \text{ugly} \]

\[ + dt \sum_{s=1}^{t \frac{dt}{dt}} D_{t-(s-1)dt} (1 - F)^{sdt}, \text{ but switch the order of summation:} \]
\[ \ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{t} N_{t-(s-1)dt} (1 - F)^{s dt} \]

\[ + \left[ 1 - (1 - F)^{dt} \right] \sum_{s=1}^{t} \sum_{j=0}^{\infty} 1_{[j,j+1)}(s dt) \ln(T_j) (1 - F)^{t-s dt} \leftarrow \text{ugly} \]

\[ + dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{s dt} , \text{ but switch the order of summation:} \]

\[ \ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{t} N_{t-(s-1)dt} (1 - F)^{s dt} \]

\[ + \ln(T_0) \left[ (1 - F)^{t-t_1} - (1 - F)^{t} \right] \]

\[ + \sum_{j=1}^{\infty} \ln(T_j) \left[ (1 - F)^{t-t_{j+1}} - (1 - F)^{t-t_{j}} \right] \leftarrow \text{after telescoping} \]

\[ + dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{s dt} , \text{ so } r_t \neq \text{lognormal, } \equiv \text{log-log-gamma?} \]
\[
\ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{t \over dt} N_{t-(s-1)dt} (1 - F)^{sdt} \\
+ \ln(T_0) \left[ (1 - F)^{(t-t_1)_+} - (1 - F)^t \right] \\
+ \sum_{j=1}^{\infty} \ln(T_j) \left[ (1 - F)^{(t-t_{j+1})_+} - (1 - F)^{(t-t_j)_+} \right] \\
+ dt \sum_{s=1}^{t \over dt} D_{t-(s-1)dt} (1 - F)^{sdt}
\]
Condition on the Times When Regimes Switch

- \( \ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{t} N_{t-(s-1)dt} (1 - F)^{sdt} \)

\[
+ \ln(T_0) \left[ (1 - F)^{(t-t_1)+} - (1 - F)^t \right]
\]

\[
+ \sum_{j=1}^{\infty} \ln(T_j) \left[ (1 - F)^{(t-t_{j+1})+} - (1 - F)^{(t-t_j)+} \right]
\]

\[
+ dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{sdt}
\]

- Conditioning on the \( t_j \) random variables and using a lognormal model (reasonable) for the random targets \( T_j \) (so each \( \ln(T_j) \) is Gaussian) we again have a (messy) Gaussian for the conditional \( \ln(r_t) \). Can that help in calculating an unconditioned \( \mathbb{E}[r_t] \) and variance?
Conditioning on the Times When Regimes Switch

\[ \ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{\frac{t}{dt}} N_{t-(s-1)dt} (1 - F)^{sd} + \ln(T_0) \left[ (1 - F)^{(t-t_1)+} - (1 - F)^t \right] + \sum_{j=1}^{\infty} \ln(T_j) \left[ (1 - F)^{(t-t_j+1)+} - (1 - F)^{(t-t_j)+} \right] + dt \sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt} (1 - F)^{sd} \]

- Conditioning on the \( t_j \) random variables and using a lognormal model (reasonable) for the random targets \( T_j \) (so each \( \ln(T_j) \) is Gaussian) we again have a (messy) Gaussian for the conditional \( \ln(r_t) \). Can that help in calculating an unconditioned \( \mathbb{E}[r_t] \) and variance?

- The answer is "Yes" ... up to an approximate expansion.
We expect the tails of $\ln(r_t)$ to be suppressed in favor of the shoulders. That suggests that $E[r_t]$, and higher moments as well, might be approximated efficiently by an Edgeworth expansion for $\ln(r_t)$. It works out to be surprisingly simple:

$$E[h(r_t)] \approx e^{(\mu + \frac{1}{2} \sigma^2)} \left[ 1 + \sum_{j=2}^{\infty} \frac{(-1)^j}{j!} \sigma^{2j} \left( \frac{\mu_{2j}}{j!} \right) \right]$$

where $\mu$ and $\sigma^2$ stand for central moments of $\ln(r_t)$ and $\mu_4$ is its mean. Conditional Gaussian ensures that odd higher moments vanish. The problem now is to calculate $\mu$, $\sigma^2$, and the $\mu_{2j}$. 

Bridgeman (University of Connecticut)
Edgeworth Expansion for the Unconditioned Moments

- We expect the tails of $\ln(r_t)$ to be suppressed in favor of the shoulders. That suggests that $\mathbb{E}[r_t]$, and higher moments as well, might be approximated efficiently by an Edgeworth expansion for $\ln(r_t)$. It works out to be surprisingly simple:

$$\mathbb{E}[(r_t)^l] \approx e^{l\mu + \frac{1}{2}(l\sigma)^2} \left\{ 1 + \frac{l^4}{4!} [\mu_4 - 3\sigma^4] \right\}$$

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The problem now is to calculate $\mu$, $\sigma^2$, and the $\mu_{2j}$.
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  \]
  Here $\sigma^2$ and $\mu_4$ stand for central moments of $\ln(r_t)$ and $\mu$ is its mean.
  \[
  \approx e^{l\mu + \frac{1}{2}(l\sigma)^2} \left\{ 1 + \frac{l^4}{4!} [\mu_4 - 3\sigma^4] \left(1 - \frac{3}{4!} (l\sigma)^2 \right) + \frac{l^6}{6!} [\mu_6 - 15\sigma^6] \right\}
  \]
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\[ = e^{l\mu + \frac{1}{2}(l\sigma)^2} \left\{ 1 + \lim_{N \to \infty} \sum_{j=2}^N \frac{l^{2j}}{(2j)!} \left[\mu_{2j} - (2j)\sigma^{2j}\right] \frac{N-j}{2n} \sum_{n=0}^\infty \frac{(-1)^n(2n)!}{(2n)!} (l\sigma)^{2n}\right\} \]

where $(2n)! = (2n - 1)(2n - 3) \cdots (1)$
Edgeworth Expansion for the Unconditioned Moments

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\]

where $(2n)! = (2n-1)(2n-3)\cdots(1)$

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Edgeworth Expansion for the Unconditioned Moments

- We expect the tails of $\ln(r_t)$ to be suppressed in favor of the shoulders. That suggests that $\mathbb{E} [r_t]$, and higher moments as well, might be approximated efficiently by an Edgeworth expansion for $\ln(r_t)$. It works out to be surprisingly simple:

$$ \mathbb{E} \left[ (r_t)^l \right] \approx e^{l\mu + \frac{1}{2}(l\sigma)^2} \left\{ 1 + \frac{l^4}{4!} [\mu_4 - 3\sigma^4] \right\} $$

- Here $\sigma^2$ and $\mu_4$ stand for central moments of $\ln(r_t)$ and $\mu$ is its mean.

$$ \approx e^{l\mu + \frac{1}{2}(l\sigma)^2} \left\{ 1 + \frac{l^4}{4!} [\mu_4 - 3\sigma^4] \left( 1 - \frac{3}{4!} (l\sigma)^2 \right) + \frac{l^6}{6!} [\mu_6 - 15\sigma^6] \right\} $$

$$ = e^{l\mu + \frac{1}{2}(l\sigma)^2} \left\{ 1 + \lim_{N \to \infty} \sum_{j=2}^{N} \frac{j^{2j}}{(2j)!} \left[ \mu_{2j} - (2j)! \sigma^{2j} \right] \sum_{n=0}^{N-j} \frac{(-1)^n (2n)!}{(2n)!} (l\sigma)^{2n} \right\} $$

where $(2n)!! = (2n - 1)(2n - 3) \cdots (1)$

- Conditional Gaussian ensures that odd higher moments vanish.

- The problem now is to calculate $\mu$, $\sigma^2$, and the $\mu_{2j}$. 
Expected Value Easy

- Remember,

\[
\ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{\frac{t}{dt}} N_{t-(s-1)dt} (1 - F)^{sdt} \\
+ \ln(T_0) \left[ (1 - F)^{(t-t_1)_+} - (1 - F)^t \right] \\
+ \sum_{j=1}^{\infty} \ln(T_j) \left[ (1 - F)^{(t-t_{j+1})_+} - (1 - F)^{(t-t_j)_+} \right] \\
+ dt \sum_{s=1}^{\frac{t}{dt}} D_{t-(s-1)dt} (1 - F)^{sdt} \text{ and condition on the } t_j
\]
Remember,

\[
\ln(r_t) = \ln(r_0) (1 - F)^t + \sigma \sqrt{dt} \sum_{s=1}^{t \frac{dt}{dt}} N_{t-(s-1)dt} (1 - F)^{sd}\n\]

\[
+ \ln(T_0) \left[ (1 - F)^{(t-t_1)+} - (1 - F)^t \right] \]

\[
+ \sum_{j=1}^{\infty} \ln(T_j) \left[ (1 - F)^{(t-t_{j+1})+} - (1 - F)^{(t-t_j)+} \right] \]

\[
+ dt \sum_{s=1}^{t \frac{dt}{dt}} D_{t-(s-1)dt} (1 - F)^{sd} \text{ and condition on the } t_j \]

So \( \mu = \mathbb{E}[\ln(r_t)] \) is given by

\[
\ln(r_0) (1 - F)^t + \ln(T_0) \left\{ \mathbb{E} \left[ (1 - F)^{(t-t_1)+} \right] - (1 - F)^t \right\} \]

\[
+ \mu_T \mathbb{E} \left[ \sum_{j=1}^{\infty} \left[ (1 - F)^{(t-t_{j+1})+} - (1 - F)^{(t-t_j)+} \right] \right] \]

\[
+ dt \sum_{s=1}^{t \frac{dt}{dt}} D_{t-(s-1)dt} (1 - F)^{sd} \text{ where } \mu_T = \mathbb{E}[\ln(T_j)] \]
So $\mu = \mathbb{E}[\ln(r_t)]$ is given by

$$\ln(r_0)(1 - F)^t + \ln(T_0) \left\{ \mathbb{E} \left[ (1 - F)^{(t-t_1)_+} \right] - (1 - F)^t \right\}$$

$$+ \mu_T \mathbb{E} \left[ \sum_{j=1}^{\infty} \left[ (1 - F)^{(t-t_{j+1})_+} - (1 - F)^{(t-t_j)_+} \right] \right]$$

$$+ dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{s dt} \text{ where } \mu_T = \mathbb{E}[\ln(T_j)]$$
So \( \mu = \mathbb{E}[\ln(r_t)] \) is given by
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\]
\[
+ \mu_T \mathbb{E} \left[ \sum_{j=1}^{\infty} \left[ (1 - F)^{(t-t_{j+1})_+} - (1 - F)^{(t-t_j)_+} \right] \right]
\]
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+ dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{sdt}
\]
where \( \mu_T = \mathbb{E}[\ln(T_j)] \)

Telescoping,
\[
= \ln(r_0) (1 - F)^t + \ln(T_0) \left\{ \mathbb{E} \left[ (1 - F)^{(t-t_1)_+} \right] - (1 - F)^t \right\}
\]
\[
+ \mu_T \left\{ 1 - \mathbb{E} \left[ (1 - F)^{(t-t_1)_+} \right] \right\} + dt \sum_{s=1}^{t} D_{t-(s-1)dt} (1 - F)^{sdt}
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So $\mu = \mathbb{E}[\ln(r_t)]$ is given by
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\ln(r_0) (1 - F)^t + \ln(T_0) \left\{ \mathbb{E} \left[ (1 - F)^{(t-t_1)_+} \right] - (1 - F)^t \right\}
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+ \mu_T \left\{ 1 - \mathbb{E} \left[ (1 - F)^{(t-t_1)_+} \right] \right\} + dt \sum_{s=1}^{t \frac{dt}{dt}} D_{t-(s-1)dt} (1 - F)^{sdt}
\]

$\mathbb{E} \left[ (1 - F)^{(t-t_1)_+} \right]$ turns out to be a Laplace transform that we can calculate (later).
Higher Moments Hard

- Remembering that the even central moments of std normal are 
  \((2n)\? = (2n - 1) (2n - 3) \cdots (1)\), the even central moments of
  \(\ln(r_t)\) are 
  \[ \mathbb{E} \left[ \{\ln(r_t) - \mathbb{E}[\ln(r_t)]\}^{2n} \right] \]

  \[ = (2n)?\mathbb{E} \left\{ \sigma^2 dt \sum_{s=1}^{t} (1 - F)^{2sd} + \sigma^2_T \sum_{j=1}^{\infty} e_j^2 \right\} \]

  \[ = (2n)?\mathbb{E} \left\{ \sigma^2 dt (1 - F)^{2dt} \frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}} + \sigma^2_T \sum_{j=1}^{\infty} e_j^2 \right\} \]
Remembering that the even central moments of std normal are 
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\[
\mathbb{E} \left[ \left\{ \ln(r_t) - \mathbb{E} [\ln(r_t)] \right\}^{2n} \right]
\]
\[
= (2n)? \mathbb{E} \left[ \left\{ \sigma^2 dt \sum_{s=1}^{t} (1 - F)^{2sd} + \sigma_T^2 \sum_{j=1}^{\infty} e_j^2 \right\}^n \right]
\]
\[
= (2n)? \mathbb{E} \left[ \left\{ \sigma^2 dt (1 - F)^{2dt} \frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}} + \sigma_T^2 \sum_{j=1}^{\infty} e_j^2 \right\}^n \right]
\]
\(\sigma_T^2\) is the common variance of the \(\ln(T_j)\) Gaussians
Higher Moments Hard

- Remembering that the even central moments of std normal are 
  \((2n)^2 = (2n - 1)(2n - 3) \cdots (1)\), the even central moments of 
  \(\ln(r_t)\) are 
  \[ \mathbb{E} \left[ \left\{ \ln(r_t) - \mathbb{E} [\ln(r_t)] \right\}^{2n} \right] \]
  \[ = (2n)^2 \mathbb{E} \left[ \left\{ \sigma^2 \frac{d}{dt} \sum_{s=1}^{t} (1 - F)^{2s} + \sigma^2_T \sum_{j=1}^{\infty} e_j^2 \right\}^n \right] \]
  \[ = (2n)^2 \mathbb{E} \left[ \left\{ \sigma^2 dt (1 - F)^{2dt} \frac{1-(1-F)^2}{1-(1-F)^{2dt}} + \sigma^2_T \sum_{j=1}^{\infty} e_j^2 \right\}^n \right] \]

- \(\sigma^2_T\) is the common variance of the \(\ln(T_j)\) Gaussians

- \(e_j = \left\{ (1 - F)^{(t-t_j+1)+} - (1 - F)^{(t-t_j)+} \right\} \) for each \(j\)
Remembering that the even central moments of std normal are 
\((2n)^2 = (2n - 1)(2n - 3) \cdots (1)\), the even central moments of 
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\[
\mathbb{E} \left[ \left\{ \ln(r_t) - \mathbb{E} [\ln(r_t)] \right\}^{2n} \right]
\]

\[
= (2n)^2 \mathbb{E} \left[ \left\{ \sigma^2 dt \sum_{s=1}^{t} (1 - F)^{2s dt} + \sigma_T^2 \sum_{j=1}^{\infty} \mathbf{e}_j^2 \right\}^n \right]
\]

\[
= (2n)^2 \mathbb{E} \left[ \left\{ \sigma^2 dt (1 - F)^{2dt} \frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}} + \sigma_T^2 \sum_{j=1}^{\infty} \mathbf{e}_j^2 \right\}^n \right]
\]

- \(\sigma_T^2\) is the common variance of the \(\ln(T_j)\) Gaussians
- \(\mathbf{e}_j = \left\{ (1 - F)^{t-t_j+1} - (1 - F)^{t-t_j} \right\}\) for each \(j\)
- The \(\{\}^n\) part can be expanded binomially, but that still leaves terms like...
... \( \mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^m \right] \) where the
\[ e_j = \left\{ (1 - F)^{(t-t_{j+1})^+} - (1 - F)^{(t-t_j)^+} \right\} \]
fail to be independent and are each complicated in their own right.
Still Need To Evaluate Terms Like

\[ \mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^m \right] \]

where the

\[ e_j = \left\{ (1 - F)^{(t-t_{j+1})} - (1 - F)^{(t-t_j)} \right\} \]

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But they do have a uniform correlation property
Still Need To Evaluate Terms Like

... \( \mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^m \right] \) where the

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fail to be independent and are each complicated in their own right.

But they do have a uniform correlation property

**Lemma:** \( \mathbb{E} \left[ e_{j_1}^{2a_1} \cdots e_{j_k}^{2a_k} \right] = \rho_{a_1,\ldots,a_k} \mathbb{E} \left[ e_{j_1}^{2a_1} \right] \cdots \mathbb{E} \left[ e_{j_k}^{2a_k} \right] \)

independent of \( \{j_1,\ldots,j_k\} \) for distinct \( \{j_1,\ldots,j_k\} \)
Still Need To Evaluate Terms Like

\[ \mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^m \right] \]

where the

\[ e_j = \left\{ (1 - F)^{(t-t_{j+1})+} - (1 - F)^{(t-t_j)+} \right\} \]

fail to be independent and are each complicated in their own right.

But they do have a uniform correlation property

Lemma: \( \mathbb{E} \left[ e_{j_1}^{2a_1} \cdots e_{j_k}^{2a_k} \right] = \rho_{a_1,\ldots,a_k} \mathbb{E} \left[ e_{j_1}^{2a_1} \right] \cdots \mathbb{E} \left[ e_{j_k}^{2a_k} \right] \)

independent of \( \{j_1, \ldots, j_k\} \) for distinct \( \{j_1, \ldots, j_k\} \)

\( \rho_{a_1,\ldots,a_k} \) can be computed using Laplace transforms and there’s even a recursive relationship \( \rho_{a_1,\ldots,a_k} = \rho_{a_1,a_2+\ldots+a_k} \rho_{a_2,\ldots,a_k} \)
... $\mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^m \right]$ where the $e_j = \left\{ (1 - F)^{(t-t_{j+1})+} - (1 - F)^{(t-t_j)+} \right\}$ fail to be independent and are each complicated in their own right.

But they do have a uniform correlation property

Lemma: $\mathbb{E} \left[ e_{j_1}^{2a_1} \cdots e_{j_k}^{2a_k} \right] = \rho_{a_1,\ldots,a_k} \mathbb{E} \left[ e_{j_1}^{2a_1} \right] \cdots \mathbb{E} \left[ e_{j_k}^{2a_k} \right]$ independent of $\{j_1,\ldots,j_k\}$ for distinct $\{j_1,\ldots,j_k\}$

$\rho_{a_1,\ldots,a_k}$ can be computed using Laplace transforms and there’s even a recursive relationship $\rho_{a_1,\ldots,a_k} = \rho_{a_1,a_2+\ldots+a_k} \rho_{a_2,\ldots,a_k}$

How does that help?
For Example

\[ \mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^2 \right] = \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^4 + \sum_{j=1}^{\infty} e_j^2 \{ \left( \sum_{i=1}^{\infty} e_i^2 \right) - e_j^2 \} \right] \]
For Example

\[ \mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^2 \right] = \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^4 + \sum_{j=1}^{\infty} e_j^2 \left\{ \left( \sum_{i=1}^{\infty} e_i^2 \right) - e_j^2 \right\} \right] \]

So \[ \mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^2 \right] = \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^4 \right] + \rho_{1,1} \left\{ \left( \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right] \right)^2 - \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \mathbb{E} \left[ e_j^2 \right] \right] \right\} \]

using monotone convergence to run expectations across \( \infty \) sums

It gets complicated fast
For Example

\[ E \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^2 \right] = E \left[ \sum_{j=1}^{\infty} e_j^4 + \sum_{j=1}^{\infty} e_j^2 \left\{ \left( \sum_{i=1}^{\infty} e_i^2 \right) - e_j^2 \right\} \right] \]

So \( E \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^2 \right] = E \left[ \sum_{j=1}^{\infty} e_j^4 \right] + \rho_{1,1} \left\{ \left( E \left[ \sum_{j=1}^{\infty} e_j^2 \right] \right)^2 - E \left[ \sum_{j=1}^{\infty} e_j^2 E \left[ e_j^2 \right] \right] \right\} \) using monotone convergence to run expectations across \( \infty \) sums

- It gets complicated fast
For $m=3$

\[
\mathbb{E} \left[ \left( \sum_{j=1}^{\infty} e_j^2 \right)^3 \right] = \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^6 + 3 \sum_{j=1}^{\infty} e_j^4 \left\{ \left( \sum_{i=1}^{\infty} e_i^2 \right) - e_j^2 \right\} \right. \\
\left. + \sum_{j=1}^{\infty} e_j^2 \left\{ \left( \sum_{i=1}^{\infty} e_i^2 \right) - e_i^2 - e_j^2 \right\} \right] \\
= \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^6 \right] + 3\rho_{2,1} \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^4 \right] \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right] \\
- (3\rho_{2,1} - \rho_{1,1,1}) \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^4 \mathbb{E} [e_j^2] \right] + \rho_{1,1,1} \left\{ \left( \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right] \right)^3 \right. \\
\left. - 3 \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \right] \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \mathbb{E} [e_j^2] \right] + \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^2 \left( \mathbb{E} [e_j^2] \right)^2 \right] \right\}
Now all you need to be able to evaluate are terms like

\[ \mathbb{E} \left[ \sum_{j=1}^{\infty} e^{2n_j} \prod_{k=1}^{n} \left( \mathbb{E} \left[ e^{2k_j} \right] \right)^{n_{k_j}} \right], \text{ where } \sum_{k=1}^{n} k n_{k} \leq m - n \]
Complicated, but each piece is simpler

- Now all you need to be able to evaluate are terms like
\[ \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^{2n} \prod_{k=1}^{n} \left( \mathbb{E} \left[ e_j^{2k} \right] \right)^{n_k} \right], \text{ where } \sum_{k=1}^{n} kn_k \leq m - n \]

- In fact, we will develop a calculation that includes the odd powers too,
\[ \mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^n \prod_{k=1}^{n} \left( \mathbb{E} \left[ e_j^k \right] \right)^{n_k} \right] \]
Complicated, but each piece is simpler

Now all you need to be able to evaluate are terms like

$$\mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^{2n} \prod_{k=1}^{n} \left( \mathbb{E} \left[ e_j^{2k} \right] \right)^{n_k} \right], \text{ where } \sum_{k=1}^{n} k n_k \leq m - n$$

In fact, we will develop a calculation that includes the odd powers too,

$$\mathbb{E} \left[ \sum_{j=1}^{\infty} e_j^{n} \prod_{k=1}^{n} \left( \mathbb{E} \left[ e_j^{k} \right] \right)^{n_k} \right]$$

Some notation: to save ink later let $\nu(x)$ stand for $x^n \prod_{k=1}^{n} \mathbb{E} \left[ x^k \right]^{n_k}$ so our expression abbreviates to

$$\mathbb{E} \left[ \sum_{j=1}^{\infty} \nu \left( e_j \right) \right]$$
Let $d_1, d_2, ..., d_j, ...$ be i.i.d inter-arrival intervals with common law $d$. 

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[Bridgeman (University of Connecticut)](https://www.urbanophile.com/)

Random Regimes
The Set-Up

- Let $d_1, d_2, \ldots, d_j, \ldots$ be i.i.d inter-arrival intervals with common law $d$
- Define $d_0$ by the relationships $0 \leq d_0 \leq d_1$ and $d_0 \sim (d_1 - d_0)$
The Set-Up

- Let \( d_1, d_2, \ldots, d_j, \ldots \) be i.i.d inter-arrival intervals with common law \( d \)
- Define \( d_0 \) by the relationships \( 0 \leq d_0 \leq d_1 \) and \( d_0 \simeq (d_1 - d_0) \)
- Let \( d \) stand for the common law of \( d_0 \) and \( (d_1 - d_0) \), the equilibrium distribution of \( d \)
The Set-Up

- Let \( d_1, d_2, \ldots, d_j, \ldots \) be i.i.d inter-arrival intervals with common law \( d \)
- Define \( d_0 \) by the relationships \( 0 \leq d_0 \leq d_1 \) and \( d_0 \sim (d_1 - d_0) \)
- Let \( d \) stand for the common law of \( d_0 \) and \( (d_1 - d_0) \), the equilibrium distribution of \( d \)
- The density \( f_d(x) = \frac{P[d \geq x]}{E[d]} \)
The Set-Up

- Let \( d_1, d_2, ..., d_j, ... \) be i.i.d inter-arrival intervals with common law \( d \)
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- Define \( d_1 = d_1 \wedge (d_0 + t) - d_0 \), so we begin at a random point in the first i.i.d. interval
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- Define \( d_1 = d_1 \wedge (d_0 + t) - d_0 \), so we begin at a random point in the first i.i.d. interval
- Set \( t_0 = 0, t_1 = d_1, \ldots, t_j = d_1 + d_2 + \ldots + d_j \)
The Set-Up

- Let \( d_1, d_2, \ldots, d_j, \ldots \) be i.i.d inter-arrival intervals with common law \( d \)
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- Let \( J = \min \{ j : t_j \geq t \} \) (a "stopping regime")
The Set-Up

- Let \( d_1, d_2, \ldots, d_j, \ldots \) be i.i.d inter-arrival intervals with common law \( d \)
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- Define random indicators \( \{1_{j < J}\}_{j \geq 1} \) by \( 1_{j < J} = 0 \) for \( j \geq J \) and \( 1_{j < J} = 1 \) for \( j < J \)
The Set-Up

Let \( \mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_j, \ldots \) be i.i.d inter-arrival intervals with common law \( \mathbf{d} \)

Define \( -\mathbf{d}_0 \) by the relationships \( 0 \leq -\mathbf{d}_0 \leq \mathbf{d}_1 \) and \( -\mathbf{d}_0 \sim (\mathbf{d}_1 - -\mathbf{d}_0) \)

Let \( \mathbf{d} \) stand for the common law of \( -\mathbf{d}_0 \) and \( (\mathbf{d}_1 - -\mathbf{d}_0) \), the equilibrium distribution of \( \mathbf{d} \)

The density \( f_{-\mathbf{d}}(x) = \frac{\Pr [\mathbf{d} \geq x]}{\mathbb{E} [\mathbf{d}]} \)

Define \( -\mathbf{d}_1 = \mathbf{d}_1 \land (\mathbf{d}_0 + t) - -\mathbf{d}_0 \), so we begin at a random point in the first i.i.d. interval

Set \( t_0 = 0, t_1 = -\mathbf{d}_1, \ldots, t_j = -\mathbf{d}_1 + \mathbf{d}_2 + \ldots + \mathbf{d}_j \)

Let \( J = \min \{j : t_j \geq t\} \) (a "stopping regime")

Define random indicators \( \{1_{j<j} \}_{j \geq 1} \) by \( 1_{j<j} = 0 \) for \( j \geq J \) and \( 1_{j<j} = 1 \) for \( j < J \)

Set \( -\mathbf{d}_j = t - t_{J-1} \) and \( -\mathbf{d}_{j+1} = t_j - t \)
The Set-Up

- Let \( d_1, d_2, \ldots, d_j, \ldots \) be i.i.d inter-arrival intervals with common law \( d \).
- Define \( d_0 \) by the relationships \( 0 \leq d_0 \leq d_1 \) and \( d_0 \sim (d_1 - d_0) \).
- Let \( d \) stand for the common law of \( d_0 \) and \( (d_1 - d_0) \), the equilibrium distribution of \( d \).
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- Let \( J = \min \{ j : t_j \geq t \} \) (a "stopping regime").
- Define random indicators \( \{1_{j<J}\}_{j \geq 1} \) by \( 1_{j<J} = 0 \) for \( j \geq J \) and \( 1_{j<J} = 1 \) for \( j < J \).
- Set \( d_j = t - t_{j-1} \) and \( d_{j+1} = t_j - t \).
- So \( t = d_1 + d_2 + \ldots + d_{J-1} + d_J \).
The Result

\[ \mathbb{E} \left[ \sum_{j=1}^{\infty} \nu (e_j) \right] = \\
K \left\{ \mathbb{E} \left[ \nu \left( (1 - F)^{d^\land t} \right) \right] - \mathbb{P} [d \geq t] \nu \left( (1 - F)^t \right) \right\} \frac{\mathbb{E} \left[ \nu (1 - (1 - F)^d) \right]}{1 - \mathbb{E} \left[ \nu (1 - F)^d \right]} \\
+ \mathbb{E} \left[ \nu \left( 1 - (1 - F)^{d^\land t} \right) \right] - \mathbb{P} [d \geq t] \nu \left( 1 - (1 - F)^t \right) \]
The Result

\[ \mathbb{E} \left[ \sum_{j=1}^{\infty} \nu(e_j) \right] = \]

\[ K \left\{ \mathbb{E} \left[ \nu \left( (1 - F)^{d^\wedge t} \right) \right] - \mathbb{P}[d \geq t] \nu \left( (1 - F)^t \right) \right\} \frac{\mathbb{E}[\nu(1-(1-F)^d)]}{1-\mathbb{E}[\nu((1-F)^d)]} \]

\[ + \mathbb{E} \left[ \nu \left( 1 - (1 - F)^{d^\wedge t} \right) \right] - \mathbb{P}[d \geq t] \nu \left( 1 - (1 - F)^t \right) \]

Where \( K = 1 - \mathbb{E} \left[ \left( \mathbb{E} \left[ \nu \left( (1 - F)^d \right) \right] \right)^{J-2} \mid J > 1 \right] \)

\[ = 1 - (1 - G)^t \frac{\mathbb{E}[(1-G)^{-d^\wedge t}]-\mathbb{P}[d \geq t](1-G)^{-t}}{\mathbb{E}[(1-G)^{-d^\wedge t}]-\mathbb{P}[d \geq t](1-G)^t} \]
The Result

\[ \mathbb{E} \left[ \sum_{j=1}^{\infty} \nu(e_j) \right] = \]
\[ K \left\{ \mathbb{E} \left[ \nu \left( (1 - F)^{d^\wedge t} \right) \right] - \mathbb{P} \left[ d \geq t \right] \nu ((1 - F)^t) \right\} \frac{\mathbb{E} \left[ \nu \left( 1 - (1 - F)^d \right) \right]} {1 - \mathbb{E} \left[ \nu ((1 - F)^d) \right]} \]
\[ + \mathbb{E} \left[ \nu \left( 1 - (1 - F)^{d^\wedge t} \right) \right] - \mathbb{P} \left[ d \geq t \right] \nu \left( 1 - (1 - F)^t \right) \]

Where \( K = 1 - \mathbb{E} \left[ \left( \mathbb{E} \left[ \nu \left( (1 - F)^d \right) \right] \right)^{J^{-2}} \mid J > 1 \right] \)
\[ = 1 - (1 - G)^t \frac{\mathbb{E} \left[ (1 - G)^{d^\wedge t} \right] - \mathbb{P} \left[ d \geq t \right] (1 - G)^{-t}} {\mathbb{E} \left[ (1 - G)^{d^\wedge t} \right] - \mathbb{P} \left[ d \geq t \right] (1 - G)^t} \]

And \( G \) is defined by \( (1 - G) = \exp \left\{ -\mathcal{L}_d^{-1} \left( \mathbb{E} \left[ \nu \left( (1 - F)^d \right) \right] \right) \right\} \), 
\( \mathcal{L}_d \) being the Laplace transform.
The Result

\[ \mathbb{E} \left[ \sum_{j=1}^{\infty} \nu(e_j) \right] = \]
\[ K \left\{ \mathbb{E} \left[ \nu \left( (1 - F)^{d^\wedge t} \right) \right] - \mathbb{P} \left[ d \geq t \right] \nu \left( (1 - F)^t \right) \right\} \frac{\mathbb{E} \left[ \nu \left( 1 - (1 - F)^d \right) \right]}{1 - \mathbb{E} \left[ \nu \left( (1 - F)^d \right) \right]} \]
\[ + \mathbb{E} \left[ \nu \left( 1 - (1 - F)^{d^\wedge t} \right) \right] - \mathbb{P} \left[ d \geq t \right] \nu \left( 1 - (1 - F)^t \right) \]

\text{Where } K = 1 - \mathbb{E} \left[ \left( \mathbb{E} \left[ \nu \left( (1 - F)^d \right) \right] \right)^{-1} \right] \text{, } J > 1
\[ = 1 - (1 - G)^t \frac{\mathbb{E} \left[ (1 - G)^{d^\wedge t} \right] - \mathbb{P} \left[ d \geq t \right] (1 - G)^t}{\mathbb{E} \left[ (1 - G)^{d^\wedge t} \right] - \mathbb{P} \left[ d \geq t \right] (1 - G)^t} \]

\text{And } G \text{ is defined by } (1 - G) = \exp \left\{ - \mathcal{L}_d^{-1} \left( \mathbb{E} \left[ \nu \left( (1 - F)^d \right) \right] \right) \right\}, \mathcal{L}_d \text{ being the Laplace transform}

\text{Meaning}
\[ \mathbb{E} \left[ (1 - G)^d \right] = \mathcal{L}_d \left\{ \mathcal{L}_d^{-1} \left( \mathbb{E} \left[ \nu \left( (1 - F)^d \right) \right] \right) \right\} = \mathbb{E} \left[ \nu \left( (1 - F)^d \right) \right] \]
Asymptotically

\[
\lim_{t \to \infty} \mathbb{E} \left[ \sum_{j=1}^{\infty} \nu(e_j) \right] = \\
\frac{\mathbb{E} \left[ \nu \left[ (1-F)^d \right] \right] \mathbb{E} \left[ \nu \left[ 1 - (1-F)^d \right] \right]}{1 - \mathbb{E} \left[ \nu \left[ (1-F)^d \right] \right]} + \mathbb{E} \left[ \nu \left( 1 - (1 - F)^d \right) \right]
\]
Everything is now of the form $P[d \geq t]$ and $E[x^v]$ for $v$ one of the random variables $d$, $d^-$ and $d^\wedge t$. 

If, for example, we take the interarrival distribution $d$ for regime-switches to be gamma $(\alpha, \beta)$, then $L_d(x) = (1 + \beta x)^{-\alpha}$, $L_1(x) = 1/\beta x^{1+\alpha}$, $L_{-d}(x) = 1/\alpha \beta x h(1 + \beta x)^{-\alpha}$, $L_{-d^t}(x) = 1/\alpha \beta x e^{xt} h(1 + \beta x)^{-\alpha}$; $t \beta$.
Everything is now of the form $\mathbb{P}[d \geq t]$ and $\mathbb{E}[x^v]$ for $v$ one of the random variables $d$, $\bar{d}$ and $\bar{d}^\wedge t$

$\mathbb{E}[x^v] = \mathcal{L}_v[-\ln(x)]$, where $\mathcal{L}_v$ is the Laplace transform of $v$
Everything is now of the form $\mathbb{P}[d \geq t]$ and $\mathbb{E}[x^v]$ for $v$ one of the random variables $d$, $\bar{d}$ and $\bar{d} \wedge t$

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If, for example, we take the interarrival distribution $d$ for regime-switches to be $\text{gamma}(\alpha, \beta)$, then
Everything is now of the form $\mathbb{P}[d \geq t]$ and $\mathbb{E}[x^v]$ for $v$ one of the random variables $d$, $\overline{d}$, and $\overline{d} \wedge t$.

$\mathbb{E}[x^v] = \mathcal{L}_v [-\ln(x)]$, where $\mathcal{L}_v$ is the Laplace transform of $v$.

If, for example, we take the interarrival distribution $d$ for regime-switches to be \textit{gamma}$(\alpha, \beta)$, then

\[
\mathcal{L}_d(x) = (1 + \beta x)^{-\alpha}, \quad \mathcal{L}^{-1}_d(y) = \frac{1}{\beta} \left( y^{-\frac{1}{\alpha}} - 1 \right),
\]

\[
\mathcal{L}_{\overline{d}}(x) = \frac{1}{\alpha \beta x} \left[ 1 - (1 + \beta x)^{-\alpha} \right], \quad \mathcal{L}_{\overline{d} \wedge t}(x) =
\]

\[
\frac{1}{\alpha \beta x} \left\{ 1 - e^{-xt} \left[ 1 - \Gamma \left( \alpha; \frac{t}{\beta} \right) \right] - (1 + \beta x)^{-\alpha} \Gamma \left( \alpha; \frac{1 + \beta x t}{\beta} \right) \right\}
\]

\[
+ e^{-xt} \left\{ 1 - \Gamma \left( \alpha + 1; \frac{t}{\beta} \right) - \frac{t}{\alpha \beta} \left[ 1 - \Gamma \left( \alpha; \frac{t}{\beta} \right) \right] \right\},
\]

$\mathbb{P}[\overline{d} \geq t] = 1 - \Gamma \left( \alpha + 1; \frac{t}{\beta} \right) - \frac{t}{\alpha \beta} \left[ 1 - \Gamma \left( \alpha; \frac{t}{\beta} \right) \right]$.
\[ \rho_{a,b} = \frac{1}{D} \left\{ \mathbb{E} \left[ (1 - F)^{2(a-b)d^\wedge t} \right] - \mathbb{P} [d^\geq t] (1 - F)^{2(a-b)t} \right\} \]
Even Those Uniform Correlation Coefficients

\[ \rho_{a,b} = \frac{1}{D} \left\{ \mathbb{E} \left[ (1 - F)^{2(a-b)d^t} \right] - \mathbb{P} \left[ d^r \geq t \right] (1 - F)^{2(a-b)t} \right\} \]

where

\[ D = \left\{ \mathbb{E} \left[ (1 - F)^{2a d^t} \right] - \mathbb{P} \left[ d^r \geq t \right] (1 - F)^{2at} \right\} \]
\[ \cdot \left\{ \mathbb{E} \left[ (1 - F)^{-2b d^t} \right] - \mathbb{P} \left[ d^r \geq t \right] (1 - F)^{-2bt} \right\} \]
Even Those Uniform Correlation Coefficients

\( \rho_{a,b} = \frac{1}{D} \left\{ \mathbb{E} \left[ (1 - F)^{2(a-b)d_t} \right] - \mathbb{P} [d_t \geq t] (1 - F)^{2(a-b)t} \right\} \)

where \( D = \left\{ \mathbb{E} \left[ (1 - F)^{2ad_t} \right] - \mathbb{P} [d_t \geq t] (1 - F)^{2at} \right\} \)
\[ \cdot \left\{ \mathbb{E} \left[ (1 - F)^{-2bd_t} \right] - \mathbb{P} [d_t \geq t] (1 - F)^{-2bt} \right\} \]

and \( \rho_{a_1,\ldots,a_k} = \rho_{a_1,a_2+\ldots+a_k} \rho_{a_2,\ldots,a_k} \) recursively
Define the Fourier Transform \( \hat{f}(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \, dx \).
Define the Fourier Transform \( \hat{f}(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \, dx \)

Let \( W \) have mean 0 and variance 1 and let \( \phi \) be std normal density
Define the Fourier Transform \( \hat{f}(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \, dx \)

Let \( W \) have mean 0 and variance 1 and let \( \phi \) be std normal density

Write \( \hat{f}_W(t) = \left[ \hat{f}_W(t) \left( \frac{1}{\phi(t)} \right) \right] \phi(t) \) and Taylor expand the bracket
Define the Fourier Transform \( \hat{f}(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \, dx \)

Let \( W \) have mean 0 and variance 1 and let \( \phi \) be std normal density

Write \( \hat{f}_W(t) = \left[ \hat{f}_W(t) \left( \frac{1}{\phi(t)} \right) \right] \phi(t) \) and Taylor expand the bracket

\[
\hat{f}_W(t) = \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \hat{f}_W(t) \left( \frac{1}{\phi(t)} \right) \right]^{(n)}_{t=0} t^n \right\} \phi(t)
\]
Define the Fourier Transform \( \hat{f}(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \, dx \)

Let \( W \) have mean 0 and variance 1 and let \( \phi \) be std normal density

Write \( \hat{f}_W(t) = \left[ \hat{f}_W(t) \left( \frac{1}{\hat{\phi}(t)} \right) \right] \hat{\phi}(t) \) and Taylor expand the bracket

\[
\hat{f}_W(t) = \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \hat{f}_W(t) \left( \frac{1}{\hat{\phi}(t)} \right) \right]_t^{(n)} t^n \right\} \hat{\phi}(t)
\]

So \( f_W(w) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \hat{f}_W(t) \left( \frac{1}{\hat{\phi}(t)} \right) \right]_t^{(n)} i^{-n} \phi^{(n)}(w) \) and use Leibniz’s rule
Where Does the Edgeworth Come From?

• Define the Fourier Transform \( \hat{f}(t) = \int_{-\infty}^{\infty} e^{-itx} f(x) \, dx \)

• Let \( W \) have mean 0 and variance 1 and let \( \phi \) be std normal density

• Write \( \hat{f}_W(t) = \left[ \hat{f}_W(t) \left( \frac{1}{\hat{\phi}(t)} \right) \right] \hat{\phi}(t) \) and Taylor expand the bracket

\[
\hat{f}_W(t) = \left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \hat{f}_W(t) \left( \frac{1}{\hat{\phi}(t)} \right) \right]^{(n)} t^n \right\} \hat{\phi}(t)
\]

• So \( f_W(w) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \hat{f}_W(t) \left( \frac{1}{\hat{\phi}(t)} \right) \right]^{(n)} i^{-n} \phi^{(n)}(w) \) and use Leibniz’s rule

\[
\left[ \hat{f}_W(t) \left( \frac{1}{\hat{\phi}(t)} \right) \right]^{(n)}_{t=0} = \left( \frac{1}{\hat{\phi}(t)} \right)^{(n)}_{t=0} + \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} \hat{f}_W^{(j)}(0) \left( \frac{1}{\hat{\phi}(t)} \right)^{(n-j)}_{t=0}
\]
So \( f_W (w) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \hat{f}_W (t) \left( \frac{1}{\phi(t)} \right) \right]^{(n)}_{t=0} i^{-n} \phi^{(n)} (w) \) and use Leibniz's rule.
Where Does the Edgeworth Come From?

- So \( f_W(w) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \hat{f}_W(t) \left( \frac{1}{\hat{\phi}(t)} \right) \right]^{(n)}_t i^{-n} \phi^{(n)}(w) \) and use Leibniz’s rule

\[
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\]
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- \( \hat{\phi}(t) \left( \frac{1}{\phi(t)} \right) \) is symmetric:

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\]

because \( W \) is mean 0 variance 1
But derivatives of Fourier transforms evaluated at 0 are just moments so
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\left[ \hat{f}_W (t) \left( \frac{1}{\phi(t)} \right) \right]^{(n)}_{t=0} = \sum_{j=3}^{n} \frac{n!(n-j)!}{j!(n-j)!} i^{-j} \left( \mathbb{E} [W^j] - j \right)
\] and

\[
\frac{\phi(2n+j)}{2n+2j} \sum_{k=0}^{2n+j} \frac{(2n+j)!(2k)!}{(2n+2j)!} \left( \frac{1}{2n+2j} \right)^{2k} - j \phi(2n+j)
\]
But derivatives of Fourier transforms evaluated at 0 are just moments so

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\]

\[
f_W(w) = \phi(w) + \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=3}^{n} \frac{n!(n-j)!}{j!(n-j)!} i^{-n-j} (\mathbb{E} [W^j] - j!) \phi^{(n)}(w)
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But derivatives of Fourier transforms evaluated at 0 are just moments so

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\]

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But derivatives of Fourier transforms evaluated at 0 are just moments so

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\]

But

\[
\phi^{(2n+j)}(w) = \left[ \sum_{k=0}^{n+\left\lfloor \frac{j}{2} \right\rfloor} \frac{(2n+j)!(2k)!}{(2n+j-2k)!(2k)!} (-1)^{2n+j-k} w^{2n+j-2k} \right] \phi(w)
\]
Finally, if $Y = \sigma W + \mu$ a change of variables gives the Edgeworth expansion

$$\phi(y) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} + \lim_{N \to \infty} \sum_{j=3}^{\infty} \frac{1}{j!} \mu_j \sigma_j y^j \sigma^j b_j N^{j-2} c_j \sum_{n=0}^{\infty} \binom{2n}{n} \binom{2n+j}{2n} \frac{1}{(2n+j)!} \frac{1}{(2n)!}$$

where $\mu_j$ is the $j$-th central moment of $Y$.
Finally, if \( Y = \sigma W + \mu \) a change of variables gives the Edgeworth expansion:

\[
f_Y(y) = \frac{1}{\sigma} \phi \left( \frac{y-\mu}{\sigma} \right) + \lim_{N \to \infty} \sum_{j=3}^{N} \frac{1}{j!} \left( \frac{\mu_j}{\sigma^j} - j? \right) \sum_{n=0}^{\left\lfloor \frac{N-j}{2} \right\rfloor} \frac{(2n)^n}{(2n)!} (-1)^n 
\sum_{k=0}^{n+\left\lfloor \frac{j}{2} \right\rfloor} \frac{(2n+j)!(2k)!}{(2n+j-2k)!(2k)!} (-1)^k \left( \frac{y-\mu}{\sigma} \right)^{2n+j-2k} \frac{1}{\sigma} \phi \left( \frac{y-\mu}{\sigma} \right)
\]

where \( \mu_j \) is the \( j \)-th central moment of \( Y \). For Esscher (aka Saddlepoint) Expansion, Taylor expand \( h_c \) around a different point than 0. For something even more flexible, use a different function than \( \phi \); try logistic, gamma, inverse gamma or inverse logistic.
Where Does the Edgeworth Come From?

- Finally, if $Y = \sigma W + \mu$ a change of variables gives the Edgeworth expansion

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$$\sum_{k=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \frac{(2n+j)! (2k)!}{(2n+j-2k)! (2k)!} (-1)^k \left( \frac{y-\mu}{\sigma} \right)^{2n+j-2k} \frac{1}{\sigma^{j}} \phi \left( \frac{y-\mu}{\sigma} \right)$$

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Where Does the Edgeworth Come From?

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\[
\sum_{k=0}^{n+\left\lfloor \frac{j}{2} \right\rfloor} \frac{(2n+j)!(2k)?}{(2n+j-2k)!(2k)!} (-1)^k \left( \frac{y-\mu}{\sigma} \right)^{2n+j-2k} \frac{1}{\sigma} \phi \left( \frac{y-\mu}{\sigma} \right)
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\[
\left[ \frac{f_W(t)}{\phi(t)} \right] \text{ around a different point than 0}
\]
Finally, if \( Y = \sigma W + \mu \) a change of variables gives the Edgeworth expansion

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f_Y(y) = \frac{1}{\sigma} \phi \left( \frac{y-\mu}{\sigma} \right) + \lim_{N \to \infty} \sum_{j=3}^{N} \frac{1}{j!} \left( \frac{\mu_j}{\sigma^j} - j? \right) \sum_{n=0}^{\frac{N-j}{2}} \frac{(2n)?}{(2n)!} (-1)^n \cdot \]

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How Do You Get The Moments?

\[ E \left[ r_t' \right] = E \left[ e^{\ln(r_t)} \right] = E \left[ e^Y \right] = \int_{-\infty}^{\infty} e^y f_Y(y) \, dy \]
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- Substitute the Edgeworth expression, complete the square to integrate just as if you were integrating for the lognormal, and expand the binomials that occur when you change variables and you wind up with...
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\[
\mathbb{E} [r_t'] = e^{\mu + \frac{1}{2} (\sigma^2)} \left\{ 1 + \lim_{N \to \infty} \sum_{j=3}^{N} \frac{j^i}{j!} \left( \mu - j?\sigma^j \right) \sum_{n=0}^{N-j} \frac{(2n)?}{(2n)!} (-1)^n (\sigma)^{2n} \right. \\
\left. \cdot \sum_{m=0}^{n+\left\lfloor \frac{j}{2} \right\rfloor} \frac{(2n+j)!}{(2n+j-2m)!} (\sigma)^{-2m} \sum_{k=0}^{m} \frac{(2k)?(2(m-k))}{(2k)!(2(m-k))!} (-1)^k \right\}
\]
How Do You Get The Moments?

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Substitute the Edgeworth expression, complete the square to integrate just as if you were integrating for the lognormal, and expand the binomials that occur when you change variables and you wind up with

\[ \mathbb{E} [r_t'] = e^{l\mu + \frac{1}{2}(l\sigma)^2} \left\{ 1 + \lim_{N \to \infty} \sum_{j=3}^{N} \frac{j^j}{j!} \left( \mu_j - j?\sigma_j \right) \sum_{n=0}^{\left[ \frac{N-j}{2} \right]} \frac{(2n)?}{(2n)!} (-1)^n (l\sigma)^{2n} \cdot \right. \]

\[ \left. \sum_{m=0}^{n+\left[ \frac{j}{2} \right]} \frac{(2n+j)!}{(2n+j-2m)!} (l\sigma)^{-2m} \sum_{k=0}^{m} \frac{(2k)?(2(m-k))?}{(2k)!(2(m-k))!} (-1)^k \right\} \]

Remarkably, \[ \sum_{k=0}^{m} \frac{(2k)?(2(m-k))?}{(2k)!(2(m-k))!} (-1)^k = 0 \text{ when } m > 0 \]
How Do You Get The Moments?

Why? 

\[ 0 = \left[ \frac{\hat{\phi}(t)}{\phi(t)} \right]_{t=0}^{(2m)} = \sum_{k=0}^{m} \frac{(2k)! (2(m-k))!}{(2k)! (2(m-k))!} (-1)^k \]
How Do You Get The Moments?

- Why? \[ 0 = \left[ \hat{\phi}(t) \left( \frac{1}{\phi(t)} \right) \right]_{t=0}^{(2m)} = \sum_{k=0}^{m} \frac{(2k)! (2(m-k))!}{(2k)! (2(m-k))!} (-1)^k \]

- Finally, \( \mathbb{E} \left[ r_t^l \right] = \)

\[
e^{l \mu + \frac{1}{2} (l \sigma)^2} \left\{ 1 + \lim_{N \to \infty} \sum_{j=3}^{N} \frac{j^i}{j!} \left( \mu_j - j \sigma^j \right) \sum_{n=0}^{\frac{N-i}{2}} \frac{(2n)!}{(2n)!} (-1)^n \left( l \sigma \right)^{2n} \right\}
\]
What About the Main Results?

- Proofs were inspired by techniques in Decoupling: From Dependence to Independence, by de la Peña and Giné
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- Conditional on \( J = j' > 1 \) the following are each independent sets: \( \{J, \overline{d}_1\} \), \( \{\overline{d}_1, d_2, \ldots, d_{j'-1}\} \), \( \{d_2, \ldots, d_{j'-1}, \overline{d}_{j'}\} \) and \( \{J, \overline{d}_{j'}\} \)
What About the Main Results?

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- This is enough independence to get a geometric series inside the main expectation \( \mathbb{E} \left[ \sum_{j=1}^{\infty} \nu(e_j) \right] \) and to pull apart the two sides of the correlation expectation for \( \rho_{a,b} \), leaving a common term involving \( \overline{d}_1 \) and \( \overline{d}_j \) which can be evaluated by writing
  \( \overline{d}_1 = t - (\overline{d}_j + d_{j-1} + \ldots + d_2) \)