## Math 210Q Final Exam Review Sheet – Answers

1. If $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{w} = -2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, then find

   (a) $\boldsymbol{v} + 2\boldsymbol{w} =< -3, 4, 6 >$ (b) $\boldsymbol{v} \cdot \boldsymbol{w} = 4$

   (c) $|\boldsymbol{v}| = 3$ and $|\boldsymbol{w}| = 3$ (d)$\theta = \cos^{-1}(4/9)$

   (e) $|\boldsymbol{v}|\cos(\theta) = 4/3$ (f) $\frac{\boldsymbol{v}\times\boldsymbol{w}}{|\boldsymbol{v}\times\boldsymbol{w}|} = \frac{1}{\sqrt{65}} < 2, -6, 5 >$

2. Let $P = (1, 2, 3)$, $Q = (2, 4, 5)$, and $R = (-1, 3, 5)$.

   (a) The line through the points $P$ and $Q$ is given by:
   $x(t) = 1 + t,\ y(t) = 2 + 2t,\ z(t) = 3 + 2t,\ 0 \leq t \leq 1.$

   (b) An equation of the plane through $P$, $Q$, and $R$ is:
   $\langle 2, -6, 5\rangle \cdot \langle x - 1, y - 2, z - 3\rangle = 0$ or $2(x - 1) - 6(y - 2) + 5(z - 3) = 0$.
   The normal vector we used is $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, -6, 5\rangle$. (Compare with 1(f)).

3. A particle moves on the curve $\boldsymbol{r}(t) = (\ln t)\boldsymbol{i} + 2t\boldsymbol{j} + t^2\boldsymbol{k}$, where $t > 0$.

   (a) $\boldsymbol{v}(t) = \left\langle \frac{1}{t}, 2, 2t \right\rangle$
   $v(t) = |\boldsymbol{v}(t)| = \sqrt{\frac{1}{t^2} + 4 + 4t^2}$
   $\boldsymbol{a}(t) = \left\langle -\frac{1}{t^2}, 0, 2 \right\rangle$

   (b) Find the arc length from $t = 1$ to $t = 2$:

$$= \int_1^2 v(t)\,dt = \int_1^2 \sqrt{\frac{1}{t^2} + 4 + 4t^2}\,dt = \int_1^2 \sqrt{\frac{1 + 4t^2 + 4t^4}{t^2}}\,dt$$

$$= \int_1^2 \sqrt{\frac{(1 + 2t^2)^2}{t^2}}\,dt = \int_1^2 \frac{(1 + 2t^2)}{t}\,dt = \ln(2) + 3.$$

   (c) Find the unit tangent vector and the unit normal vector at $t = 1$:

$$\boldsymbol{T}(t) = \frac{\boldsymbol{v}(t)}{v(t)} = \frac{\left\langle \frac{1}{t}, 2, 2t \right\rangle}{\sqrt{\frac{1}{t^2} + 4 + 4t^2}} = \left\langle \frac{1}{1 + 2t^2}, \frac{2t}{1 + 2t^2}, \frac{2t^2}{1 + 2t^2} \right\rangle$$

$$\boldsymbol{T}(1) = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle$$

$$\boldsymbol{T}'(t) = \left\langle \frac{-4t}{(1 + 2t^2)^2}, \frac{2(1 + 2t^2) - 2t(4t)}{(1 + 2t^2)^2}, \frac{4t(1 + 2t^2) - 2t^2(4t)}{(1 + 2t^2)^2} \right\rangle$$

$$\boldsymbol{T}'(1) = \left\langle -\frac{4}{9}, -\frac{2}{9}, \frac{4}{9} \right\rangle$$

$$\boldsymbol{N}(1) = \frac{\boldsymbol{T}'(1)}{|\boldsymbol{T}'(1)|} = \frac{\left\langle -\frac{4}{9}, -\frac{2}{9}, \frac{4}{9} \right\rangle}{2/3} = \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$$

4. Let $w = \ln(x^2 + y^2 + z^2)$, where $x = \sin(st)$, $y = \cos(st)$, and $z = s^2t^2$. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ **in terms of $s$ and $t$.**

$$\frac{\partial w}{\partial s} = \frac{4s^3t^4}{1 + s^4t^4} \qquad \frac{\partial w}{\partial t} = \frac{4s^4t^3}{1 + s^4t^4}$$

5. Find and classify all the critical points of $f(x,y) = 12x + 27y - x^3 - y^3$.

(2,3) is a local maximum, (-2,3) is a saddle point, (2,-3) is a saddle point, (-2,-3) is a local minimum.

6. Find the equation of the tangent plane to the surface $z = xe^y$ at the point where $x = 0$ and $y = 1$.

$$z - 0 = e(x-0) + 0(y-1) \quad \text{or} \quad z = ex.$$

7. Find the directional derivative of $f(x,y,z) = x^y + x\sqrt{1+z}$, at the point $(1,2,3)$ in the direction of the vector $\boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$.

$$D_{\boldsymbol{u}}f(1,2,3) = \nabla f(1,2,3) \cdot \frac{\boldsymbol{v}}{|\boldsymbol{v}|} = \left\langle 4, 0, \frac{1}{4} \right\rangle \cdot \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle = \frac{5}{2}$$

8. Evaluate $\displaystyle\iint_D e^{-x^2-y^2}\, dA$, where $D$ is the region inside the unit circle $x^2 + y^2 = 1$.

$$\pi\left(1 - \frac{1}{e}\right)$$

9. Find the volume of the solid $E$ which lies below the graph of $z = x^2 + y^2$, and above the region $D$ on $xy-$plane, where $D$ is bounded by $y = x^2$ and $x = y^2$.

$$\text{Volume} \; = \; \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)\, dy\, dx \; = \; \frac{6}{35}$$

10. Evaluate $\displaystyle\iiint_E \frac{1}{x^2+y^2+z^2}\, dV$, where $E$ is the region between the two spheres $x^2 + y^2 + z^2 = 1, x^2 + y^2 + z^2 = 9$, and inside the cone $z = \sqrt{3x^2 + 3y^2}$.

$$= \int_0^{\pi/6} \int_0^{2\pi} \int_1^3 \left(\frac{1}{\rho^2}\right) \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi \; = \; 4\pi\left(1 - \frac{\sqrt{3}}{2}\right)$$

11. Calculate the work done by the force field $\boldsymbol{F}(x,y) = xy\boldsymbol{i} + x^2\boldsymbol{j}$ in moving a particle along the curve $C$ described by the vector function $\boldsymbol{r}(t) = \langle \sin(t), 1+t \rangle$, $0 \le t \le \pi$.

$$\text{Work} \; = \; \int_0^{\pi} \langle (\sin t)(1+t), \sin^2 t \rangle \cdot \langle \cos t, 1 \rangle \, dt \; = \; \frac{\pi}{4}$$

12. Evaluate the line integral $\displaystyle\int_C x^3 z\, ds$ where $C$ is the curve described by the vector function $\boldsymbol{r}(t) = \langle 2\sin(t), t, 2\cos(t) \rangle$, $0 \le t \le \pi/2$.

$$= \int_0^{\pi/2} \Big(2\sin t\Big)^3 \Big(2\cos t\Big) \sqrt{5}\, dt \; = \; 4\sqrt{5}$$

13. Evaluate the line integral $\int_C (e^x + xy)dx + (x + y^5 + \sin(y^2))dy$ where $C$ is the rectangle with vertices $(0,0), (0,2), (2,4), (0,4)$, traversed counterclockwise.

$$\text{Answer} = \int_0^4 \int_0^2 (x-1)\,dx\,dy = 0.$$

14. Let $\boldsymbol{F}(x,y) = (x^2 + 3y\cos(x))\boldsymbol{i} + (3\sin(x) + y)\boldsymbol{j}$ and let $C$ be the line segment from $(0,0)$ to $(1,2)$.

   (a) $\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \frac{7}{3} + 6\sin(1)$

   (b) The vector field is conservative ($P_y = Q_x$), so the line integral is independent of path. Since $C$ and $C_1$ have the same initial and terminal points, the values are equal.

15. Evaluate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where $\boldsymbol{F}(x,y,z) = (3x^2 + yz)\boldsymbol{i} + (xz + 2z^3)\boldsymbol{j} + (2x^2z + 3yz^2)\boldsymbol{k}$, and $C$ is the curve with vector equation $\boldsymbol{r}(t) = t^2\boldsymbol{i} + t^3\boldsymbol{j} + t\boldsymbol{k},\ 0 \le t \le 1$.

$$= \int_0^1 22t^5\,dt = \frac{22}{6} = \frac{11}{3}$$

16. Let $f(x,y,z) = x^2yz$. Calculate the following:

   (a) $\nabla f = \langle 2xyz, x^2z, x^2y \rangle$

   (b) $\nabla \cdot (\nabla f) = 2yz$

   (c) $\nabla \times (\nabla f) = \vec{0}$ (the zero vector)

17. Let $\boldsymbol{F}(x,y,z) = \langle x^2 + y, yz, xy \rangle$. Calculate the following:

   (a) $\nabla \cdot \boldsymbol{F} = 2x + z$

   (b) $\nabla \times \boldsymbol{F} = \langle x - y, -y, -1 \rangle$

   (c) $\nabla(\nabla \cdot \boldsymbol{F}) = \langle 2, 0, 1 \rangle$

   (d) $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$

18. Let $\boldsymbol{G}(x,y,z) = x^2y\boldsymbol{i} + yz^2\boldsymbol{j} + y^2z\boldsymbol{k}$. Determine if $\boldsymbol{G}$ is conservative. If so, find a potential for $\boldsymbol{G}$.

$$\nabla \times \boldsymbol{G} = \langle 0, 0, -x^2 \rangle \neq \vec{0} \text{ (not conservative)}$$

19. Let $\boldsymbol{F}(x,y,z) = (y + ze^x)\boldsymbol{i} + (x + y)\boldsymbol{j} + (e^x - \cos(z))\boldsymbol{k}$. Determine if $\boldsymbol{F}$ is conservative. If so, find a potential for $\boldsymbol{F}$.

Potential function for $\boldsymbol{F}$ is $f(x,y,z) = xy + \frac{y^2}{2} + e^x z - \sin(z)$.

20. Find the area of the surface $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

$$\text{Area} = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\, r\,dr\,d\theta = \frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right)$$