

As of June 1, 2011 here are some more corrections and extensions of the 12-21-2007 paper in ARCH2008.1

Section (c) on pages 11 and 12 of the 8-12-2008 corrections needs to be replaced with the following:

(c) **Some of the incomplete gamma functions in section 3.3 for calculating $\mathcal{L}_{\bar{d}\wedge t}(x)$ and $\mathcal{L}_{\bar{d}}(x)$ are undefined when $x \leq 0$**

In this case, the definition of $\mathcal{L}_{\bar{d}\wedge t}(x)$ gives one of four results:

(i) If $x = 0$ L'Hôpital's rule applied to the expressions in section 3.3 gives

$$\mathcal{L}_{\bar{d}\wedge t}(0) = 1 \text{ and } \mathcal{L}_{\bar{d}}(0) = 1.$$

(ii) If $-\frac{1}{\beta} < x < 0$ the expression in section 3.3 for $\mathcal{L}_{\bar{d}\wedge t}(x)$ remains valid and can be used in calculations for finite t .

$$\begin{aligned} \mathcal{L}_{\bar{d}\wedge t}(x) &= \frac{1}{\alpha\beta x} \left\{ 1 - e^{-xt} \left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right) \right] \right. \\ &\quad \left. - (1 + \beta x)^{-\alpha} \Gamma\left(\alpha; \frac{(1 + \beta x)t}{\beta}\right) \right\} + e^{-xt} \mathbb{P}[\bar{d} \geq t]. \end{aligned} \tag{e.5}$$

An application of L'Hôpital's rule on the two exponential terms above is enough to show that

$$\mathcal{L}_{\bar{d}}(x) = \frac{1}{\alpha\beta x} \left\{ 1 - (1 + \beta x)^{-\alpha} \right\}.$$

(iii) If $x = -\frac{1}{\beta}$ L'Hôpital's rule applied to the third term in brackets in (e.5) gives

$$\mathcal{L}_{\bar{d}\wedge t}\left(-\frac{1}{\beta}\right) = -\frac{1}{\alpha} \left\{ 1 - e^{\frac{1}{\beta}t} \left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right) \right] - \frac{t^\alpha}{\Gamma(\alpha + 1)\beta^\alpha} \right\} + e^{\frac{1}{\beta}t} \mathbb{P}[\bar{d} \geq t] \tag{e.6}$$

which can be used in calculations for finite t and makes $\mathcal{L}_{\bar{d}}(-\frac{1}{\beta}) = \infty$ obvious.

(iv) If $x < -\frac{1}{\beta}$ the integral that defines the incomplete gamma function in (e.5) is still available and can be integrated by parts repeatedly, on powers of s :

$$\begin{aligned} \mathcal{L}_{\bar{d}\wedge t}(x) &= \frac{1}{\alpha\beta x} \left\{ 1 - e^{-xt} \left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right) \right] \right. \\ &\quad \left. - \int_0^t \frac{s^{\alpha-1} e^{-s(x+\frac{1}{\beta})}}{\Gamma(\alpha)\beta^\alpha} ds \right\} + e^{-xt} \mathbb{P}[\bar{d} \geq t] \tag{e.7} \\ &= \frac{1}{\alpha\beta x} \left\{ 1 - e^{-xt} \left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right) \right] \right. \\ &\quad \left. - \frac{t^\alpha}{\Gamma(\alpha)\beta^\alpha} e^{-t(x+\frac{1}{\beta})} \sum_{n=0}^{\infty} \frac{(x+\frac{1}{\beta})^n t^n}{\alpha(\alpha+1)\cdots(\alpha+n)} \right\} + e^{-xt} \mathbb{P}[\bar{d} \geq t] \end{aligned} \tag{e.7.1}$$

Note that for $t = \infty$ and $x < -\frac{1}{\beta} < 0$ (e.7) already is enough to show that $\mathcal{L}_{\bar{d}}(x) = \infty$.

For finite t how many terms of the sum in (e.7.1) are needed to calculate $\mathcal{L}_{\bar{d} \wedge t}(x)$ within roughly an absolute accuracy of ϵ ? Note that the terms of the sum are alternating in sign ($x + \frac{1}{\beta} < 0$) so compare the sum of two successive terms with $\frac{\epsilon}{2}$:

$$\begin{aligned} & \left| \frac{1}{\alpha\beta x} \frac{t^\alpha}{\Gamma(\alpha)\beta^\alpha} e^{-t(x+\frac{1}{\beta})} \frac{\left((\alpha+n) + \left(x + \frac{1}{\beta}\right)t\right) \left(x + \frac{1}{\beta}\right)^{n-1} t^{n-1}}{\alpha(\alpha+1) \cdots (\alpha+n)} \right| < \frac{\epsilon}{2} \\ & \left| \frac{t^\alpha}{\alpha\beta^{\alpha+1} x \Gamma(\alpha+n+1)} e^{-t(x+\frac{1}{\beta})} \left((\alpha+n) + \left(x + \frac{1}{\beta}\right)t\right) \left(x + \frac{1}{\beta}\right)^{n-1} t^{n-1} \right| < \frac{\epsilon}{2} \text{ so} \\ & (n-1) \ln \left(\left| \left(x + \frac{1}{\beta}\right)t \right| \right) + \ln \left(\left| \frac{\left((\alpha+n) + \left(x + \frac{1}{\beta}\right)t\right)}{\Gamma(\alpha+n+1)} \right| \right) < -t \left(x + \frac{1}{\beta}\right) \ln \left(\left| \frac{\epsilon}{2} \alpha \beta^{\alpha+1} x t^{-\alpha} \right| \right) \end{aligned} \tag{e.7.2}$$

Now set the condition for the ratio between two successive pairs of sums of two successive terms to be less than $\frac{1}{2}$ in absolute value.

$$\left| \frac{\alpha + n + 2 + \left(x + \frac{1}{\beta}\right)t}{\alpha + n + \left(x + \frac{1}{\beta}\right)t} \cdot \frac{\left(x + \frac{1}{\beta}\right)^2 t^2}{(\alpha + n + 1)(\alpha + n + 2)} \right| < \frac{1}{2}$$

which approximately follows when

$$\begin{aligned} & \frac{\left|x + \frac{1}{\beta}\right|t}{\alpha + n} < \frac{1}{\sqrt{2}} \\ & \sqrt{2} \left|x + \frac{1}{\beta}\right|t - \alpha < n. \end{aligned} \tag{e.7.3}$$

If n is chosen large enough to satisfy both (e.7.2) and (e.7.3) then the approximation in using n terms of the sum in (e.7.1) should be accurate within an absolute value of ϵ .

Going back to (e.7) we can get some more information out of integrating by

parts on the exponential rather than on the powers of s :

$$\begin{aligned}
\mathcal{L}_{\bar{d}\wedge t}(x) &= \frac{1}{\alpha\beta x} \left\{ \begin{array}{l} 1 - e^{-xt} \left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right) \right] \\ - \int_0^t \frac{s^{\alpha-1} e^{-s(x+\frac{1}{\beta})}}{\Gamma(\alpha)\beta^\alpha} ds \end{array} \right\} + e^{-xt} \mathbb{P}[\bar{d} \geq t] \\
&= \mathcal{L}_{\bar{d}\wedge t}(x) = \frac{1}{\alpha\beta x} \left\{ \begin{array}{l} 1 - e^{-xt} \left[1 - \Gamma\left(\alpha; \frac{t}{\beta}\right) \right] \\ + \frac{1}{\Gamma(\alpha)\beta^\alpha} e^{-t(x+\frac{1}{\beta})} \sum_{j=0}^k \frac{(\alpha-1)\cdots(\alpha-j)}{(x+\frac{1}{\beta})^{j+1}} t^{\alpha-j-1} \\ - \frac{(\alpha-1)\cdots(\alpha-k-1)}{\Gamma(\alpha)\beta^\alpha (x+\frac{1}{\beta})^{k+1}} e^{-t(x+\frac{1}{\beta})} \int_0^t s^{n-k-2} e^{-(s-t)(x+\frac{1}{\beta})} ds \end{array} \right\} + e^{-xt} \mathbb{P}[\bar{d} \geq t]
\end{aligned} \tag{e.8}$$

for any $k \leq \lfloor \alpha - 1 \rfloor$. If α happens to be an integer this provides an exact calculation for $\mathcal{L}_{\bar{d}\wedge t}(x)$ since the sum terminates with a zero error term when $k = \alpha - 1$. In this case, one need not use the approximation in (e.7.1) through (e.7.3). In addition to providing this exact calculation for integer α , (e.8) also helps to provide estimates for some later expressions involving $\mathcal{L}_{\bar{d}\wedge t}(x)$. For now, note that the integral in the error term in (e.8) is $O(t^{\alpha-k-2})$ because $-(s-t)(x+\frac{1}{\beta}) < 0$ and $e^{-(s-t)(x+\frac{1}{\beta})}$ makes the integrand irrelevant until s is close to t .

General Formulae To Calculate (2.8) and Related Expected Values

Corresponding to (2.10), (2.11) corrected, and page 7 of the 8-12-2008 corrections the following general formula is available:

$$\begin{aligned}
&\mathbb{E} \left[\left(\sum_{j=1}^{\infty} c_j^2 \right)^n \right] = \\
&= \sum_{\{i_l: \sum_l l \cdot i_l = n\}} \frac{n!}{\prod_l l^{i_l}} \rho_{\{i_l\}} \left\{ \sum_{\{j_m, i_{l,m}: \sum_m j_m \cdot i_{l,m} = i_l\}} \prod_m \frac{1}{j_m!} \left[(-1)^{\sum_l i_{l,m-1}} \cdot \right. \right. \\
&\quad \left. \left. \cdot \frac{(\sum_l i_{l,m} - 1)!}{\prod_l i_{l,m}!} \sum_{j=1}^{\infty} \left(\prod_l \mathbb{E} [e_j^{2l}]^{i_{l,m}} \right) \right]^{j_m} \right\} \tag{2.8}
\end{aligned}$$

defining $\rho_{\{i_l\}} = \rho_{\dots, l+1, \dots, l+1, l, \dots, l, l-1, \dots, l-1, \dots}$

where each l , for $n \geq l \geq 1$, appears i_l times.

Corresponding to what's needed to compute (2.6) in the 8-12-2008 corrections:

$$\begin{aligned}
& \mathbb{E} \left[\left(\sum_{j=1}^{\infty} e_j^2 \right)^n (1-F)^{-k\mathbf{t}_1 \wedge t} \right] = \\
& = \sum_{\left\{ i_l : \sum_l l \cdot i_l = n \right\}} \frac{n!}{\prod_l l^{i_l}} \rho_{\{i_l\}, -\frac{k}{2}} \left\{ \sum_{\left\{ j_m, i_{l,m} : \sum_m j_m \cdot i_{l,m} = i_l \right\}} \prod_m \frac{1}{j_m!} \left[(-1)^{\sum_l i_{l,m-1}} \cdot \right. \right. \\
& \quad \left. \left. \cdot \frac{(\sum_l i_{l,m} - 1)!}{\prod_l i_{l,m}!} \sum_{j=0}^{\infty} \left(\prod_l \mathbb{E} [e_j^{2l}]^{i_{l,m}} \right) \right]^{j_m} \right\} \cdot \mathbb{E} [(1-F)^{-k\mathbf{t}_1 \wedge t}]
\end{aligned}$$

(2.6.f)

By the rules for ρ at (4.5.8) $\rho_{\{i_l\}, -\frac{k}{2}} = \rho_{\{i_l\}} \rho_{\sum_l l \cdot i_l, -\frac{k}{2}} = \rho_{\{i_l\}} \rho_{n, -\frac{k}{2}}$, so we can simplify (2.6.f) slightly to

$$\begin{aligned}
& \mathbb{E} \left[\left(\sum_{j=1}^{\infty} e_j^2 \right)^n (1-F)^{-k\mathbf{t}_1 \wedge t} \right] = \\
& = \sum_{\left\{ i_l : \sum_l l \cdot i_l = n \right\}} \frac{n!}{\prod_l l^{i_l}} \rho_{\{i_l\}} \left\{ \sum_{\left\{ j_m, i_{l,m} : \sum_m j_m \cdot i_{l,m} = i_l \right\}} \prod_m \frac{1}{j_m!} \left[(-1)^{\sum_l i_{l,m-1}} \cdot \right. \right. \\
& \quad \left. \left. \cdot \frac{(\sum_l i_{l,m} - 1)!}{\prod_l i_{l,m}!} \sum_{j=0}^{\infty} \left(\prod_l \mathbb{E} [e_j^{2l}]^{i_{l,m}} \right) \right]^{j_m} \right\} \rho_{n, -\frac{k}{2}} \mathbb{E} [(1-F)^{-k\mathbf{t}_1 \wedge t}]
\end{aligned}$$

(2.6.g)

A complete formula for computing (2.6)

If we take the formula for (2.6) in the 8-12-2008 corrections, expand the two binomials, and interchange the order of summations so that powers of $\sum_{j=1}^{\infty} e_j^2$ are summed over first, followed by powers of $(1-F)^{-\mathbf{t}_1 \wedge t}$, and finally substitute the expression (2.6.g) just derived into each occurrence of $\mathbb{E} \left[\left(\sum_{j=1}^{\infty} e_j^2 \right)^l (1-F)^{-k\mathbf{t}_1 \wedge t} \right]$, the resulting expression is:

$$\begin{aligned}
& \mathbb{E}[(\ln(r_t) - \mathbb{E}[\ln(r_t)])^n] = \\
& = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{l!} \sigma_T^{2l} \mathbb{E} \left[\left(\sum_{j=1}^{\infty} e_j^2 \right)^l \sum_{k=0}^{n-2l} \frac{1}{k!} (1-F)^{-k\mathbf{t}_1 \wedge t} \cdot \right. \\
& \quad \left. \sum_{m=0}^{\lfloor \frac{n-k-2l}{2} \rfloor} \frac{n!(2(m+l))^{?}}{(n-2(m+l))!(2(m+l))!} (-1)^{n-2(m+l)-k} (\ln(T_0) - \mu_T)^{n-2(m+l)} \cdot \right. \\
& \quad \left. (1-F)^{(n-2(m+l))t} \frac{(n-2(m+l))!}{(n-2(m+l)-k)!} \mathbb{E}[(1-F)^{-\mathbf{t}_1 \wedge t}]^{n-2(m+l)-k} \cdot \right. \\
& \quad \left. \frac{(m+l)!}{m!} \sigma^{2m} dt^m (1-F)^{2mdt} \left(\frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}} \right)^m \right] \\
& = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{l!} \sigma_T^{2l} \left(\sum_{\left\{ i_g : \sum_g i_g = l \right\}} \frac{l!}{\prod_g i_g!} \rho_{\{i_g\}} \right) \left\{ \sum_{\left\{ j_h, i_{g,h} : \sum_h j_h \cdot i_{g,h} = i_g \right\}} \prod_h \frac{1}{j_h!} \cdot \right. \\
& \quad \left. \left[(-1)^{\left(\sum_g i_{g,h-1} \right)} \frac{\left(\sum_g i_{g,h} - 1 \right)!}{\prod_g i_{g,h}!} \sum_{j=1}^{\infty} \left(\prod_g \mathbb{E} \left[e_j^{2g} \right]^{i_{g,h}} \right)^{j_h} \right] \right\} \left(\sum_{k=0}^{n-2l} \frac{1}{k!} \rho_{l, -\frac{k}{2}} \cdot \right. \\
& \quad \mathbb{E}[(1-F)^{-k\mathbf{t}_1 \wedge t}] (\ln(T_0) - \mu_T)^k (1-F)^{kt} \left[\sum_{m=0}^{\lfloor \frac{n-k-2l}{2} \rfloor} \frac{n!(2(m+l))^{?}(m+l)!}{(n-2(m+l)-k)!(2(m+l))!m!} \cdot \right. \\
& \quad \left. [-(\ln(T_0) - \mu_T)(1-F)^t \mathbb{E}[(1-F)^{-\mathbf{t}_1 \wedge t}]]^{n-2(m+l)-k} \sigma^{2m} dt^m (1-F)^{2mdt} \cdot \right. \\
& \quad \left. \left. \left(\frac{1-(1-F)^{2t}}{1-(1-F)^{2dt}} \right)^m \right] \right)
\end{aligned}$$

(2.6)

A Slight Restatement of (3.5) and (3.6)

The closed form just presented for (2.6) requires the ability to evaluate expressions such as

$$\sum_{j=1}^{\infty} \left(\prod_{k=1}^n \mathbb{E} [e_j^k]^{n_k} \right). \quad (3.5.a)$$

This can be evaluated by (3.5) in the original paper (or (3.6) for $t = \infty$). This is not immediately apparent because (3.5) is written in a slightly different form. Here is how to restate (3.5) to a form consistent with the form of (3.5.a).

Given the expression (3.5.a), for any variable x (whether random or not) define

$$\nu(x) = \prod_{k=1}^n \mathbb{E} [x^k]^{n_k}$$

Obviously if x is a non-random variable then

$$\nu(x) = \prod_{k=1}^n x^{k \cdot n_k}$$

This definition of $\nu(x)$ means that (3.5.a) looks like

$$\sum_{j=1}^{\infty} \left(\prod_{k=1}^n \mathbb{E} [e_j^k]^{n_k} \right) = \sum_{j=0}^{\infty} \nu(e_j).$$

This definition of $\nu(x)$ is equivalent to the one on page 11 of the original paper. The only change is that the original paper had

$$\mathbb{E} \left[\sum_{j=1}^{\infty} \left(e_j^n \prod_{k=1}^n \mathbb{E} [e_j^k]^{n_k} \right) \right] \text{ and } \nu(x) = x^n \prod_{k=1}^n \mathbb{E} [x^k]^{n_k}.$$

So long as monotone or dominated convergence applies (monotone applies in all of our applications) then

$$\begin{aligned} \mathbb{E} \left[\sum_{j=1}^{\infty} \left(e_j^n \prod_{k=1}^n \mathbb{E} [e_j^k]^{n_k} \right) \right] &= \sum_{j=1}^{\infty} \left(\mathbb{E} [e_j^n]^{n_n+1} \prod_{k=1}^{n-1} \mathbb{E} [e_j^k]^{n_k} \right) \text{ and} \\ \mathbb{E} \left[x^n \prod_{k=1}^n \mathbb{E} [x^k]^{n_k} \right] &= \mathbb{E} [x^n]^{n_n+1} \prod_{k=1}^{n-1} \mathbb{E} [x^k]^{n_k} \text{ for random } x \text{ and} \\ x^n \prod_{k=1}^n x^{k \cdot n_k} &= x^{n \cdot (n_n+1)} \prod_{k=1}^{n-1} x^{k \cdot n_k} \text{ for non-random } x. \end{aligned}$$

Therefore (3.5) and (3.5.a), together with the corresponding versions of $\nu(x)$, differ only in how to count the exponent on $\mathbb{E} [e_j^n]$ or $\mathbb{E} [x^n]$, where n is the highest power of e_j or x that appears.

With this notational understanding we can now rewrite (3.5) as

$$\begin{aligned}
& \sum_{j=1}^{\infty} \left(\prod_{k=1}^n \mathbb{E} [e_j^k]^{n_k} \right) = \sum_{j=1}^{\infty} \nu(e_j) \\
& = \left(1 - (1-G)^t \frac{\mathbb{E} [(1-G)^{-\bar{\mathbf{d}} \wedge t}] - \mathbb{P} [\bar{\mathbf{d}} \geq t] (1-G)^{-t}}{\mathbb{E} [(1-G)^{\bar{\mathbf{d}} \wedge t}] - \mathbb{P} [\bar{\mathbf{d}} \geq t] (1-G)^t} \right) \cdot \\
& \quad \left\{ \nu \left((1-F)^{\bar{\mathbf{d}} \wedge t} \right) - \mathbb{P} [\bar{\mathbf{d}} \geq t] \nu \left((1-F)^t \right) \right\} \frac{\nu \left(1 - (1-F)^{\bar{\mathbf{d}}} \right)}{1 - \nu \left((1-F)^{\bar{\mathbf{d}}} \right)} + \\
& \quad + \nu \left(1 - (1-F)^{\bar{\mathbf{d}} \wedge t} \right) - \mathbb{P} [\bar{\mathbf{d}} \geq t] \nu \left(1 - (1-F)^t \right) \tag{3.5.a}
\end{aligned}$$

and (3.6) as

$$\begin{aligned}
& \lim_{t \rightarrow \infty} \sum_{j=1}^{\infty} \left(\prod_{k=1}^n \mathbb{E} [e_j^k]^{n_k} \right) = \lim_{t \rightarrow \infty} \sum_{j=1}^{\infty} \nu(e_j) \\
& = \frac{\nu \left((1-F)^{\bar{\mathbf{d}}} \right) \nu \left(1 - (1-F)^{\bar{\mathbf{d}}} \right)}{1 - \nu \left((1-F)^{\bar{\mathbf{d}}} \right)} + \nu \left(1 - (1-F)^{\bar{\mathbf{d}}} \right) \tag{3.6.a}
\end{aligned}$$

Now (3.5.a) and (3.6.a) are suitable for evaluating the expression above for (2.6).

On page 10 of the 8-12-2008 corrections there is an error in the asymptotic expression for drift compensation term D_t as $t \rightarrow \infty$. The correct formula is

$$\begin{aligned}
& \lim_{t \rightarrow \infty} D_t = \\
& = -\frac{1}{2} \sigma^2 \frac{(1-F)^{dt}}{1 + (1-F)^{dt}} + \frac{1}{2} \sigma_T^2 \frac{1 - (1-F)^{dt}}{dt(1-F)^{dt}} \left(1 - \lim_{t \rightarrow \infty} \mathbb{E} \left[\sum_{j=1}^{\infty} e_j^2 \right] \right)
\end{aligned}$$