Math 210Q **Name:** SOLUTIONS

Practice Exam 3

Directions: Show all of your work and clearly indicate your answers. Correct answers with no work may not receive credit.

1. Calculate the iterated integrals

a. $\displaystyle\int_1^2 \int_0^2 (y + 2xe^y)\ dx\ dy$

$$= \int_1^2 (2y + 4e^y)\ dy = 3 + 4e^2 - 4e$$

b. $\displaystyle\int_0^1 \int_0^x \cos(x^2)\ dy\ dx$

$$= \int_0^1 \cos(x^2)\, x\ dx = \frac{1}{2}\sin(x^2)\Big|_0^1 = \frac{1}{2}\sin(1)$$

c. $\displaystyle\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} y\sin(x)\ dz\ dy\ dx$

$$= \int_0^\pi \int_0^1 \sin(x)\, y\sqrt{1-y^2}\ dy\ dx = \int_0^\pi \sin(x)\ dx \cdot \int_0^1 y\sqrt{1-y^2}\ dy$$

$$= \Big(-\cos(x)\Big)\Big|_0^\pi \cdot \Big(-\frac{1}{2}\Big)\frac{2}{3}\Big(1-y^2\Big)^{3/2}\Big|_0^1 = \frac{2}{3}$$

2. Evaluate $\displaystyle\iint_D xy\ dA$, where $D = \{(x,y) : 0 \le y \le 1, y^2 \le x \le y+2\}$.

$$= \int_0^1 \int_{y^2}^{y+2} xy\ dx\ dy = \int_0^1 \frac{1}{2}\Big(y^3 + 4y^2 + 4y - y^5\Big)\ dy = \frac{41}{24}$$

3. Evaluate $\displaystyle\iint_D \frac{y}{1+x^2}\ dA$, where D is bounded by $y = \sqrt{x}, y = 0, x = 1$.

$$= \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\ dy\ dx = \frac{1}{2}\int_0^1 \frac{x}{1+x^2}\ dx = \frac{1}{4}\ln(1+x^2)\Big|_0^1 = \frac{1}{4}\ln(2)$$

4. Calculate the iterated integral: $\int_0^1 \int_x^1 \cos(y^2)\ dy\ dx$. [Change order of integration.]

$$= \int_0^1 \int_0^y \cos(y^2)\ dx\ dy = \int_0^1 \cos(y^2) \cdot y\ dy = \frac{1}{2}\sin(y^2)\Big|_0^1 = \frac{1}{2}\sin(1)$$

5. Calculate the iterated integral: $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\ dx\ dy$. [Change order of integration.]

$$= \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3}\ dy\ dx = \int_0^1 \frac{e^{x^2}}{x^3} \cdot \frac{y^2}{2}\Big|_0^{x^2}\ dx = \int_0^1 \frac{e^{x^2}}{x^3} \cdot \frac{x^4}{2}\ dx$$

$$= \frac{1}{2}\int_0^1 e^{x^2}\ x\ dx = \frac{1}{4}e^{x^2}\ \Big|_0^1 = \frac{1}{4}(e-1)$$

6. Find the volume of the solid under the paraboloid $z = x^2 + 4y^2$ and above the rectangle $R = [0, 2] \times [1, 4]$.

$$= \int_1^4 \int_0^2 (x^2 + 4y^2)\ dx\ dy = \int_1^4 \left(\frac{8}{3} + 8y^2\right)\ dy = \frac{528}{3} = 176$$

7. Calculate the volume of the solid under the surface $z = x^2 y$ that lies above the triangle in the xy-plane with vertices $(1, 0), (2, 1)$ and $(4, 0)$. [Warning: tedious algebra ahead.]

$$= \int_0^1 \int_{y+1}^{-2y+4} x^2 y\ dx\ dy = \frac{1}{3}\int_0^1 \Big(-9y^4 + 45y^3 - 99y^2 + 63y\Big)\ dy = \frac{53}{20}$$

Alternatively,

$$= \int_1^2 \int_0^{x-1} x^2 y\ dy\ dx + \int_2^4 \int_0^{-\frac{x}{2}+2} x^2 y\ dy\ dx = \frac{31}{60} + \frac{32}{15} = \frac{53}{20}$$

8. Evaluate $\int\int_D \sqrt{x^2 + y^2}\ dA$, where D is the region in the first quadrant bounded by the circle $x^2 + y^2 = 2$ and the $x-$ and the $y-$axes.

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} r \cdot\ r\ dr\ d\theta = \frac{r^3}{3}\Big|_0^{\sqrt{2}} \cdot \theta\Big|_0^{\frac{\pi}{2}} = \frac{2\sqrt{2}}{3} \cdot \frac{\pi}{2} = \frac{\pi\sqrt{2}}{3}$$

9. Evaluate $\int\int_D x\ dA$, where D is the region above the x-axis bounded by the circle $x^2 + y^2 = 9$.

$$= \int_0^{\pi} \int_0^3 r\cos(\theta) \cdot\ r\ dr\ d\theta = \frac{r^3}{3}\Big|_0^3 \cdot \sin(\theta)\Big|_0^{\pi} = 9 \cdot 0 = 0$$

10. Calculate the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 3$.

$$= \int_0^{2\pi} \int_0^2 \Big(3 - r\sin(\theta)\Big) \cdot r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \Big(3r - r^2 \sin(\theta)\Big) \, dr \, d\theta = \int_0^{2\pi} \left(6 - \frac{8}{3}\sin(\theta)\right) d\theta$$

$$= \left(12\pi + \frac{8}{3}\right) - \left(0 + \frac{8}{3}\right) = 12\pi$$

11. Evaluate $\displaystyle\int\int\int_E xy \, dV$, where $E = \{(x, y, z) : 0 \le x \le 3,\ 0 \le y \le x,\ 0 \le z \le x + y\})$.

$$= \int_0^3 \int_0^x \int_0^{x+y} xy \, dz \, dy \, dx = \int_0^3 \int_0^x (x^2 y + xy^2) \, dy \, dx = \int_0^3 \frac{5}{6} x^4 \, dx = \frac{1}{6} x^5 \Big|_0^3 = \frac{81}{2}$$

12. Evaluate $\displaystyle\int\int\int_E z \, dV$, where E is the solid region bounded by the planes $z = 0$, $z = 1 + x + y$, and the cylinder $x^2 + y^2 = 1$ in the first octant. [Warning : highly involved and technical algebra here.]

$$= \int\int_D \int_0^{1+x+y} z \cdot \, dz \, dA = \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{1 + r(\cos(\theta) + \sin(\theta))} z \cdot r \, dz \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{2} \Big(1 + r(\cos(\theta) + \sin(\theta))\Big)^2 \cdot r \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^1 \Big(r + 2r^2(\cos(\theta) + \sin(\theta)) + r^3(\cos(\theta) + \sin(\theta))^2\Big) \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{2}{3}(\cos(\theta) + \sin(\theta)) + \frac{1}{4}(1 + \cos(\theta)\sin(\theta))\right) d\theta$$

$$= \frac{1}{2} \left(\frac{1}{2}\theta + \frac{2}{3}(\sin(\theta) - \cos(\theta)) + \frac{1}{4}\left(\theta + \frac{1}{2}\sin^2(\theta)\right)\right) \Bigg|_0^{\frac{\pi}{2}} = \frac{3\pi}{16} + \frac{19}{24}$$

13. Find the volume of the solid inside the cylinder $x^2 + y^2 = 4$ under the paraboloid $z = x^2 + y^2$ and above the plane $z = 0$.

$$= \int\int_D \int_0^{x^2+y^2} 1 \, dz \, dA = \int_0^{2\pi} \int_0^2 \int_0^{r^2} 1 \cdot \, r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^2 r^3 \, dr \, d\theta$$

$$= \frac{r^4}{4} \Big|_0^2 \cdot \theta \Big|_0^{2\pi} = 8\pi$$

14. Evaluate $\displaystyle\iiint_E (x^2+y^2)\,dV$, where E is the solid region above the xy-plane between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$.

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_1^2 \Big(\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta\Big)\cdot\rho^2\sin\phi\;d\rho\;d\theta\;d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_1^2 \rho^4\sin^3\phi\;d\rho\;d\theta\;d\phi$$

$$= \frac{\rho^5}{5}\Big|_1^2\cdot\theta\Big|_0^{2\pi}\cdot\int_0^{\pi/2}\sin^3\phi\;d\phi = \frac{31}{5}\cdot 2\pi\cdot\frac{2}{3} = \frac{124\pi}{15}$$

To evaluate the last integral, do the following:

$$\int_0^{\pi/2}\sin^3\phi\;d\phi = \int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\;d\phi = \int_0^{\pi/2}\Big(\sin\phi-\cos^2\phi\sin\phi\Big)\;d\phi$$

$$= \Big(-\cos\phi+\frac{1}{3}\cos^3\phi\Big)\Big|_0^{\pi/2} = \frac{2}{3}$$

15. Evaluate $\displaystyle\iiint_E y^2\sqrt{x^2+y^2+z^2}\;dV$, where E is the solid region in the first octant inside the sphere $x^2+y^2+z^2=1$.

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \Big(\rho^2\sin^2\phi\sin^2\theta\cdot\rho\Big)\cdot\rho^2\sin\phi\;d\rho\;d\theta\;d\phi$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^5\sin^3\phi\sin^2\theta\;d\rho\;d\theta\;d\phi$$

$$= \int_0^1\rho^5\;d\rho\cdot\int_0^{\frac{\pi}{2}}\sin^2\theta\;d\theta\cdot\int_0^{\frac{\pi}{2}}\sin^3\phi\;d\phi$$

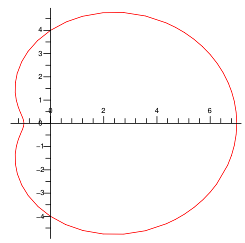
$$= \Big[\frac{1}{6}\rho^6\Big]_0^1\cdot\Big[\frac{1}{2}\theta-\frac{1}{4}\sin(2\theta)\Big]_0^{\pi/2}\cdot\Big[-\cos\phi+\frac{1}{3}\cos^3\phi\Big]_0^{\pi/2}$$

$$= \frac{1}{6}\cdot\frac{\pi}{4}\cdot\frac{2}{3} = \frac{\pi}{36}$$

To evaluate the third integral, see above. To do the second, use the half-angle identity:

$$\int_0^{\pi/2}\sin^2\theta\;d\theta = \int_0^{\pi/2}\frac{1}{2}\Big(1-\cos(2\theta)\Big)\theta\;d\theta = \frac{1}{2}\Big(\theta-\frac{1}{2}\sin(2\theta)\Big)\Big|_0^{\pi/2} = \frac{\pi}{4}$$

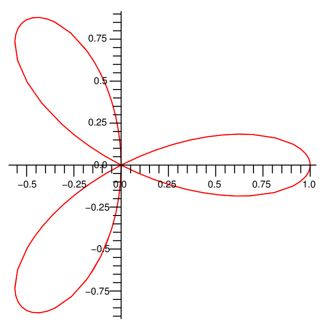
16. Find the area of the cardioid $r = 4 + 3\cos(\theta)$.



$$\text{Area} = \iint_D 1\, dA = \int_0^{2\pi} \int_0^{4+3\cos\theta} r\, dr d\theta = \int_0^{2\pi} \frac{1}{2}(4 + 3\cos\theta)^2\, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} \left(16 + 24\cos\theta + 9 \cdot \frac{1+\cos 2\theta}{2}\right) d\theta = \frac{41\pi}{2}.$$

17. Find the area of one petal of the polar rose $r = \cos(3\theta)$.

For this problem, it is important to get your bounds right. It is easy to see that $0 \le r \le \cos(3\theta)$. To find the bounds for θ, we look at the graph of the curve.



We want only one petal, so we need to find an interval of θ values where r starts at zero and then goes back to zero. That is, we need to find the angles θ where $r = 0$ and our range will be the interval between two consecutive such θ values.

Consider $\cos(t)$. We know that $\cos(t) = 0$ when $t = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$. In our case, $t = 3\theta$, so we have $\theta = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \cdots$. The petal on the right is the one where θ goes from $-\frac{\pi}{6}$ to $\frac{\pi}{6}$, so we will use this interval for θ. (You can use any consecutive θ values, however.) Therefore,

$$\text{Area} = \iint_D 1\, dA = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \int_0^{\cos(3\theta)} r\, dr d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(\cos 3\theta)^2\, d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{4}(1 + \cos 6\theta)\, d\theta = \frac{\pi}{12}.$$

18. Find the volume of the solid region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 1$. (This region is sometimes called an *ice cream cone*.)

$$\text{Volume} = \iiint_E 1\, dV = \int_0^{\pi/4} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$$

$$= \left(\frac{\rho^3}{3}\right)\Big|_0^1 \cdot \left(\theta\right)\Big|_0^{2\pi} \cdot \left(-\cos\phi\right)\Big|_0^{\pi/4} = \frac{2\pi}{3}\left(-\frac{\sqrt{2}}{2} + 1\right)$$