Pick yourself a number: ______________

Midterm no. 2

3160 Probability, Fall 2009

November 19, 2009

50 min.

Your name: Sample solutions

Score (out of 52)	Letter grade

Score: ______

1. *(7 points)* A textbook contains, on average, 1.1 typos per page. The author offers you a bet: you pick a random page; if it contains no typo, you will lose 10\$, and if it contains more than one typo, you will win 10\$. Should you take this bet? Compute your expected winnings if you take the bet.

$P(X=i) = e^{-1.1} \cdot \frac{1.1^i}{i!}$, X = number of typos

Lose 10\$ with prob. $P(X=0) = e^{-1.1} \approx 0.3329...$

Win 10\$ with prob. $P(X>1) = 1 - P(X=0) - P(X=1) = 1 - e^{-1.1} - 1.1 \cdot e^{-1.1} \approx 0.3009$

W = winnings; $E[W] = -10 \cdot 0.3329 + 10 \cdot 0.3009 \approx -0.319...$

$\Rightarrow$ Do not ~~pl~~ make this bet; epect to lose 32 cents.

2. *(5 points)* In one round of a game, player A has to roll two dice. If the sum of the two rolls is 7, he wins the game. Otherwise, he has to play another round, until he has won the game. What is the probability mass function of the number of rounds A has to play?

$P(X=7) = \frac{6}{36} = \frac{1}{6}$

X = sum of two rolls
$X=7$: 6 possibilities
$(6,1),(5,2),(4,3),(3,4),(2,5),(1,6)$

N = number of rounds

$N=n$ if A rolls
- "not a 7" $(p=\frac{5}{6})$ $n-1$ times in a row, an
- then a 7 $(p=\frac{1}{6})$

$P(N=n) = \left(\frac{5}{6}\right)^{n-1} \cdot \frac{1}{6}$

3. *(6 points)* A basketball team's offense has a shooting percentage of 55%, i.e. whenever someone attempts a shot, they will score a basket with probability $p = 0.55$.

   (a) Over the course of a game, the team attempts 80 shots. What is, approximately, the probability that they will score less than 40 baskets in this game?

   (b) Over the 20 games in a season, the team attempts 1600 shots. What is, approximately, the probability that they will score on less than 800 of their 1600 attempts?

(a) $P(X < 39.5) = P\left(\frac{X - 0.55 \cdot 80}{\sqrt{80 \cdot 0.45 \cdot 0.55}} < \frac{39.5 - 0.55 \cdot 80}{\sqrt{80 \cdot 0.45 \cdot 0.55}}\right)$

$\approx P(Z < -1.011) \approx 1 - \phi(1.01) \approx 1 - 0.8438$

$= 0.1562$

(b) $P(X < 800) \approx P(Z < -4.02) \approx 0$

4. *(5 points)* SAT scores are normally distributed with a mean of $\mu = 500$ and a standard deviation of $\sigma = 100$. What is the probability that among 10 random test participants, at most one student will have an SAT score better than 550?

$P(X > 550) = \ldots = 1 - \phi(\frac{1}{2}) = 0.3085$

S= number of students with $X > 550$:

$P(S \le 1) = \cancel{0.3085^{10} + \binom{10}{1} 0.3085^{9} \cdot 0.6915^{1}} \approx$

$= \binom{10}{0} 0.6915^{10} + \binom{10}{1} 0.6915^{9} \cdot 0.3085^{1}$

≈ 0.1365

Score: ________

5. *(8 points)* A continuous random variable has a probability density function of $f_X(x) = cx^3$ for $0 < x < 1$.

(a) Determine the constant c.

$$\int_0^1 c x^3 dx = c \cdot \frac{1}{4}$$

$$\Rightarrow c = 4$$

(b) Find the expectation value $E[X]$.

$$\int_0^1 4 x^3 \cdot x \, dx = \frac{4}{5}$$

(c) Find the variance of X.

$$E[X^2] = \int_0^1 4 x^3 \cdot x^2 dx = \frac{2}{3}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75}$$

6. *(5 points)* A ball is dropped from a random height between 0 and 100 meters. If $\cancel{X}$ H is the height, the time T in seconds until the ball hits the ground is given by $T = \sqrt{\frac{H}{5}}$ (e.g., when the ball is dropped from the maximum height 100m, it takes $\sqrt{20} \approx 4.47$ seconds to hit the ground). Find the probability density function of T.

① $P(T \le t) = P\left(\sqrt{\frac{H}{5}} \le t\right) = P(H \le 5t^2)$

$= \int_0^{5t^2} \frac{1}{100} dt = \frac{5t^2}{100}$ for $0 \le t \le \sqrt{20}$

$f_T(t) = \frac{d}{dt} \frac{5t^2}{100} = \frac{t}{10}$

② $g(H) = \sqrt{\frac{H}{5}} \Leftarrow$ monotone increasing!

$g^{-1}(t) = 5t^2$

$f_T(t) = f_H(g^{-1}(t)) \cdot \left|\frac{d}{dt} g^{-1}(t)\right|$

$= \frac{1}{100} \cdot 10t$ if $0 \le 5t^2 \le 100$, i.e. $0 \le t \le \sqrt{20}$

Score: ______

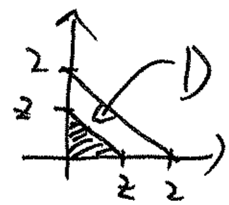
7. *(9 points)* X, Y are jointly continuous random variables whose joint probability density function is given by

$$f(x,y) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x \le 2, 0 \le y \le 2, 0 \le x+y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X, Y independent? Give a reason for your answer.

No, condition "$x+y \le 2$" not consistent with independence.

(E.g. $f_X(1.5) \neq 0$, $f_Y(1.5) \neq 0$, but $f(1.5, 1.5) = 0$)



(b) Find the probability density function of $Z = X + Y$.

① $P(Z \le z) = \iint_D f(x,y)\, dA = \frac{1}{2} \cdot \text{Area}(D)$

$= \frac{1}{2} z^2/2 = \frac{z^2}{4}$ for $0 \le z \le 2$

$\Rightarrow f_Z(z) = \frac{2z}{4} = \frac{z}{2}$

② $= \int_0^z \int_0^{z-x} \frac{1}{2}\, dy\, dx = \ldots = \frac{z^2}{4}$

(c) Find the conditional density of X, given that $Y = y$ for some $0 < y < 2$. Interpret the result.

$$f_Y(y) = \int_0^{2-y} \frac{1}{2}\, dx = \frac{2-y}{2}$$

$$f_{X|Y}(x|y) = \frac{\frac{1}{2}}{\frac{2-y}{2}} = \frac{1}{2-y}$$

for $0 \le x \le 2$ and $x+y \le 2$ i.e. $x \le 2-y$

$\Leftrightarrow 0 \le x \le 2-y$

* $X|Y$ is a uniform random variable distributed over $(0, 2-y)$

8. *(7 points)* Marta and Gregory decide to meet as soon as possible. For Marta, this means she will arrive at a time uniformly distributed between 12.00 and 1pm. For Gregory, this means his arrival time in hours after 12 pm is an exponential random variable with $\lambda = 2$.

   (a) *(5 points)* What is the probability that Gregory arrives before Marta?

Y = hours after 12pm Marta is arriving

X = hours ... " " Gregory " ——— "

$f_Y(y) = 1$ for $0 \le y \le 1$

$f_X(x) = 2e^{-2x}$ for $0 \le x$

$$P(Y \geq X) = \int_0^1 1 \int_0^y 2 \cdot e^{-2x} \, dx \, dy = \int_0^1 \left[-e^{-2x}\right]_0^y dy$$

$$= \int_0^1 (-e^{-2y}+1) \, dy = 1 + \left[\tfrac{1}{2}e^{-2y}\right]_0^1 = 1 + \tfrac{1}{2}e^{-2} - \tfrac{1}{2} \approx 0.56$$

   (b) *(2 points)* In case Marta arrives before Gregory, how long will she have to wait for Gregory on average?

Memory-less property of exp. distribution: Same as expected time for Gregory to arrive after 12pm, i.e.

$$E[X] = \frac{1}{\lambda} = \frac{1}{2} \text{ hours}$$