

ERRATA

Updated August 19, 2008

Page 17, line 1: The integral should be

$$\int_0^t g'_n(|X_s - X'_s|) \operatorname{sgn}(X_s - X'_s) [b(X_s) - b(X'_s)] ds$$

Page 19, line 4: $\nu \geq 2$

Page 19, lines 6,7: Insert between lines 6 and 7: We can also define Bessel processes of order $\nu \in [0, 2)$ by (7.3) for t up to the first hit of 0; at 0 a local time term needs to be included.

Page 21, line 2: and $X_t \rightarrow \infty$ a.s.

Page 21, line 9: ... is a martingale up to the first hit of 0.

Page 24, lines 5-15: where $|e_s|$ is bounded, say by c_2 (cf. PTA, theorem I.5.11]). Let $\tau = \inf\{t > 0 : |X_{B_t}^1| > N\}$. Since

$$|X_{B_{n+1}}^1 - X_{B_n}^1| \geq |\widetilde{W}_{n+1} - \widetilde{W}_n| - c_2,$$

then $|X_{B_{n+1}}^1| > N$ on the set $\{|X_{B_n}^1| < N, |\widetilde{W}_{n+1} - \widetilde{W}_n| > c_2 + 2N\}$. There exists $c_3 \in (0, 1)$ such that $\mathbb{P}(|\widetilde{W}_{n+1} - \widetilde{W}_n| > c_2 + 2N) > c_3$. By the independent increment property of Brownian motion,

$$\begin{aligned} \mathbb{P}(\tau > n+1) &= \mathbb{P}(\tau > n+1, \tau > n) = \mathbb{E}[\mathbb{P}(\tau > n+1 \mid \mathcal{F}_n); \tau > n] \\ &\leq (1 - c_3)\mathbb{P}(\tau > n). \end{aligned}$$

By induction, $\mathbb{P}(\tau > n) \leq (1 - c_3)^n$; hence $\tau < \infty$ a.s. Since $d\langle M \rangle_t/dt$ is

Page 25, lines 11-12:

Theorem 8.4. Let W_t be a d -dimensional Brownian motion and let X_t be a d -dimensional process such that

$$X_t^i = x_0^i + \int_0^t \sum_{j=1}^d H_{ij}(s) dW_s^j + \int_0^t B_i(s) ds, \quad i = 1, \dots, d,$$

where H_{ij} and B_i are predictable and bounded. For each s and ω let $K(s)$ be the matrix that is the inverse of $H(s)$. Suppose there exists M such that

for all s, i , and j , $|H_{ij}(s)|$, $|K_{ij}(s)|$, and $|B_i(s)|$ are bounded by M a.s. Let $\varepsilon > 0$, $t_0 > 0$. There exists c_1 depending only on M such that

Page 44, line 7: $\Lambda(x) > 0$

Page 44, Display (1.2): $\sum_{i=1}^d y_i^2$

Page 45, line -3: $e^{-\lambda s} \lambda u(X_s) ds$

Page 61, line 3 to Page 62, line 5: Replace with the following.

Let us say $f \in C^{2+\alpha}(D)$ if f , $\partial_i f$ and $\partial_{ij} f$ are bounded in D for all i, j and

$$\|f\|_{C^{2+\alpha}(D)} = \sup_{x \in D} |f(x)| + \sup_i \sup_{x \in D} |\partial_i f(x)| + \sup_{i,j} \|\partial_{ij} f\|_{C^\alpha(D)}$$

is finite. We define $C^{1+\alpha}$ analogously.

(2.2) Proposition. (a) If $f \in C^\alpha$ on B , then $P_B f \in C^\alpha$ on B and there exists c_1 independent of f such that

$$\|P_B f\|_{C^\alpha(B)} \leq c_1 \|f\|_{C^\alpha}.$$

(b) If $f \in C^{2+\alpha}$ on B , then $P_B f \in C^{2+\alpha}$ on B and there exists c_2 independent of f such that

$$\|P_B f\|_{C^{2+\alpha}(B)} \leq c_2 \|f\|_{C^{2+\alpha}}.$$

Proof. Clearly $|P_B f(x)| \leq \|f\|_\infty$, and $P_B f$ in B depends only on the values of f on ∂B . By [PTA, Proposition II.1.3], $P_B f$ is C^∞ in B . Using this and rotational invariance it suffices to obtain an estimate on $|P_B f(y) - P_B f(x)|$ for $x, y \in B(e_1, 1/4)$, where e_i is the unit vector in the x_i direction.

Let us write $f = f_1 + f_2$, where the C^α norms of f_1, f_2 are bounded by a constant times the C^α norm of f , f_1 is supported in $B(e_1, 1/2)$ and f_2 is 0 in $B(e_1, 3/8)$.

If we use the explicit formula for the Poisson kernel in B (see [PTA, Theorem II.1.17], for example) and differentiate it, we deduce that

$$|\nabla P_B f_2(x)| \leq c_3 \|f_2\|_\infty, \quad x \in B(e_1, 1/4).$$

We are using here the fact that f_2 is zero in $B(e_1, 3/8)$. Therefore by the mean value theorem,

$$|P_B f_2(y) - P_B f_2(x)| \leq c_3 \|f_2\|_\infty |y - x| \leq c_4 \|f\|_\infty |y - x|^\alpha.$$

We thus only need to consider $P_B f_1$ in $B(e_1, 1/4)$. Let us map $B(0, 1)$ to $B(e_d, 1)$ by a translation, map $B(e_d, 1)$ to $H_{1/2} = \{(x_1, \dots, x_d) : x_d > 1/2\}$

by inversion through the unit sphere, that is, the map $x \rightarrow I(x) = x/|x|^2$, and finally map $H_{1/2}$ to $H = \{(x_1, \dots, x_d) : x_d > 0\}$ by a translation. The composite map is nonsingular in $B(e_1, 1/4)$ and the inversion map has the property that if u is harmonic in a domain D , then $|x|^{2-d}u(I(x))$ is harmonic in $I(D)$ (see [PTA, Lemma II.1.18] for these facts). Since f_1 is supported in $B(e_1, 1/2)$, it suffices to show

$$\|P_H g\|_{C^\alpha(H)} \leq c_5 \|g\|_{C^\alpha(H)}, \quad (2.3.1)$$

where $P_H g$ is the harmonic extension of g in H , that is, $P_H g(x) = \mathbb{E}^x g(W_{\tau_H})$, where W is a Brownian motion and τ_H is the first exit time of W from H .

Write $\tilde{x} = (x_1, \dots, x_{d-1})$ so that $x = (\tilde{x}, x_d)$. Define $g_{\tilde{z}}(\tilde{x}, x_d) = g(\tilde{x} + \tilde{z}, x_d)$. We have

$$\begin{aligned} |P_H g(\tilde{x}, x_d) - P_H g(\tilde{y}, x_d)| &= |P_H g_{\tilde{x}}(0, x_d) - P_H g_{\tilde{y}}(0, x_d)| \quad (2.3.2) \\ &= |P_H(g_{\tilde{x}} - g_{\tilde{y}})(0, x_d)| \\ &\leq \|g_{\tilde{x}} - g_{\tilde{y}}\|_{L^\infty(\partial H)} \leq \|g\|_{C^\alpha(H)} |\tilde{x} - \tilde{y}|^\alpha. \end{aligned}$$

By Stein [1], Proposition V.7,

$$|\partial_d P_H g(\tilde{x}, t)| \leq c_6 \|g\|_{C^\alpha(H)} t^{-1+\alpha}.$$

So if $y_d > x_d$,

$$\begin{aligned} |P_H g(\tilde{x}, y_d) - P_H g(\tilde{x}, x_d)| &= \left| \int_{x_d}^{y_d} \partial_d(P_H g)(\tilde{x}, t) dt \right| \quad (2.3.3) \\ &\leq c_6 \|g\|_{C^\alpha(H)} \int_{x_d}^{y_d} t^{-1+\alpha} dt \\ &= c_7 \|g\|_{C^\alpha(H)} (y_d^\alpha - x_d^\alpha) \\ &\leq c_8 \|g\|_{C^\alpha(H)} (y_d - x_d)^\alpha. \end{aligned}$$

Combining (2.3.2) and (2.3.3) proves (2.3.1).

(b) We decompose $f = f_1 + f_2$ as in (a), and handle $P_B f_2$ similarly to what was done in (a). By the same transformations of the state space as in (a), it is enough to show

$$\|P_H g\|_{C^{2+\alpha}(H)} \leq c_9 \|g\|_{C^{2+\alpha}(H)}. \quad (2.3.4)$$

We may handle partials with respect to x_1, \dots, x_{d-1} as in (2.3.2), so it suffices to consider $\partial_{id} P_H g$ for $i \neq d$ and $\partial_{dd} P_H g$. Since $P_H g$ is harmonic in H , then $\partial_{dd} P_H g = -\sum_{i=1}^{d-1} \partial_{ii} P_H g$, which can be handled as in (2.3.2), so we are left to consider $\partial_{id} P_H g$ with $i \neq d$. The operators ∂_i and P_H commute if $i \neq d$ by the argument in (2.3.2), and writing G for $\partial_i g$, it therefore suffices to show

$$\|P_H G\|_{C^{1+\alpha}} \leq c_{10} \|G\|_{C^{1+\alpha}}. \quad (2.3.5)$$

Differences in the x_1, \dots, x_{d-1} directions are handled as in (2.3.2), so we need to look at differences in the x_d direction. By Stein [1], Proposition V.9,

$$|\partial_{dd}P_H G(\tilde{x}, t)| \leq c_{11}\|G\|_{C^\beta} t^{-2+\beta},$$

where we take $\beta = 1 + \alpha$. Therefore

$$\begin{aligned} |\partial_d P_H G(\tilde{x}, y_d) - \partial_d P_H G(\tilde{x}, x_d)| &= \left| \int_{x_d}^{y_d} \partial_{dd} P_H G(\tilde{x}, t) dt \right| \\ &\leq c_{11}\|G\|_{C^{1+\alpha}(H)} \int_{x_d}^{y_d} t^{-1+\alpha} dt \\ &\leq c_{12}\|G\|_{C^{1+\alpha}(H)} (y_d - x_d)^\alpha, \end{aligned}$$

similarly to (2.3.3). This is what we need to complete the proof of (b). \square

Page 62, line -8:

$$\partial_{ij} G_{B(x_0, R)} f = \partial_{ij} U f - \partial_{ij} P_{B(x_0, R)}(U f),$$

Page 77, line -2: > 0 .

Page 160, lines 15–21: Replace by the following.

If $x, y \in Q$,

$$u(x) - u(y) = \int_0^{|y-x|} \partial_r u(y + rv) dr, \quad v = (x - y)/|y - x|.$$

Integrating with respect to y ,

$$|Q|[u(x) - u_Q] = \int_Q \int_0^{|y-x|} \partial_r u(x + rv) dr dy.$$

Set $V(z)$ equal to $|\nabla u(z)|$ if $z \in Q$ and 0 otherwise. Then

$$\begin{aligned} |u(x) - u_Q| &\leq \frac{1}{|Q|} \int_{|y-x| \leq 2\sqrt{d}} \int_0^\infty V(x + rv) dr dy \\ &\leq c_2 \int_Q |y - x|^{1-d} V(y) dy. \end{aligned}$$

Now apply this inequality together with Theorem IV.5.1 of [PTA] where we set $p = 2$ and we set $K(x, y) = |y - x|^{1-d}$ if $x, y \in Q$ and 0 otherwise. We obtain

$$\int_Q |u(x) - u_Q|^2 dx \leq c_2^2 \int_Q \left[\int_Q K(x, y) V(y) dy \right]^2 dx \leq c_3 \int_Q |\nabla u(x)|^2 dx.$$

Page 188. Replace the proof of Theorem 8.4 by the following.

Proof. As in the proof of Theorem 7.5, we may assume without loss of generality that $d \geq 3$. As in the proof of Theorem I.8.5, it suffices to consider the case where ψ is differentiable. By Proposition 6.7 there exists c_2 and c_3 such that if $|x - x_0| \leq c_2 r^{1/2}$ and $|x - y| \leq c_2 r^{1/2}$, then $p_{B(x_0, r^{1/2})}(x, y) \geq c_3 r^{-d/2}$. Choose n large so that if $r = t/n$, then $r^{1/2} \leq \varepsilon/8$ and $r \|\psi'\|_\infty \leq (c_2/2)r^{1/2}$. Let $y_i = \psi(ir)$. Let $c_4 = c_2/4$.

If $x \in B(y_i, c_4 r^{1/2})$ and $y \in B(y_{i+1}, c_4 r^{1/2})$, then

$$|x - y| \leq 2c_4 r^{1/2} + |y_i - y_{i+1}| \leq 2c_4 r^{1/2} + r \|\psi'\|_\infty \leq c_2 r^{1/2}.$$

Taking $x_0 = y_i$, we see $p_{B(y_i, r^{1/2})}(x, y) \geq c_3 r^{-d/2}$. It follows that

$$\mathbb{P}^x(X_r \in B(y_{i+1}, r^{1/2}), \sup_{s \leq r} |X_s - X_0| \leq \varepsilon/4) \geq c_2 r^{-d/2} |B(y_i, r^{1/2})| \geq c_5.$$

Note

$$\begin{aligned} & \mathbb{P}^{\psi(0)}(\sup_{s \leq t} |X_s - \psi(s)| < \varepsilon) \\ & \geq \mathbb{P}^{\psi(0)}(X_{ir} \in B(y_i, c_4 r^{1/2}), \sup_{s \leq r} |X_s - X_{ir}| \leq \varepsilon/4, i = 0, \dots, n), \end{aligned}$$

and applying the Markov property n times, this is greater than $c_5^n > 0$. \square

Page 202, lines -10, -9: Replace by the following:

Since

$$\begin{aligned} 0 &= I - I = V(W + \varepsilon H_j) V^{-1}(W + \varepsilon H_j) - V(W) V^{-1}(W) \\ &= (V(W + \varepsilon H_j) - V(W)) V^{-1}(W) \\ &\quad + V(W + \varepsilon H_j) (V^{-1}(W + \varepsilon H_j) - V^{-1}(W)), \end{aligned}$$

then

$$V^{-1}(W + \varepsilon H_j) - V^{-1}(W) = -V^{-1}(W + \varepsilon H_j) (V(W + \varepsilon H_j) - V(W)) V^{-1}(W).$$

Dividing both sides by ε and letting $\varepsilon \rightarrow 0$,