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Probability Theory and Related Fields

Uniqueness in Law for Pure Jump Markov Processes

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Summary. Let A be the operator defined on C^2 functions by

$$Af(x)=\int[f(x+h)-f(x)-f'(x)\,h\,1_{([-1,1])}(h)]\,\nu(x,dh).$$

Sufficient conditions are given for existence and uniqueness for the martingale problem associated with A. In the case of stable-like processes, where $\nu(x,dh)$ is equal to the Lévy measure for the stable symmetric process of index $\alpha(x)$ for each x, the conditions reduce to $\alpha(x)$ continuous for existence and $\alpha(x)$ Dini continuous for uniqueness.

1. Introduction

The simplest pure jump Markov processes on the line are the Lévy processes which have no Gaussian component. These are characterized by a drift (which we assume to be 0 for the time being) and a Lévy measure $\nu(dh)$. For an appropriate class of functions, the infinitesimal generator of such a process is given by

$$Af(x)=\int[f(x+h)-f(x)-f'(x)\,h\,1_{([-1,1])}(h)]\,\nu(dh).$$

The next most complicated examples that one would naturally consider are processes that at each point x behave like a Lévy process, but which Lévy process will depend on the point x. So we suppose we are given a kernel $\nu(x,dh)$ and an operator A given by

$$Af(x)=\int[f(x+h)-f(x)-f'(x)\,h\,1_{([-1,1])}(h)]\,\nu(x,dh), \tag{1.1}$$

and we ask, when is there a process corresponding to the operator A, and can there be more than one?

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We phrase this problem in terms of a martingale problem: If X_t is the canonical coordinate process and $x_0 \in \mathbb{R}$, when does there exist a probability P such that

(1.2) (i) $P(X_0 = x_0) = 1$, and

(ii) for all $f \in C^2$, $f(X_t) - f(X_0) - \int_0^t Af(X_s)\, ds$ is a P-local martingale.

Can there exist more than one such P?

The purpose of this paper is to give sufficient conditions for the existence and uniqueness of a solution to the martingale problem (1.2).

To explain our results, let us first look at a class of examples that we refer to as stable-like processes. Here we are given a function $\alpha(x)$ and $\nu(x, dh) = \zeta_{\alpha(x)} |h|^{-(1+\alpha(x))}\, dh$, where $\zeta_{\alpha(x)}$ is a constant chosen so that $A(\exp(iux)) = -|u|^{\alpha(x)} \exp(iux)$. Thus, at each point x the process behaves like a symmetric stable process of index $\alpha(x)$. For the stable-like processes our conditions are quite simple: we require the regularity condition $0 < \inf_x \alpha(x) \leq \sup_x \alpha(x) < 2$; for existence we need the continuity of $\alpha(x)$; for uniqueness Dini continuity of $\alpha(x)$ is sufficient (see Corollary 2.3 for the precise statement.)

Our conditions in the general case are the natural analogs of the ones for the stable-like case. First we require some mild regularity of $\nu(x, dh)$. Existence will hold under a continuity condition on ν. Uniqueness holds under slightly more stringent conditions, i.e., an integral condition replaces the assumption of Dini continuity.

Besides their intrinsic interest, pure jump Markov processes arise in the study of diffusions. For example, if X_t is a diffusion on a manifold with a boundary, the boundary process will usually be a pure jump process.

There is also a connection with pseudodifferential operators. The stable-like processes, for example, correspond to pseudodifferential operators with symbol $\sigma(x, u) = -|u|^{\alpha(x)}$. Such symbols are of variable order, and the usual machinery of pseudodifferential operators does not seem to be sufficient to study them (cf. Sect. 1 of [3]).

Previous studies of weak uniqueness for pure jump Markov processes have concentrated almost exclusively on perturbations of stable processes (with a single fixed α); see [6, 7, 11, 12, 13]. If one removes the restriction to pure jump processes and allows a continuous component, then this continuous component becomes the dominant term; this situation has been studied by Komatsu [5] and Stroock [9]. Under considerably stronger smoothness assumptions, e.g., in the stable-like case $\alpha(x)$ must be Lipschitz continuous, Ito's theory of stochastic differential equations as extended to processes with jumps by Skorokhod [8] will give pathwise uniqueness.

In Sect. 2 we state our main theorems and results. Existence of a solution to the martingale problem is established in Sect. 3. Section 4 contains the key estimate used in proving uniqueness, and the remainder of the proof of uniqueness is done in Sect. 5 and 6; the Markov property is also discussed in Sect. 6. Section 7 consists of some remarks concerning some extensions, connections with singular integrals, and applications to local time.

We denote the Fourier transform of f by $\hat{f}$ and $\sup_x |f(x)|$ by $\|f\|$. We denote the bounded functions in C^2 by C_b^2. The letter c denotes positive constants whose values are unimportant and may change from line to line.

2. Statement of Results

Let $\Omega = D[0, \infty)$, the space of paths that are right continuous with left limits, and let $X_t: \Omega \to \mathbb{R}$ be defined by $X_t(\omega) = \omega(t)$. Let $\mathscr{F}_t$ be the smallest right continuous σ-field containing $\sigma(X_s, s \leqq t)$.

Given a kernel $\nu(x, dh)$, define the operator A on C_b^2 functions by

(2.1) $$Af(x) = \int [f(x+h) - f(x) - f'(x)\, h\, 1_{([-1,1])}(h)]\, \nu(x, dh).$$

We say that a probability measure P solves the martingale problem for A starting at x_0 if

(2.2) (i) $P(X_0 = x_0) = 1$, and
(ii) for every $f \in C_b^2$,
$$f(X_t) - (X_0) - \int_0^t Af(X_s)\, ds \text{ in a } P\text{-local martingale.}$$

Occasionally, when it is necessary to use other processes, we will say that (P, X_t) solves the martingale problem.

Concerning existence, we have

Theorem 2.1. *Suppose*
(i) $\sup_x \int (1 \wedge h^2)\, \nu(x, dh) < \infty$, *and*
(ii) *for each* $f \in C_b^2$, $Af(x)$ *is uniformly continuous in* x.

Then for every $x_0 \in \mathbb{R}$ *there exists a solution to the martingale problem for* A *starting from* x_0.

The proof of Theorem 2.1 is given in Sect. 3.

We introduce some notation. Let

(2.3) $$\Phi_x(u) = \int [e^{iuh} - 1 - iuh\, 1_{([-1,1])}(h)]\, \nu(x, dh),$$

let $\lambda > 0$, and let

(2.4) $$W_\lambda(x, y, u) = \frac{\lambda - \Phi_x(u)}{\lambda - \Phi_y(u)}.$$

Primes, e.g., $W_\lambda''(x, y, u)$, refer to derivatives with respect to u.

We need some mild regularity assumptions.

(2.5) There exists $\delta \in (0, 1)$ such that
(i) $\inf_x |\Phi_x(u)| \geqq c|u|^\delta$ for $|u| \geqq 1$;

(ii) $\sup_x \int (1 \wedge h^2)\, \nu(x, dh) < \infty$

(iii) there exists $\rho > 0$ such that
$\sup_{|x-y|<\rho} u^2 |W_\lambda''(x, y, u)| \leqq c_\lambda |u|^\delta$; and

(iv) for all $\rho < 1$,

$$\limsup_{\varepsilon \downarrow 0} \sup_{\substack{|x-y| \leqq \varepsilon \\ \rho \leqq |u| \leqq \rho^{-1}}} |W_\lambda''(x, y, u)| = 0.$$

As we shall when we consider the stable-like processes below (Corollary 2.3), these assumptions are quite mild.

Let

$$M^*(z) = \sup_x |z|^{-2} \int (1 \wedge u^2 z^2) |W_\lambda''(x, x+z, u)|\, du.$$

Our uniqueness theorem then says,

Theorem 2.2. *Suppose* (2.5) *holds. Suppose also that*

(i) $\limsup_{\varepsilon \downarrow 0} \sup_{|u| \in [\varepsilon^{1/2}, \varepsilon^{-1}]} \sup_{\substack{x, y \\ |y| \leqq 1}} \left| W_\lambda\left(x, x+\varepsilon y, \frac{u}{\varepsilon}\right) - 1 \right| = 0$, *and*

(ii) $\limsup_{\eta \downarrow 0} \int_{-\eta}^{\eta} M^*(z)\, dz = 0.$

Then for each x_0 the martingale problem for A starting at x_0 has a unique solution.

A consequence of Theorem 2.2 is that if we denote the unique solution by P^{x_0}, then (P^{x_0}, X_t) forms a strong Markov, Feller process (see Sect. 6).

To understand what these hypotheses say, let us look at a special case. Suppose

$$\nu(x, dh) = \zeta_{\alpha(x)} |h|^{-(1+\alpha(x))}\, dh, \tag{2.6}$$

where $\zeta_{\alpha(x)}$ is a constant chosen so that $A \exp(iux) = -|u|^{\alpha(x)} \exp(iux)$. Let $\beta(z) = \sup_{|x-y| \leqq z} |\alpha(x) - \alpha(y)|$. We then have the following corollary.

Corollary 2.3. *Suppose*

(i) $0 < \inf_x \alpha(x) \leqq \sup_x \alpha(x) < 2$;

(ii) $\beta(z) = o(1/|\ln z|)$ *as* $z \to 0$; *and*

(iii) $\int_0^1 \frac{\beta(z)}{z}\, dz < \infty.$

Then for each x_0 the martingale problem for A starting at x_0 has a unique solution.

Proof of Corollary 2.3, given Theorem 2.2. We know that for the stable-like processes $\Phi_x(u) = -|u|^{\alpha(x)}$. If we let

$$\delta = \min(\inf_x \alpha(x), 2 - \sup_x \alpha(x), 1)/2,$$

then (2.5) (i, ii) are obvious. A straightforward calculation shows that, letting $a=\alpha(x)$, $b=\alpha(y)$.

$$
\begin{aligned}
(2.7)\quad W_\lambda''(x,y,u) &= ((a-b)^2-(a-b))\,u^{a+2b-2}/(\lambda+u^b)^3 \\
&\quad +\lambda[(2a^2-2a-2ab-b^2+b)\,u^{a+b-2}+(b^2+b)\,u^{2b-2}]/(\lambda+u^b)^3 \\
&\quad +\lambda^2[(a^2-a)\,u^{a-2}+(b-b^2)\,u^{b-2}]/(\lambda+u^b)^3 \\
&= I_1+I_2+I_3,
\end{aligned}
$$

if $u>0$ and a similar expression for $u<0$. Now (2.5) (iii, iv) follow.

For $|u|$ large, $W_\lambda(x,y,u)\sim|u|^{\alpha(x)-\alpha(y)}$, and hence Corollary 2.3 (ii) implies Theorem 2.2 (i). Note also that in the presence of Corollary 2.3 (ii), Corollary 2.3 (iii) is equivalent to

$$\int_0^1 \frac{\beta(z)}{z^{1+\beta(z)}}\,dz<\infty.$$

By a change of variables,

$$(2.8)\qquad M^*(z)=\sup_x |z|^{-3}\int(1\wedge u^2)\left|W_\lambda''\left(x,x+z,\frac{u}{z}\right)\right|du.$$

The first term on the right of (2.7) may be bounded by

$$|I_1|\leqq c|a-b|\,u^{a-b-2}.$$

If $|z|$ is small, then $|a-b|\leqq\beta(|z|)$ is small, and the contribution of I_1 to $M^*(z)$ is bounded by

$$c|a-b||z|^{-3}\int(1\wedge u^2)\left(\frac{u}{|z|}\right)^{a-b-2}du=c|a-b||z|^{-(1+(a-b))}.$$

As for I_2, if $|z|$ is small, then $|a-b|\leqq\beta(|z|)<b-\delta$, and so

$$u^{a+b-2}/(\lambda+u^b)^3\leqq c\,u^{-\delta-2},$$

and similarly for $u^{2b-2}/(\lambda+u^b)^3$. Hence the contribution of I_2 to $M^*(z)$ is bounded by

$$c|z|^{-3}\int\left(\frac{u}{|z|}\right)^{-\delta-2}(1\wedge u^2)\,du=c|z|^{-(1-\delta)}.$$

The contribution of I_3 to $M^*(z)$ is similar. Hence for $|z|$ small,

$$M^*(z)\leqq\frac{c\beta(|z|)}{|z|^{1+\beta(|z|)}}+\frac{c}{|z|^{1-\delta}},$$

which establishes Theorem 2.2. (ii). Applying Theorem 2.2 now establishes the corollary. □

3. Existence

In this section we prove Theorem 2.1. Let $\eta \in (0,1)$ and let

$$\tau_\eta = \inf\{s: |X_s - X_0| \geqq \eta\}.$$

The first step towards tightness is

Proposition 3.1. *Suppose $P(X_0 = x_0) = 1$ and that for every $f \in C_b^2$ there exists a constant c_f depending only on $\|f\|$ and $\|f''\|$ such that $f(X_t) - f(X_0) - c_f t$ is a supermartingale. Then $P(\tau_\eta \leqq t) \leqq ct/\eta^2$, c independent of x_0*

Proof. Let $f_0(x) = x^2$ for $|x| \leqq 2$, $f_0(x) \geqq 4$ for $|x| \geqq 2$, $f_0(x) \leqq 8$ for all x, and $f_0 \in C^2$. Let $f(x) = f_0(x - x_0)$. If $|x - x_0| \geqq \eta$, then $f(x) \geqq \eta^2$. So by optional stopping,

$$\eta^2 P(\tau_\eta \leqq t) \leqq Ef(X_{t \wedge \tau_\eta}) - Ef(X_0) \leqq c_f E(t \wedge \tau_\eta) \leqq c_f t. \quad \square$$

We now give a tightness criterion.

Proposition 3.2. *Suppose for each n that $P_n(X_0 = x_0) = 1$ and that for every $f \in C_b^2$ there exists c_f (depending only on $\|f\|$ and $\|f''\|$) such that $f(X_t) - f(X_0) - c_f t$ is a P_n-supermartingale. Then the sequence P_n is tight on $D[0, t_0]$ for each t_0.*

Proof. Let τ_n be a bounded stopping time, let Q_ω^n be a regular conditional probability (r. c. p.) for $E_n(\cdot | \mathscr{F}_{\tau_n})$, and let $Y_t = X_{\tau_n + t}$. Standard arguments show that Y_t satisfies the hypotheses of Proposition 3.1. For example, if $f \in C_b^2$, $Z_t = f(Y_t) - f(Y_0) - c_f t$, $s \leqq t$, $B \in \mathscr{F}_{\tau_n}$, and $A \in \mathscr{F}_{\tau_n + s}$, then

$$\begin{aligned}\int_B Q_\omega^n(Z_t; A)\, P_n(d\omega) &= E_n(Z_t; A \cap B) \leqq E_n(Z_s; A \cap B) \\ &= \int_B Q_\omega^n(Z_s; A)\, P_n(d\omega).\end{aligned}$$

Since B is arbitrary in $\mathscr{F}_{\tau_n}$

$$Q_\omega^n(Z_t; A) \leqq Q_\omega^n(Z_t; A), \qquad a.s.(P_n)$$

A routine argument handle the null sets shows that for almost all ω, Z_t is a Q_ω^n-supermartingale. Similarly $Q_\omega^n(Y_0 = X_{\tau_n}(\omega)) = 1$ for almost all ω. Hence by Proposition 3.1 applied to Q_ω^n, if $\delta \in (0,1)$,

(3.1) $$P_n\Big(\sup_{\tau_n \leqq s \leqq \tau^n + \delta} |X_s - X_{\tau_n}| \geqq \eta\Big) = E_n\, Q_\omega^n\big(\sup_{s \leqq \delta} |Y_s - Y_0| \geqq \eta\big) \leqq c\delta/\eta^2.$$

This estimate gives tightness by [1]. $\square$

Proof of Theorem 2.1. We first construct appropriate P_n. Define P_n to be the probability measure such that for all $k \leqq n2^n$ and all $f \in C_b^2$,

$$f(X_{\frac{k+1}{2^n}\wedge t}) - f(X_{\frac{k}{2^n}\wedge t}) - Af(X_{\frac{k}{2^n}})\left(\frac{k+1}{2^n}\wedge t - \frac{k}{2^n}\wedge t\right)$$

is a P_n-martingale and such that $P_n(X_0 = x_0) = 1$. The P_n may be constructed by piecing together Lévy processes: from $k/2^n$ to $(k+1)/2^n$ take the process that behaves like the Lévy process with Lévy measure $\nu(X_{k/2^n}, dh)$. One way of making this precise is to set up a stochastic differential equation with a Poisson point process as the driving term; we leave this to the interested reader.

If $f \in C_b^2$, then

$$\begin{aligned}(3.2)\qquad |Af(x)| &\leqq 2\|f\|\,\nu(x, [-1,1]^c) + \|f''\| \int_{|h|\leqq 1} h^2\,\nu(x, dh)\\ &\leqq c_f \sup_x \int (1\wedge h^2)\,\nu(x, dh)\end{aligned}$$

So hypothesis (i) and Proposition 3.2 yield tightness on $D[0, t_0]$ for all t_0.

Relabeling if necessary, let P_n be a subsequence which converges, say to P. It remains to show that P solves the martingale problem for A. To do that, it suffices to show that if $f \in C_b^2$,

$$(3.3)\qquad E_n\left[Y\int_t^u B_n(s)\,ds\right] \to E\left[Y\int_t^u B(s)\,ds\right] \quad \text{as} \quad n\to\infty,$$

where $Y = \prod_{i=1}^m g_i(X_{r_i})$, $r_1 \leqq r_2 \leqq \ldots \leqq r_m \leqq t$, the g_i are continuous functions bounded by 1, $B(s) = Af(X_s)$, and $B_n(s) = Af(X_{k/2^n})$ if $k/2^n \leqq s < (k+1)/2^n$.

We have

$$\begin{aligned}(3.4)\qquad E_n\left[Y\int_t^u B_n(s)\,ds\right] &= E_n\left[Y\int_t^u B(s)\,ds\right]\\ &\quad + E_n\left[Y\int_t^u (B_n(s) - B(s))\,ds\right].\end{aligned}$$

The first term on the right of (3.4) converges to $E\left[Y\int_t^u B(s)\,ds\right]$ since P_n converges weakly to P, $B(s) = Af(X_s)$ and Af is continuous by hypothesis. The second term on the right is bounded by $E_n \int_t^u |B_n(s) - B(s)|\,ds$. By the continuity of Af, we know $B_n(s) \to B(s)$ for each $\omega \in D[0, u]$, but we need a bit more than that.

Let $\varepsilon>0$, and choose δ small so that $|Af(x)-Af(y)|<\varepsilon$ if $|x-y|\leqq\delta$. For each n, let $\Gamma_n=\{k\leqq u2^n\colon \sup_{k/2^n\leqq r,s\leqq(k+1)/2^n}|X_s-X_r|>\delta\}$ and let $I_n=\bigcup_{k\in\Gamma_n}[k/2^n,(k+1)/2^n]$.

A consequence of the estimate (3.1) is that X_t has no fixed discontinuities under P. Hence for m sufficiently large, $E\int_t^u 1_{I_m}(X_s)\,ds\leqq\varepsilon$ by dominated convergence. Since $P_n\xrightarrow{w}P$ and $I_n\subseteq I_m$ if $n\geqq m$, we have

$$\begin{aligned}\limsup_n E_n\int_t^u 1_{I_n}(X_s)\,ds&\leqq\limsup_n E_n\int_t^u 1_{I_m}(X_s)\,ds\\&\leqq E\int_t^u 1_{I_m}(X_s)\,ds\leqq\varepsilon.\end{aligned}$$

Therefore

$$\begin{aligned}E_n\int_t^u|B_n(s)-B(s)|\,ds&\leqq\varepsilon(u-t)+\|Af\|\,E_n\int_t^u 1_{I_n}(X_s)\,ds\\&\leqq c\varepsilon\end{aligned}$$

if n is sufficiently large. Using this with (3.4) completes the proof. □

4. The Key Estimate

In this section we derive our key estimate, Proposition 4.2.

Let φ be an even nonnegative C^∞ function with support in $[-\frac{1}{2},\frac{1}{2}]$ and with $\int\varphi(x)\,dx=1$. Define $\varphi_\varepsilon(x)=\varepsilon^{-1}\,\varphi(x/\varepsilon)$. Let $\lambda\in[1,2]$ be fixed. Define

$$r_z^\varepsilon(x)=E^x\int_0^\infty e^{-\lambda t}\,\varphi_\varepsilon(X_t^z)\,dt,$$

where X_t^z is the Lévy process with Lévy measure $\nu(z,dh)$, no drift and no Gaussian component. If we hold z fixed, then

$$\hat{r}_z^\varepsilon(u)=\frac{\hat{\varphi}(\varepsilon u)}{\lambda-\Phi_z(u)},\tag{4.1}$$

where $\Phi_z(u)$ is defined by (2.3). Since $\operatorname{Re}\Phi_z(u)=\int(\cos uh-1)\;\nu(z,dh)\leqq0$, the denominator of (4.1) is bounded away from 0.

Suppose $g\in C^\infty$ with compact support and let

$$h_\varepsilon(x)=\int r_y^\varepsilon(x-y)\,g(y)\,dy.\tag{4.2}$$

We first look at

$$(4.3)\qquad J_\varepsilon(x)=\int_{x-\varepsilon}^{x+\varepsilon}(\lambda-A_x)\,r_y^\varepsilon(x-y)\,g(y)\,dy,$$

where

$$(4.4)\qquad A_x f(z)=\int[f(z+h)-f(z)-f'(z)\,h\,1_{([-1,1])}(h)]\,\nu(x,dh).$$

Proposition 4.1. *$J_\varepsilon(x)\to g(x)$ uniformly in x as $\varepsilon\to 0$.*

Proof. Since φ is in the Schwartz class, so is $\hat{\varphi}$, hence $((\lambda-A_x)\ r_y^\varepsilon)^\wedge(u)$ $=\dfrac{\lambda-\Phi_x(\mathrm{u})}{\lambda-\Phi_y(\mathrm{u})}\hat{\varphi}(\varepsilon u)$ is in L_1. Since φ is in the Schwartz class, $r_y^\varepsilon\in L_1$; and r_y^ε is also in C^∞. Therefore $\|A_x r_y^\varepsilon\|<\infty$.

Note that if $|z|\geqq 2$ and $\varepsilon\leqq 1$,

$$(4.5)\qquad |A_x\,\varphi_\varepsilon(z)|\leqq 2\,\|\varphi_\varepsilon\|\int_{z-1}^{z+1}\nu(x,dh)$$

since φ has support in $[-\frac{1}{2},\frac{1}{2}]$. By Fubini

$$\int_{|z|\geqq 2}\int_{z-1}^{z+1}\nu(x,dh)\,dz\leqq 2\,\nu(x,[-1,1]^c)<\infty,$$

hence $A_x\,\varphi_\varepsilon$ is in L_1. Since the resolvents of Lévy processes map L_p into L_p boundedly for all $p\in[1,\infty]$, then $A_x\,r_y^\varepsilon(z)=E^z\int_0^\infty e^{-\lambda t}A_x\,\varphi_\varepsilon(X_t^y)\,dt$ is in L_1.

We may therefore use the Fourier inversion formula to write

$$(4.6)\qquad J_\varepsilon(x)=(2\pi)^{-1}\int_{x-\varepsilon}^{x+\varepsilon}\int e^{-iu(x-y)}\frac{\lambda-\Phi_x(u)}{\lambda-\Phi_y(u)}\hat{\varphi}(\varepsilon u)\,du\,g(y)\,dy.$$

Using (2.5) (i) and Theorem 2.2 (i), we now follow the proof of Proposition 3.2 of [3] (starting at (3.5) of that proof). □

For $\varepsilon\in(0,1)$, let

$$(4.7)\qquad \Gamma_\varepsilon(u)=\int_0^u e^{-it}\,\hat{\varphi}(\varepsilon t)\,dt$$

and

$$(4.8)\qquad \Lambda_\varepsilon(u)=\int_0^u \Gamma_\varepsilon(t)\,dt.$$

By Proposition 3.3 of [3], we have

(4.9) (i) $|\Gamma_\varepsilon(u)| \leqq c_\varepsilon(1 \wedge |u|^{-2})$;
(ii) $|\Gamma_\varepsilon(u)| \leqq c(1 \wedge |u|)$, c independent of ε; and
(iii) $|\Lambda_\varepsilon(u)| \leqq c(1 \wedge u^2)$, c independent of ε.

From the proof of that proposition we have

$$(4.10) \qquad \Lambda_\varepsilon(u) = -\int \frac{1-\cos uy}{y^2}\, \varphi_\varepsilon(1-y)\, dy.$$

Since φ_ε is an approximation to the identity with support in $[-\varepsilon/2, \varepsilon/2]$, it follows that

$$(4.11) \qquad \Lambda_\varepsilon(u) \to \cos u - 1 \quad \text{boundedly pointwise as} \quad \varepsilon \to 0.$$

We now look at

$$(4.12) \qquad K_\varepsilon(x) = \int_{[x-\varepsilon, x+\varepsilon]^c} (\lambda - A_x)\, r_y^\varepsilon(x-y)\, g(y)\, dy.$$

Note that

$$(4.13) \qquad (\lambda - A)\, h_\varepsilon(x) = (\lambda - A_x)\, h_\varepsilon(x) = J_\varepsilon(x) + K_\varepsilon(x).$$

We can write $K_\varepsilon(x)$ as

$$K_\varepsilon(x) = M_\varepsilon\, g(x) = \int M_\varepsilon(x, y)\, g(y)\, dy,$$

where

$$(4.14) \qquad M_\varepsilon(x, y) = 1_{([x-\varepsilon, x+\varepsilon]^c)}(y)(\lambda - A_x)\, r_y^\varepsilon(x-y).$$

Fix a point w. Define the kernel v^η by

$$(4.15) \qquad v^\eta(x, dh) = \begin{cases} v(x, dh) & |x-w| \leqq \eta \\ v(w-\eta, dh) & x \leqq w-\eta \\ v(w+\eta, dh) & x \geqq w+\eta \end{cases}$$

Define A_x^η, Φ_x^η, $W_\lambda^\eta(x, y, u)$, M_ε^η, and K_ε^η by the analogs of (4.4), (2.3), (2.4), (4.12), and (4.14). Define A^η by $A^\eta f(x) = A_x^\eta f(x)$.

Proposition 4.2. *There exists η_0 and $\gamma < 1$ independent of w such that*

$$\sup_x \int \sup_{\varepsilon \leqq 1} M_\varepsilon^\eta(x, y)\, dy \leqq \gamma$$

whenever $\eta \leqq \eta_0$.

Proof. It is easy to see that (2.5) and the hypotheses of Theorem 2.2 still hold for Φ_x^η, W_λ^η. By taking η sufficiently small, (2.5) (iii) will hold with the sup taken over all x and y.

As in the proof of Proposition 4.1,

$$(4.16) \qquad B_\varepsilon^\eta(x, y) = (\lambda - A_x^\eta)\, r_x^\varepsilon(z) = (2\pi)^{-1} \int e^{-iuz}\, W_\lambda^\eta(x, y, u)\, \hat{\varphi}(\varepsilon u)\, du,$$

where $z=x-y$. By a change of variables,

$$B_\varepsilon^\eta(x,y)=(2\pi)^{-1}|z|^{-1}\int e^{-iu}\,W_\lambda^\eta\left(x,y,\frac{u}{z}\right)\hat{\phi}(\tilde{\varepsilon}u)\,du,$$

where $\tilde{\varepsilon}=\varepsilon/z$.

Integrate by parts twice, looking at positive and negative u separately and real and imaginary parts separately. By (4.9) and (2.5) (iii), we get (cf. proof of Proposition 3.3 in [3])

$$\begin{aligned}(4.17)\qquad B_\varepsilon^\eta(x,y)&=(2\pi)^{-1}|z|^{-3}\int \Lambda_{\tilde{\varepsilon}}(u)\,W_\lambda^{\eta\prime\prime}\left(x,y,\frac{u}{z}\right)du\\ &\leqq c|z|^{-3}\int(1\wedge u^2)\left|W_\lambda^{\eta\prime\prime}\left(x,y,\frac{u}{z}\right)\right|du.\end{aligned}$$

By Theorem 2.2 (ii),

$$\sup_x \int_{x-\theta}^{x+\theta} \sup_\varepsilon |B_\varepsilon^\eta(x,y)|\,dy \leqq 1/8,$$

if we take θ sufficiently small. By (2.5) (iii),

$$|B_\varepsilon^\eta(x,y)|\leqq c|z|^{-3}\int(1\wedge u^2)\left|\frac{u}{z}\right|^{-2+\delta}du\leqq c|z|^{-(1+\delta)},$$

which is integrable at $\pm\infty$. Therefore, taking θ smaller if necessary,

$$\sup_x \int_{|y-x|\geqq\theta^{-1}} \sup_\varepsilon |B_\varepsilon^\eta(x,y)|\,dy\leqq 1/8.$$

So to complete the proof it suffices to show

$$(4.18)\qquad \sup_x \int_{\theta\leqq|x-y|\leqq\theta^{-1}} \sup_\varepsilon |B_\varepsilon^\eta(x,y)|\,dy\leqq 1/2.$$

Fix x. By (2.5) (iii),

$$\begin{aligned}|B_\varepsilon^\eta(x,y)|\leqq c_\lambda &\int_{|u|\leqq\rho \text{ or } |u|\geqq\rho^{-1}} (1\wedge u^2)|u|^{-2+\delta}\,du\\ &+c_\lambda \int_{\rho<|u|<\rho^{-1}} (1\wedge u^2)|W_\lambda^{\eta\prime\prime}(x,y,u)|\,du,\end{aligned}$$

provided $\theta\leqq|x-y|\leqq\theta^{-1}$.

Since $\int_{-\rho}^{\rho} u^\delta\, du + \int_{|u| \geqq \rho^{-1}} |u|^{-2+\delta}\, du \leqq 1/8(\theta^{-1}-\theta)\, c_\lambda$ provided ρ is sufficiently small, it only remains to show

$$\int_{\rho \leqq |u| \leqq \rho^{-1}} (1 \wedge u^2)\, |W_\lambda^{\eta''}(x, y, u)|\, du \leqq 1/8(\theta^{-1}-\theta)\, c_\lambda. \tag{4.19}$$

But by (2.5) (iv), we can choose η small enough so that (4.19) is satisfied. □

5. Uniqueness – Local Case

In this section we suppose η_0 and γ have been chosen as in Proposition 4.2, and we prove uniqueness for the martingale problem for A^η starting at x_0 if $\eta \leqq \eta_0$, where $w = x_0$. For typographical convenience we drop the η's from our notation throughout this section.

Let E_1 and E_2 be two solutions to the martingale problem for A starting at x_0.

Proposition 5.1. *For all f bounded and Borel and all t, $E_1 f(X_t) = E_2 f(X_t)$.*

Proof. If $f \in C_b^2$,

$$E_i f(X_t) - f(x_0) = E_i \int_0^t Af(X_s)\, ds, \qquad i = 1, 2. \tag{5.1}$$

Multiplying by $\lambda e^{-\lambda t}$, integrating t from 0 to ∞, and using Fubini gives

$$\lambda E_i \int_0^\infty e^{-\lambda t} f(X_t)\, dt - f(x_0) = E_i \int_0^\infty e^{-\lambda s} Af(X_s)\, ds, \qquad i = 1, 2. \tag{5.2}$$

Let S_λ^i be the operator defined on L_∞ by

$$S_\lambda^i f = E_i \int_0^\infty e^{-\lambda t} f(X_t)\, dt, \qquad i = 1, 2. \tag{5.3}$$

Then (5.2) says

$$f(x_0) = S_\lambda^i((\lambda - A) f), \qquad i = 1, 2. \tag{5.4}$$

Apply (5.3) to $f = h_\varepsilon$, where h_ε is defined by (4.2). Using (4.13),

$$h_\varepsilon(x_0) = S_\lambda^i(g + (J_\varepsilon - g) + M_\varepsilon g), \qquad i = 1, 2. \tag{5.5}$$

Let $S_\lambda^A = S_\lambda^1 - S_\lambda^2$, and let $\theta = \sup_{\|f\| \leqq 1} |S_\lambda^A f|$. Clearly, $\theta \leqq 2/\lambda$.

Now take the difference of (5.5) for $i = 1, 2$:

$$S_\lambda^A(g + (J_\varepsilon - g) + M_\varepsilon g) = 0.$$

Then

$$(5.6)\qquad \begin{aligned} |S_\lambda^4 g| &\leqq |S_\lambda^4(J_\varepsilon - g)| + |S_\lambda^4 M_\varepsilon g| \\ &\leqq \theta \| J_\varepsilon - g \| + \theta \| M_\varepsilon g \| \\ &\leqq \theta \| J_\varepsilon - g \| + \theta\gamma \| g \|, \end{aligned}$$

where $\gamma < 1$ by Proposition 4.2. Letting $\varepsilon \to 0$ and using Proposition 4.1, we get finally

$$(5.7)\qquad |S_\lambda^4 g| \leqq \theta\gamma \| g \|.$$

Now, since S_λ^i is an operator induced by the measure $S_\lambda^i(A) = S_\lambda^i 1_A$, we have $\theta = \sup\{|S_\lambda^4 g| : \| g \| \leqq 1,\ g \in C^\infty$ with compact support$\}$. But taking the sup over such g in (5.7) gives $\theta \leqq \theta\gamma$. Since $\gamma < 1$ and $\theta \leqq 2/\lambda$, we must have $\theta = 0$, or $S_\lambda^1 = S_\lambda^2$ for $\lambda \in [1, 2]$.

By the uniqueness of the Laplace transform and the right continuity of X_t, $E_1 f(X_t) = E_2 f(X_t)$ if f is continuous and bounded. The proposition is now immediate. □

The proof is the next proposition is essentially the same argument as that in [10], Sect. 6.2, and so we only sketch it.

Proposition 5.2. *The finite dimensional distributions of X_t under E_1 and E_2 agree.*

Proof. Let $\psi_i(X_s) = E_i(g(X_{t+s}) | X_s)$, $i = 1, 2$, g bounded and continuous. If Q_ω^i is a r.c.p. for $E_i(\cdot | X_s)$, it is easy to see that (Q_ω^i, X_{s+t}) satisfies the martingale problem for A starting at $X_s(\omega)$ (cf. proof of Proposition 3.2). By Proposition 5.1, $\psi_1(X_s) = Q_\omega^1 g(X_{s+t}) = Q_\omega^2 g(X_{s+t}) = \psi_2(X_s)$. Denote the common value by ψ.

If f is continuous and bounded,

$$\begin{aligned} E_1 g(X_{t+s}) f(X_s) &= E_1(f(X_s) E_1(g(X_{t+s}) | X_s)) = E_1((\psi f)(X_s)) \\ &= E_2((\psi f)(X_s)) = E_2 g(X_{t+s}) f(X_s). \end{aligned}$$

This shows the two dimensional distributions agree. Now repeat, using induction. □

Remark. Suppose v is such that $h_\varepsilon(x) \to Rg(x) = \int r_y(x-y)\, g(y)\, dy$ *as* $\varepsilon \to 0$ for some kernel r_y (what is needed is that the resolvent for X_t^y has a sufficiently nice density). By our bounds on B_ε in Sect. 4 and (4.11), $M_\varepsilon g \to Mg$, where

$$(5.8)\qquad Mg(x) = \int g(y)\, M(x, y)\, dx$$

and

$$(5.9)\qquad M(x, y) = -(2\pi)^{-1} |x-y|^{-3} \int (1 - \cos u)\, W_\lambda^{y\prime\prime}\left(x, y, \frac{u}{x-y}\right) du.$$

Taking the limit in (5.5) as $\varepsilon \to 0$, then gives us

$$(5.10)\qquad Rg(x_0) = S_\lambda(g + Mg),$$

or

$$S_\lambda = R(I+M)^{-1} = \sum_{i=0}^{\infty} R(-M)^i. \tag{5.11}$$

The expansion is possible since $\|Mg\| \leqq \gamma \|g\|$, $\gamma < 1$. It is interesting to compare (5.11) with formulas for resolvents used by [5–7, 9–13]. There the resolvent S_λ is written as a perturbation of the resolvent of Brownian motion or a stable process. Here S_λ is a perturbation of R; R, however, does not correspond, to any Markov process.

6. Uniqueness – Global Case

In this section we complete the proof of Theorem 2.2 as well as prove the strong Markov and Feller properties. The proofs are the same as those in [10], and so are only sketched.

Proof of Theorem 2.2. Let η be chosen as in Proposition 4.2. Let $\tau = \inf\{t: |X_t - x_0| \geqq \eta\}$. By Sect. 5, we have uniqueness for the martingale problem for A^η starting at x for each x; call the solution P_x^η.

If P is a solution to the martingale problem for A starting at x_0, define Q by

$$Q(A \cap B \circ \theta_\tau) = E(P_{X_\tau}^\eta(B); A), \qquad A \in \mathscr{F}_\tau, \qquad B \in \mathscr{F}_\infty, \tag{6.1}$$

where θ_t is the usual shift operator. Check that such $A \cap B \circ \theta_\tau$ generate $\mathscr{F}_\infty$ and that Q solves the martingale problem for A^η. Hence $Q = P_{x_0}^\eta$, and so if $A \in \mathscr{F}_\tau$, $P(A) = Q(A) = P_{x_0}^\eta(A)$. We thus have uniqueness for the martingale problem for A for events in $\mathscr{F}_\tau$.

Let $\tau_1 = \tau$, $\tau_{i+1} = \inf\{t > \tau_i: |X_t - X_{\tau_i}| \geqq \eta\}$. By Proposition 3.1, $\tau_i \to \infty$ as $i \to \infty$ (cf. [3], Sect. 5). To show uniqueness for the martingale problem for A for events in $\mathscr{F}_\infty$, it suffices to show uniqueness for events in $\mathscr{F}_{\tau_i}$ for each i.

We use induction. Suppose we have uniqueness on $\mathscr{F}_{\tau_i}$. If $A \in \mathscr{F}_\tau$, and P is a solution to the martingale problem, then

$$P(A \cap B \circ \theta_{\tau_i}) = E(E(B \circ \theta_{\tau_i} | \mathscr{F}_{\tau_i}); A). \tag{6.2}$$

If Q_ω is a r.c.p. for $E(\cdot | \mathscr{F}_{\tau_i})$, check that Q_ω, $X_{\tau_i + t}$) solves the martingale problem for A starting at $X_{\tau_i}(\omega)$. By the above, $\Lambda = E(B \circ \theta_{\tau_i} | \mathscr{F}_{\tau_i}) = Q_\omega(B \circ \theta_{\tau_i})$ is then uniquely determined. But by the induction hypotheses $E(\Lambda; A)$ is uniquely determined. Hence, P is uniquely determined on the collection of $A \cap B \circ \theta_{\tau_i}$, $A \in \mathscr{F}_{\tau_i}$, $B \in \mathscr{F}_{\tau_i}$, and this collection generates $\mathscr{F}_{\tau_{i+1}}$. □

For each x, then, the martingale problem for A starting at x has a unique solution P^x.

Proposition 6.1. (X_t, P^x) is a strong Markov process.

Proof. Let τ be a bounded stopping time and let Q_ω be a r.c.p. for $E(\cdot|\mathscr{F}_\tau)$. As before, (Q_ω, X_{r+t}) solves the martingale problem for A starting at $X_\tau(\omega)$. Hence, by the uniqueness of such a solution,

$$E^{X_\tau(\omega)} f(X_t) = Q_\omega f(X_{t+\tau}) = E[f(X_{t+\tau})|\mathscr{F}_\tau]$$

for f bounded and Borel. □

We also have

Proposition 6.2. *(X_t, P^x) is a Feller process.*

Proof. As in Sect. 3, if $x_n \to x$, then the sequence P^{x_n} is tight, and any limit point solves the martingale problem for A starting at x. So, if f is bounded and continuous, $E^{x_n} f(X_t) \to E^x f(X_t)$. □

7. Remarks

1. Extensions. There is no difficulty whatsoever is extending our method to pure jump processes in $\mathbb{R}^d$ with $d > 1$. We restricted our attention to $d = 1$ only for simplicity.

Similarly, our method allows for the inclusion of a drift term. Thus, instead of A, one considers the operator

$$\tilde{A} f(x) = b(x) f'(x) + A f(x).$$

Existence holds if in addition to the hypotheses on A we require b bounded and continuous. Uniqueness will hold if b is bounded and continuous and if (2.5) and the hypotheses of Theorem 2.2 hold with $\Phi_x(u)$ replaced by $\tilde{\Phi}_x(u) = ib(x) u + \Phi_x(u)$.

2. Local times. Suppose for some $\delta \in (0, 1)$

$$\inf_x |\Phi_x(u)| \geqq c|u|^{1+\delta} \quad \text{for } |u| \geqq 1 \tag{7.1}$$

For stable-like processes this corresponds to $\inf_x \alpha(x) > 1$. Then the expansion (5.11) holds and by [3], Proposition 3.1, R maps L_1 to L_∞. Moreover, an examination of the derivation of the bounds on M show that $\sup_y \int M(x, y)\, dx \leqq \gamma < 1$ as well as $\sup_x \int M(x, y)\, dy \leqq \gamma$. So by the generalized Young's inequality ([4], Theorem 0.10), M maps L_p to L_p with norm bounded by γ for $p \in [1, \infty]$. Therefore $(I + M)^{-1}$ maps L_1 to L_1 and by (5.11) S_λ maps L_1 to L_∞. This fact, a localization argument, and Sect. 5 of [3] prove

Theorem 7.1. *Suppose* (2.5), *the hypotheses of Theorem 2.2, and (7.1) hold. Then* (X_t, P) *has an occupation time density* $L_t(y)$.

This is a slight improvement on [3], Theorem 6.1, since there is no need to suppose the existence of an approximating sequence v_n.

3. Singular integrals. Suppose

$$K(x,y) \sim \frac{\alpha(x)-\alpha(y)}{|x-y|^{1+\alpha(x)-\alpha(y)}} \quad \text{as } |x-y| \to 0 \tag{7.2}$$

and

$$|K(x,y)| \leqq C|x-y|^{-(1+\delta)} \qquad \text{for some } \delta > 0.$$

In the stable-like case we showed that if the hypotheses of Corollary 2.3 hold, then K is integrable and the operator defined by $Kf(x) = \int K(x,y)\,dy$ maps L_∞ to L_∞.

It would be interesting to examine the properties of the operator K when α fails to be Dini continuous but perhaps satisfies some weaker modulus of continuity. One would have to define Kf in a principal value sense, as is done in the theory of singular integrals, and one could ask, does K map L_p to L_p for $p \in (1, \infty)$? If so, could one use this to prove uniqueness for the corresponding martingale problem? The kernel for K seems to be of a type not previously considered by analysis.

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