Math 210Q **Name:** SOLUTIONS
Practice Exam 1

Directions: Show all of your work and clearly indicate your answers. Correct answers with no work may not receive credit. **Good Luck!**

Question 1. (25 points) Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(a) Find $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$

$\mathbf{a} \cdot \mathbf{b} = 10$

$\mathbf{a} \times \mathbf{b} = -4\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$

(b) Determine the angle between $\mathbf{a}$ and $\mathbf{b}$.

Let θ be the angle in question. Then

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right) = \cos^{-1}\left(\frac{10}{14}\right).$$

Notice that $|\mathbf{a}| = |\mathbf{b}| = \sqrt{1+4+9} = \sqrt{14}$.

(c) Find the unit vector in the direction of $\mathbf{a}$.

$$\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}}{\sqrt{14}} = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}.$$

(d) Find parametric equations for the line in $\mathbf{R}^3$ that passes through the point $(0, 1, 0)$ in the direction of $\mathbf{a}$. Do the same in the direction of $\mathbf{b}$.

First $\mathbf{a}$: The vector equation for the line is

$$\mathbf{r}(t) =< 0, 1, 0 > +t < 1, 2, 3 >=< t, 1 + 2t, 3t >,$$

so the parametric equations for the line are

$$\begin{aligned} x &= t \\ y &= 1 + 2t \\ z &= 3t \end{aligned}$$

Now for $\mathbf{b}$: The vector equation for the line is

$$\mathbf{r}(t) =< 0, 1, 0 > +t < 3, 2, 1 >=< 3t, 1 + 2t, t >,$$

so the parametric equations for the line are

$$\begin{aligned} x &= 3t \\ y &= 1 + 2t \\ z &= t \end{aligned}$$

(e) Find the equation of the plane passing through $(1, 1, 1)$ and with normal vector $\mathbf{b} - \mathbf{a}$.

The vector equation is

$$(\mathbf{b} - \mathbf{a}) \cdot < x - 1, y - 1, z - 1 >= 0,$$

or

$$< 2, 0, -2 > \cdot < x - 1, y - 1, z - 1 >= 0.$$

The scalar equation is simply

$$2x - 2z = 0.$$

Question 2. (5 points) Find the area of the triangle with vertices $P(1, 4, 6)$, $Q(-2, 5, -1)$, and $R(1, -1, 1)$.

$$\frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}| < -3, 1, -7 > \times < 0, -5, -5 > | = \frac{1}{2}\sqrt{2050}.$$

Question 3. (5 points) Suppose one angle of a parallelogram is $\pi/3$. If all the sides of the parallelogram have length 1, what is the area of the parallelogram?

$$\text{Area} = (1)(1)\sin(\pi/3) = \frac{\sqrt{3}}{2}.$$

Question 4. (5 points) Find the point in which the line with parametric equations

$$\begin{aligned} x &= t \\ y &= 3 - 2t \\ z &= 3 + 3t \end{aligned}$$

intersects the plane

$$z = 3x + 2y.$$

Use the values for x, y, z given by the equation for the line and plug them into the equation for the plane. Solve for t and you will get $t = 3/4$. Put this value back into the equations for x, y, z. You should get the point $(3/4, 3/2, 21/4)$.

Question 5. (5 points) Find the Parametric equation of the line described by the intersection of these planes.

$$3x - 6y - 2z = 15$$
$$y - 2z = 5$$

We wish to write x, y, and z as functions of another variable, say t. There are many ways to do this, I will give the one I think is the easiest. Begin by letting $z = t$. Solving for y in the second equation, you should get $y = 5 + 2t$. Plug these into the first equation and solve for x:

$$x = \frac{1}{3}(6y + 2z + 15) = \frac{45}{3} + \frac{14}{3}t.$$

Question 6. (5 points) Find a vector perpendicular to the plane that passes through the points $P(1,0,1)$, $Q(0,1,0)$, and $R(0,0,1)$. What is the equation of the plane containing these points?

The vector will be

$$\vec{PQ} \times \vec{PR} = <0,1,1> .$$

This is the normal vector of the required plane, so the vector equation will be:

$$<0,1,1> \cdot <x-1, y, z-1> = 0,$$

so the scalar equation is

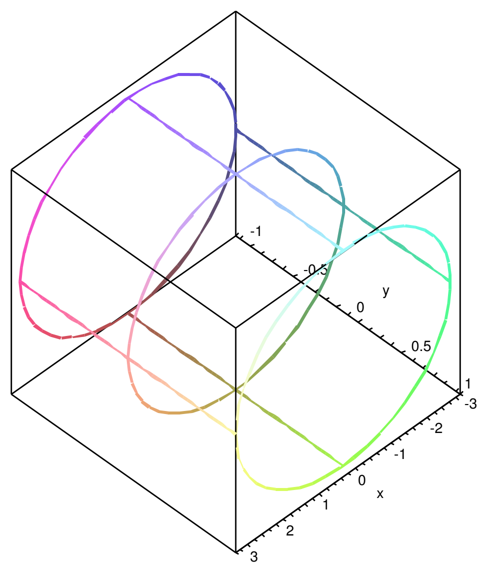
$$y + z = 1.$$

Question 7. (5 points) Consider the surface whose cartesian equation is given by

$$x^2 + z^2 = 9.$$

Sketch a graph of the equation above by drawing some traces.

For any $y = k$, the trace is the circle $x^2 + z^2 = 9$. This is a cylinder, and the graph should look something like this:

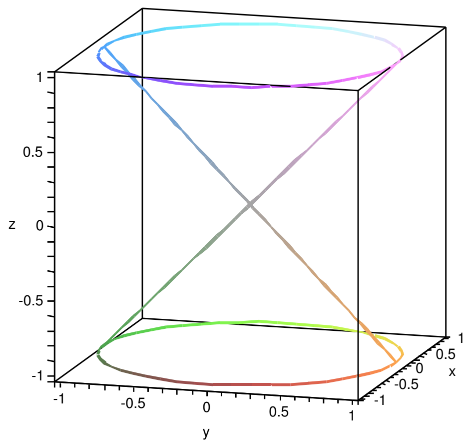


Question 8. (5 points) Consider the quadric surface whose cartesian equation is given by

$$z^2 = x^2 + y^2.$$

Sketch a graph of the equation above by drawing some traces.

For any $z = k$, the trace is the circle $x^2 + y^2 = k^2$. For $x = 0$ or $y = 0$, the trace is a pair of lines. If $x = k$ or $y = k$ for some $k \neq 0$, the trace is a hyperbola. This is a cone and the graph should look something like this:



Question 9. (20 points) A particle in space moves according to the position function:

$$\mathbf{r}(t) = t\ \mathbf{i} + \sin(2t)\ \mathbf{j} + \cos(2t)\ \mathbf{k}.$$

1. Find formulas for the velocity, speed and acceleration at any time t.

   $\mathbf{v}(t) = \mathbf{r}'(t) =< 1, 2\cos(2t), -2\sin(2t) >$,

   $v(t) = |\mathbf{v}(t)| = \sqrt{1 + 4\cos^2(2t) + 4\sin^2(2t)} = \sqrt{5}$,

   $\mathbf{a}(t) = \mathbf{r}''(t) =< 0, -4\sin(2t), -4\cos(2t) >$.

2. Find the tangential and normal components of acceleration.

   $a_T = v'(t) = 0$,

   $a_N = v(t)|\mathbf{T}'(t)| = \left(\sqrt{5}\right)\left(\frac{4}{\sqrt{5}}\right) = 4$.

3. Find the unit normal vector $\mathbf{N}(t)$.

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} =< 0, -\sin(2t), -\cos(2t) >$$

Question 10. (5 points) Suppose $\mathbf{r}(t)$ describes a curve. Show that the unit tangent vector, $\mathbf{T}(t)$, and unit normal vector, $\mathbf{N}(t)$, are orthogonal to each other.

By definition, $|\mathbf{T}(t)| = 1$, we see that

$$0 = \frac{d}{dt}(|\mathbf{T}(t)|^2) = \frac{d}{dt}(\mathbf{T}(t) \cdot \mathbf{T}(t)) = 2\mathbf{T}(t) \cdot \mathbf{T}'(t).$$

It follows that $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal. The result follows from the fact that

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

Question 11. (5 points) Find the angles (approximately) of the triangle with vertices $A(1,0), B(3,6), C(-1,4)$.

I will find one angle, the others can be found in a similar manner. Consider the vectors

$$\overrightarrow{AB} = <2,6>, \quad \overrightarrow{AC} = <-2,4>.$$

Then the angle between the two vectors is

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}\right) = \cos^{-1}\left(\frac{20}{\sqrt{800}}\right) = \frac{\pi}{4} \approx 0.7853981635.$$

Question 12. (5 points) Find a unit vector orthogonal to the vectors $\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $4\mathbf{j}+4\mathbf{k}$

The cross product will be orthogonal to both vectors:

$$<1,-1,1> \times <0,4,4> = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 4 & 4 \end{vmatrix} = <-8,-4,4>.$$

The magnitude of this vector is $\sqrt{64+16+16} = \sqrt{96}$, so the unit vector will be

$$<\frac{-8}{\sqrt{96}}, \frac{-4}{\sqrt{96}}, \frac{4}{\sqrt{96}}>.$$

Question 13. (5 points) True or false? The *cross product* of two unit vectors is also a unit vector.

This is false. Recall the formula

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta,$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. If $|\mathbf{a}|=1$ and $|\mathbf{b}|=1$, then

$$|\mathbf{a}\times\mathbf{b}| = \sin\theta,$$

and this is only equal to 1 when $\theta = \frac{\pi}{2}$.

Question 14. (10 points) In each of the problems below, determine weather the given lines are perpendicular, parallel, or neither:

For these problems, we look at the vector equations for the lines, which have the form $\mathbf{r_1} = \mathbf{r_{10}} + t\mathbf{v_1}$ and $\mathbf{r_2} = \mathbf{r_{20}} + s\mathbf{v_2}$. The lines are perpendicular or parallel if the vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are perpendicular or parallel, respectively. If the vectors are not perpendicular or parallel, then neither are the lines.

1. $L_1 : x = 3-t, \quad y = 1+2t, \quad z = 5+3t$

   $L_2 : x = 2s, \quad y = 4-4s, \quad z = 1-6s$

   We have $\mathbf{v_1} = <-1,2,3>$ and $\mathbf{v_2} = <2,-4,-6>$. Since $\mathbf{v_1}\times\mathbf{v_2} = \mathbf{0}$, the vectors are parallel, hence the lines L_1 and L_2 are also parallel. (In fact, $\mathbf{v_2} = -2\mathbf{v_1}$.)

2. $L_1 : x = 3 - t, \quad y = 1 + 2t, \quad z = 5 + 3t$

$L_2 : x = -3s, \quad y = 4 + 6s, \quad z = 1 - 5s$

We have $\mathbf{v_1} =< -1, 2, 3 >$ and $\mathbf{v_2} =< -3, 6, -5 >$. Since $\mathbf{v_1} \cdot \mathbf{v_2} = \mathbf{0}$, the vectors are perpendicular, hence the lines L_1 and L_2 are also perpendicular.

Question 15. (5 points) Find the equation of the plane containing the vectors $< 1, 0, 1 >$ and $< 1, -2, 2 >$ and passing through the point $(1, 2, 3)$

The normal vector $\mathbf{n}$ is orthogonal to the plane, so we can let it be the cross product of two vectors in the plane, i.e.

$$\mathbf{n} =< 1, 0, 1 > \times < 1, -2, 2 >= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & -2 & 2 \end{vmatrix} =< 2, -1, -2 > .$$

So the vector equation of the plane is

$$< 2, -1, -2 > \cdot < x - 1, y - 2, z - 3 >= 0,$$

or

$$2x - y + 2z + 6 = 0.$$

Question 16. (5 points) Find the parametric equations of the line passing through the point $(2, 3, 1)$ and perpendicular to the plane $2x - y + 4z = 12$.

The vector equation for the line will have the form $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$, where $\mathbf{r_0} =< 2, 3, 1 >$ and $\mathbf{v}$ is some vector parallel to the line, hence perpendicular to the plane. The normal vector of the plane is perpendicular to the plane, so we can let $\mathbf{v}$ be the normal vector, which is $< 2, -1, 4 >$. Therefore, the vector equation of the line is

$$< x, y, z >=< 2, 3, 1 > +t < 2, -1, 4 >,$$

so the parametric equations are

$$\begin{cases} x = 2 + 2t, \\ y = 3 - t, \\ z = 1 + 4t. \end{cases}$$

Question 17. (5 points) Find the parametric equations of a line passing through the point $(2, 0, 2)$ and *contained in* the plane $2x - y + 4z = 12$. [Think about it for a minute.]

The vector equation for the line will have the form $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$, where $\mathbf{r_0} =< 2, 0, 2 >$ and $\mathbf{v}$ is some vector parallel to the line, hence *parallel* to the plane. The normal vector of the plane is *perpendicular* to the plane, so we let $\mathbf{v}$ be *any* vector orthogonal to the normal vector, $\mathbf{n} =< 2, -1, 4 >$. There are infinitely many such vectors. How do we find one? Let $\mathbf{v} =< a, b, c >$ and solve $\mathbf{n} \cdot \mathbf{v} = 0$. That is, solve the equation $2a - b + 4c = 0$ for a, b, c. There are many choices, so just pick an easy one, like $a = 2, b = 0, c = -1$. Then the vector equation of the line is

$$< x, y, z >=< 2, 0, 2 > +t < 2, 0, -1 >,$$

so the parametric equations are

$$x = 2 + 2t, \quad y = 0, \quad z = 2 - t.$$

You can check that this line is in the plane by verifying that these values for x, y, and z satisfy the equation of the plane.