Math 210Q **Name:** SOLUTIONS
Practice Exam 2

Directions: Show all of your work and clearly indicate your answers. Correct answers with no work may not receive credit.

1. Find all second partial derivatives of the function $f(x,y) = xe^{-2y}$.

$$f_x(x,y) = e^{-2y} \qquad f_y(x,y) = -2xe^{-2y}$$
$$f_{xx}(x,y) = 0 \qquad f_{yy}(x,y) = 4xe^{-2y}$$
$$f_{xy}(x,y) = -2e^{-2y} \qquad f_{yx}(x,y) = -2e^{-2y}$$

2. Find the gradient of the function $f(x,y,z) = z^2 e^{x\sqrt{y}}$ at the point $P = (0,4,3)$. Find the maximum rate of change of f at P. In which direction does it occur?

$$\nabla f(x,y,z) =< \sqrt{y}z^2 e^{x\sqrt{y}},\ \frac{xz^2}{2\sqrt{y}} e^{x\sqrt{y}},\ 2ze^{x\sqrt{y}} >$$
$$\nabla f(0,4,3) =< 18, 0, 6 >$$
Maximum is $|\nabla f(0,4,3)| = \sqrt{18^2 + 6^2} = \sqrt{360}$ in direction of $< 18, 0, 6 >$.

3. Let $g(x,y) = 2\sqrt{x} - y^2$ and let $P = (1,5)$. Find the directional derivative of g at the given point in the direction of the vector $\boldsymbol{v} =< 4, 1 >$. Find the maximum rate of change of g at P. In which direction does it occur?

$$\nabla g(x,y) = \left\langle \frac{1}{\sqrt{x}},\ -2y \right\rangle, \text{so } \nabla g(1,5) =< 1, -10 > .$$

Let $\boldsymbol{u} = \frac{\mathbf{v}}{|\mathbf{v}|} =< \frac{4}{\sqrt{17}}, \frac{1}{\sqrt{17}} >$. Then

$$D_u g(1,5) = \nabla g(1,5) \cdot \boldsymbol{u} = \frac{4}{\sqrt{17}} + \frac{-10}{\sqrt{17}} = \frac{-6}{\sqrt{17}}.$$

The maximum is

$$|\nabla g(1,5)| = \sqrt{101}$$

in the direction of $< 1, -10 >$.

4. Find the equation of the tangent plane to the surface $z = e^x(\sin(y) + \cos(y))$ at the point where $x = 0$ and $y = \pi/2$.

$$z - 1 = (1)(x - 0) + (-1)(y - \frac{\pi}{2})$$

5. Find the equation of the tangent plane to the surface $\sin(xyz) = x + 2y + 3z$ at the point $(2, -1, 0)$.

Let $F(x, y, z) = x + 2y + 3z - \sin(xyz)$. Then the equation will be

$$1(x - 2) + 2(y + 1) + 5(z - 0) = 0.$$

6. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ where the tangent plane is parallel to the plane $2x + y - 3z = 2$.

The planes are parallel if the normal vectors are parallel. At any given point (x_0, y_0, z_0), a vector normal to the plane tangent to the sphere is the gradient of $F(x, y, x) = x^2 + y^2 + z^2 - 1$ at (x_0, y_0, z_0), that is

$$< 2x_0,\ 2y_0,\ 2z_0 > .$$

The vector normal to the given plane is $< 2, 1, -3 >$. These normal vectors are parallel if

$$2x_0 = 2K,\ 2y_0 = K,\ 2z_0 = -3K,$$

for some constant K. The only (x_0, y_0, z_0) that satisfy these equations and lie on the sphere are

$$\left(\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right) \quad \text{and} \quad \left(-\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right),$$

that is, when $K = \pm 2/\sqrt{14}$.

7. Find the linear approximation of the function $f(x, y, z) = \frac{e^x}{y^2+z^2}$ at the point $(0, 1, 1)$ and use it to estimate the number $\frac{e^{0.02}}{(1.01)^2+(0.97)^2}$.

For (x, y, z) near $(0, 1, 1)$, we have

$$f(x, y, z) \approx \frac{1}{2} + \frac{1}{2}(x - 0) + (-\frac{1}{2})(y - 1) + (-\frac{1}{2})(z - 1).$$

Then

$$f(0.02, 1.01, 0.97) \approx 0.52.$$

Using a calculator, I got the estimate 0.5202454564.

8. Suppose that $f(x, y, z) = x\sqrt{y} + \frac{z^2}{x}$, where $x = t^3 - 4t,\ y = e^{-2t},\ z = t^2 - 4$. Use the Chain Rule to find $\frac{\mathrm{d}f}{\mathrm{d}t}$.

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left(\sqrt{y} - \frac{z^2}{x^2}\right)\left(3t^2 - 4\right) + \left(\frac{x}{2\sqrt{y}}\right)\left(-2e^{-2t}\right) + \left(\frac{2z}{x}\right)\left(2t\right)$$

9. Suppose that $w = x^2y + xy^2$, where $x = u^2 + v,\ y = u - v^2$. Use the Chain Rule to find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

$$w_u = (2xy + y^2)(2u) + (x^2 + 2xy)(1)$$
$$w_v = (2xy + y^2)(1) + (x^2 + 2xy)(-2v)$$

10. Suppose $u(x, y) = e^{ax} \sin(ay)$, where a is a constant. Show that f is harmonic, that is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$u_x = ae^{ax} \sin(ay) \qquad u_y = ae^{ax} \cos(ay)$$
$$u_{xx} = a^2e^{ax} \sin(ay) \qquad u_{yy} = -a^2e^{ax} \sin(ay)$$

So

$$u_{xx} + u_{yy} = a^2e^{ax} \sin(ay) - a^2e^{ax} \sin(ay) = 0.$$

11. If $z = y + f(x^2 - y^2)$, where f is differentiable, show that

$$y\,\frac{\partial z}{\partial x} + x\,\frac{\partial z}{\partial y} = x.$$

$$z_x = f'(x^2 - y^2) \cdot (2x)$$
$$z_y = 1 + f'(x^2 - y^2) \cdot (-2y)$$

So

$$yz_x + xz_y = 2xyf'(x^2 - y^2) + x - 2xyf'(x^2 - y^2) = x.$$

12. Find all the critical points of the function $f(x, y) = x^3 - 6xy + 8y^3$. Use the Second Derivatives Test to classify them as maxima, minima or saddle points.

The critical points are $(0, 0)$ and $(1, 1/2)$.

$$D(0, 0) = (0)(0) - (-6)^2 = -36 < 0 \text{ (saddle point)},$$

$$D(1, 1/2) = (6)(24) - (-6)^2 = 108 > 0 \text{ (local minimum)}.$$

13. Find all the critical points of the function $f(x, y) = (x^2+y)e^y$. Use the Second Derivatives Test to classify them as maxima, minima or saddle points.

The only critical point is $(0, -1)$.

$$D(0, -1) = \left(\frac{2}{e}\right)\left(\frac{1}{e}\right) - (0)^2 = \frac{2}{e^2} > 0,$$

so the point $(0, -1)$ is a local minimum.

14. Find the points on the surface $z^2 = xy + 1$ that are closest to the origin

The distance from a point (x, y, z) to the origin is given by $d = \sqrt{x^2 + y^2 + z^2}$. Since this point must be on the surface, we have $z^2 = xy + 1$, so the distance is given by

$$d = \sqrt{x^2 + y^2 + xy + 1}.$$

To minimize this, we will minimize the thing under the radical (the radicand),

$$f(x, y) = x^2 + y^2 + xy + 1.$$

This has only one critical point, $(0, 0)$, and a quick check shows this is a local minimum. Since there is no boundary to check, this must be our absolute minimum. Therefore, the point on the surface closest to the origin is $(0, 0, 1)$.

15. If the length of the diagonal of a rectangular box must be L, what is the largest possible volume?

Let the sides have lengths x, y, z. Then the length of the diagonal is $L = \sqrt{x^2 + y^2 + z^2}$. This is **FIXED**, you are not trying to maximize this quantity. What you are trying to maximize is the volume $V = xyz$. You could solve for z and plug that value into V to get a function of two variables,

$$V(x, y) = xy\sqrt{L^2 - x^2 - y^2},$$

and then maximize this in the usual way. What I like to do, is use implicit differentiation. Either way, the answer you get is $\left(L/\sqrt{3}\right)^3$.

16. Find the absolute maximum and minimum values of f on the set D, where $f(x, y) = xy^2$ and $D = \{(x, y) : x^2 + y^2 \leq 3\}$.

The first derivatives are $f_x(x, y) = y^2$ and $f_y(x, y) = 2xy$. These are both zero when $y = 0$, regardless of x. It follows that the critical points of f on D are all points of the form $(x, 0)$, where $x^2 \leq 3$. Since we are looking for absolute maxima and minima, the type of these critical points is irrelevant (which is handy since the second derivatives test fails in this case), so we need only evaluate f at these points. For any x, $f(x, 0) = 0$.

Now it remains to check the boundary. On the boundary of D, we have $x^2 + y^2 = 3$, so $y^2 = 3 - x^2$. Plug this into f to get a function of x:

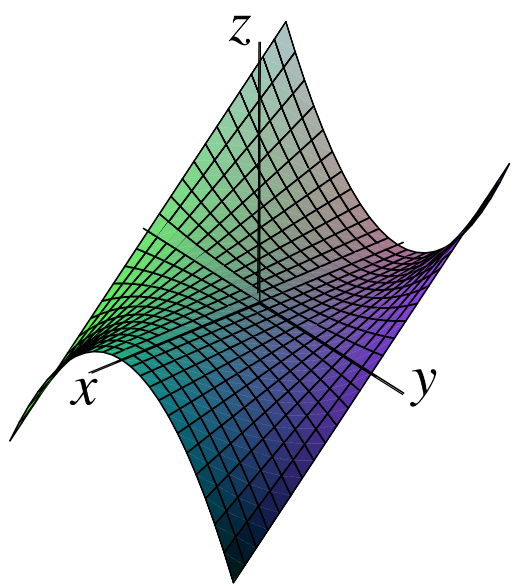
$$g(x) = f(x, y(x)) = x(3 - x^2) = 3x - x^3.$$

We wish to find the maximum and minimum values of this function, so we find the critical points:

$$g'(x) = 3 - 3x^2.$$

This is zero when $x = \pm 1$. Plugging this into g, we get $g(1) = 2$ and $g(-1) = -2$. These are the maximum and minimum values, respectively.

We didn't need to know what kind of critical points the points $(x, 0)$ were, and the second derivatives test wouldn't have identified them anyway. To satisfy our curiosity, we look at a graph of the function:



17. Suppose f is a differentiable function of x and y, and $g(u,v) = f(ue^v, u^2 + \cos v)$, Use the table of values to calculate $g_u(1,0)$ and $g_v(1,0)$.

	f	g	f_x	f_y
$(1,0)$	3	6	4	8
$(1,2)$	6	3	2	5

To do this problem, we treat f as a function of x and y, where $x = ue^v$ and $y = u^2 + \cos v$. Then we use the chain rule:

$$g_u(u,v) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = f_x(x,y)\cdot e^v + f_y(x,y)\cdot 2u,$$

and

$$g_v(u,v) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = f_x(x,y)\cdot ue^v + f_y(x,y)\cdot(-\sin v).$$

Now we evaluate when $u = 1$ and $v = 0$. But when $u = 1$ and $v = 0$, we have $x = 1$ and $y = 2$, by the formulas above. Therefore,

$$g_u(1,0) = f_x(1,2)\cdot(1) + f_y(1,2)\cdot 2 = 12,$$

and

$$g_v(1,0) = f_x(1,2)\cdot 1 + f_y(1,2)\cdot(0) = 2.$$

18. Find all of the points (x,y) where the maximum rate of change of the function $f(x,y) = x^3 + y^2 + x - y + 32$ is in the same direction as the vector $\boldsymbol{v} =< 1,1 >$.

At any point (x,y), the maximum rate of change is in the direction of the gradient vector at that point $\nabla f(x,y,z)$. Calculate the gradient at (x,y):

$$\nabla f(x,y,z) =< 3x^2 + 1, 2y - 1 > .$$

We want to find the values of x and y so that this vector is in the same *direction* as $\boldsymbol{v}$. Size doesn't matter, so we want

$$\nabla f(x,y,z) = k\,\boldsymbol{v}$$

for some constant k. Therefore,

$$< 3x^2 + 1, 2y - 1 >=< k, k > .$$

It follows that

$$3x^2 + 1 = 2y - 1,$$

so

$$y = \frac{3}{2}x^2 + 1.$$

So, to answer the question, all points on the parabola $y = \frac{3}{2}x^2 + 1$.