

## The Most Visited Site of Brownian Motion and Simple Random Walk

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**Summary.** Let  $L(t, x)$  be the local time at  $x$  for Brownian motion and for each  $t$ , let  $\bar{V}(t) = \inf \{x \geq 0: L(t, x) \vee L(t, -x) = \sup_y L(t, y)\}$ , the absolute value of the most visited site for Brownian motion up to time  $t$ . In this paper we prove that  $\bar{V}(t)$  is transient and obtain upper and lower bounds for the rate of growth of  $\bar{V}(t)$ . The main tools used are the Ray-Knight theorems and William's path decomposition of a diffusion. An invariance principle is used to get analogous results for simple random walks. We also obtain a law of the iterated logarithm for  $\bar{V}(t)$ .

### 1. Introduction

Let  $S_n$  be a simple random walk in  $\mathbb{R}^1$  started at 0. The basic question asked in this paper is as follows: is 0 the most visited site infinitely often? To formulate this precisely define

$$\begin{aligned} N(n, k) &= \sum_{j=0}^n 1_{\{k\}}(S_j) \\ \bar{N}(n) &= \sup_k N(n, k) \\ \bar{U}(n) &= \inf \{k \geq 0: N(n, k) \vee N(n, -k) = \bar{N}(n)\}. \end{aligned}$$

The question then becomes, does  $P(\bar{U}(n) = 0 \text{ i.o.}) = 1$ ? Rather surprisingly the answer is no as we show in Sect. 5.

Our approach to this problem is to consider the analogous question for a one dimensional Brownian motion  $W(t)$  and then use an invariance principle. Thus let  $L(t, x)$  be the local time of  $W$  at  $x$  up to time  $t$  (for a precise definition

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see (2.3)) and define

$$L^*(t) = \sup_{x \geq 0} L(t, x)$$

$$V(t) = \inf \{x \geq 0: L(t, x) = L^*(t)\}.$$

(Observe that  $V(t)$  is the “most visited site” on the positive  $x$ -axis, the results for the entire  $x$ -axis will follow easily from our analysis of this case.) We will prove that  $V(t)$  is a transient process and furthermore obtain estimates on how quickly it escapes to infinity (see Theorems 3.6 and 3.7). Set

$$T_r = \inf \{t: L(t, 0) > r\}.$$

Our main tool is a theorem of Knight [12] which describes  $L(T_r, z)$  as a Markov process in the space variable, (for closely related results see Ray [14]). This enables us to obtain information about  $V(t)$  at the random times  $T_r$ , from which the results follow fairly easily. To give some indication of how the proof goes, define for any  $K > 0$

$$I(t, K) = \sup_{0 \leq x \leq K} L(t, x). \tag{1.1}$$

Obviously, for any  $t$ ,  $L(t, 0) \leq I(t, K) \leq L^*(t)$ . It is not too difficult to show that for any  $K > 0$

$$\liminf_{r \rightarrow \infty} \frac{L^*(T_r)}{r} = 1, \quad \limsup_{r \rightarrow \infty} \frac{I(T_r, K)}{r} = 1$$

and so to distinguish between them, we investigate the second order terms. We prove that for any  $\varepsilon > 0$  and  $K > 0$ , if  $r$  is sufficiently large

$$L^*(T_r) - r \geq r^{1-\varepsilon}, \quad I(T_r, K) - r \leq r^{(1/2)+\varepsilon}$$

(see Propositions 3.3 and 3.5 for more precise results). From this it is immediate that  $V(T_r)$  is transient, and a little further analysis shows that  $V(t)$  is also. The main difficulty in this approach is to obtain a good upper bound for  $P(L^*(T_r) - r \leq \lambda$  for some  $r \in [1, 2]$ ), which is done in Lemma 3.2.

In addition to investigating the small values of  $V(t)$  we also investigate its large values. By the Law of the Iterated Logarithm (L.I.L.) for  $W(t)$ , it is immediate that

$$\limsup_{t \rightarrow \infty} \frac{V(t)}{(2t \log \log t)^{1/2}} \leq 1 \quad \text{a.s.} \tag{1.2}$$

We prove in Sect. 4 that there is in fact equality in (1.2). To convince oneself that this should indeed be the case recall Strassen’s L.I.L. [16]. Given  $\varepsilon \in (0, 1)$  let

$$x(s) = \begin{cases} s & \text{if } 0 \leq s \leq 1 - \varepsilon \\ 1 - \varepsilon & \text{if } 1 - \varepsilon \leq s \leq 1 \end{cases}$$

Then  $x(s)$  is a limit point of  $W(st)/(2t \log \log t)^{1/2}$  with probability one. Now at those times  $t$  for which  $W(st)/(2t \log \log t)^{1/2} \sim x(s)$ , it is intuitively clear that  $V(t)$  should be approximately  $(1 - \varepsilon)(2t \log \log t)^{1/2}$ . The proof given in Sect. 4 makes this idea precise.

In concluding the introduction we should point out that other properties of  $L^*(t)$  have been studied by several authors, for example see [1, 4, 7, 10].

**2. Preliminaries**

Let  $Y(t)$  be a Bessel process of index 0, abbreviated BES(0), [11]. That is,  $Y(t)$  is a Markov process on  $[0, \infty)$  with 0 as a trap and with generator  $\frac{1}{2}y'' - \frac{1}{2x}y'$ .

Set  $X(t) = Y^2(t)$ , so that  $X(t)$  is a (BES(0))<sup>2</sup>. One readily checks that  $X(t)$  is a diffusion on natural scale with 0 as a trap and with generator  $2xy''$ . Further its transition densities are given by ([11] page 100)

$$p_t(x, y) = (x/y)^{1/2} (2t)^{-1} \exp(-(x+y)/2t) I_1((xy)^{1/2}/t) \quad x, y, t > 0 \quad (2.1)$$

where  $I_1$  is the modified Bessel function of the first kind and

$$I_1(x) \sim e^x (2\pi x)^{-1/2} \quad \text{as } x \rightarrow \infty. \quad (2.2)$$

Let  $W(t)$  be a one-dimensional Brownian motion. The local time of  $W(t)$  defined by

$$L(t, x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(x-\varepsilon, x+\varepsilon)}(W(s)) ds \quad (2.3)$$

exists and is jointly continuous in  $t$  and  $x$  except on a set of probability zero [17]. Define, for  $r \geq 0$ ,

$$T_r = \inf \{t: L(t, 0) > r\}.$$

It is well-known that  $\{T_r: r \geq 0\}$  is a stable subordinator of index 1/2.

**Theorem 2.1** (Knight). *For any  $r \geq 0$ , the process  $\{L(T_r, z): z \geq 0\}$  is a (BES(0))<sup>2</sup> started at  $L(T_r, 0) = r$ .*

We now state three results which give most of the properties of  $L(T_r, z)$  that are needed in the remainder of the paper.

**Lemma 2.2.** (a) *For each  $z \geq 0$ ,  $\{L(T_r, z): r \geq 0\}$  is a stationary independent increment process.*

(b) *For each  $r \geq 0, z \geq 0, u < (2z)^{-1}$ ,*

$$E \exp(uL(T_r, z)) = \exp(ur(1 - 2uz)^{-1}).$$

(c) *For each  $r \geq 0, \{L(T_r, z): z \geq 0\}$  is a martingale.*

(d) *For each  $z \geq 0, \{L(T_r, z) - r: r \geq 0\}$  is a martingale.*

*Proof.* (a) This follows immediately from the strong Markov and stationary independent increment properties of Brownian motion.

(b) Fix  $z \geq 0$ . We begin by computing the Lévy measure  $n(dy)$  of the Lévy process  $L(T_r, z)$ . Let  $X_i$  be a sequence of i.i.d. random variables with distribution  $P(X_i > \gamma) = \exp(-\gamma/2z)$  and  $N(r)$  an independent Poisson process of intensity  $(2z)^{-1}$ . Then by comparing with (2.1) a direct calculation shows that  $X_1 + \dots + X_{N(r)}$  has the same distribution as  $L(T_r, z)$ , and since they both have

stationary independent increments, they are identical as processes. Now for  $\gamma \geq 0$

$$\begin{aligned} h n((\gamma, \infty)) &= E \sum_{r \leq h} 1_{(\gamma, \infty)}(L(T_r, z) - L(T_{r-}, z)) \\ &= P(N(h) = 1) P(X_1 > \gamma) + O(h^2) \quad \text{as } h \rightarrow 0, \end{aligned}$$

thus

$$n((\gamma, \infty)) = (2z)^{-1} \exp(-\gamma/2z).$$

Hence

$$\begin{aligned} E \exp(u L(T_r, z)) &= \exp \left( r \int_0^\infty (e^{u\gamma} - 1) n(d\gamma) \right) \\ &= \exp(ur(1 - 2uz)^{-1}) \end{aligned}$$

provided  $u < (2z)^{-1}$ .

(c) By (b),  $EL(T_r, z) < \infty$  and by Theorem 2.1,  $L(T_r, z)$  is a diffusion on natural scale, hence it is a martingale.

(d) By (c)  $EL(T_r, z) = EL(T_r, 0) = r$ . Then for each  $z$ ,  $L(T_r, z) - r$  is a zero mean, stationary independent increment process and hence a martingale.  $\square$

*Remark.* The explanation for why  $L(T_r, z)$  can be represented as the sum of a Poisson number of independent exponentials is part of the ‘‘folklore’’ of excursion theory. We very briefly sketch the argument. By Itô’s theory of excursions [9], the number  $N(r)$ , of excursions from 0 that reach the level  $z$  by time  $T_r$  is Poisson. By the strong Markov property, one can show that the  $i$ th excursion that reaches  $z$  will increase local time at  $z$  by  $X_i$ , where  $X_i$  is exponential and independent of  $X_1, \dots, X_{i-1}$  (cf. [13]).

To find the parameters of  $X_i$  and  $N(r)$ , use Tanaka’s formula. If  $\tau_z = \inf\{t: W(t) = z\}$ ,

$$E(W(\tau_z \wedge t))^+ = \frac{1}{2} EL(\tau_z \wedge t, 0)$$

for each  $t$ . Letting  $t \rightarrow \infty$ , using bounded convergence on the left, and monotone convergence on the right, we get

$$EL(\tau_z, 0) = 2 E(W(\tau_z))^+ = 2z.$$

By the translation invariance and symmetry of Brownian motion,

$$E(L(\tau_0, z) | W(0) = z) = EL(\tau_z, 0) = 2z,$$

from which one can determine the parameters of the  $X_i$  and of  $N(r)$ .

A slight modification of the above argument enables one to compute the distribution of  $L(T_r, z)$  given  $L(T_r, y)$  for  $0 \leq y \leq z$ . This, together with the first eight pages of [18] give, in fact, an excursion theory proof of Knight’s theorem.

The following result, which we will later apply to the process of Theorem 2.1, gives a path decomposition of a  $(BES(0))^2$  at its maximum and is a corollary of Theorem 2.4 of Williams [19].

**Theorem 2.3.** *Consider a probability space  $(\Omega, \mathcal{F}, Q)$  on which the following independent random elements are defined,*

- (a)  $Y(t)$  a diffusion on  $[0, \infty)$  with generator  $2xy'' + 4y'$  started at  $x=1$ ,
  - (b) a random variable  $M$  with distribution satisfying  $P(M > m) = m^{-1}$  for  $m \geq 1$ ,
  - (c) for each  $m \geq 1$  a diffusion  $Z^m(t)$  on  $[0, m]$  with generator  $2xy'' - 4x(m-x)^{-1}y'$  started at the entrance boundary point  $m$  and absorbed at 0.
- Define  $\rho = \inf\{t: Y(t) = M\}$  and

$$X(t) = \begin{cases} Y(t) & t \leq \rho \\ Z^M(t - \rho) & t \geq \rho. \end{cases}$$

Then  $X(t)$  is a  $(\text{BES}(0))^2$  started at  $x=1$ .

*Remark.* We will only use the premaximum representation of a  $(\text{BES}(0))^2$ . It is easy to check that in fact  $Y(t)$  is a  $(\text{BES}(4))^2$ .

**Lemma 2.4.** For all  $a > 0$

- (a)  $\{L(t, x): t \geq 0, x \in \mathbb{R}\} \stackrel{d}{=} \{a^{-1/2} L(at, a^{1/2}x): t \geq 0, x \in \mathbb{R}\}$ ,
- (b)  $\{L(T_r, z): r \geq 0, z \in \mathbb{R}\} \stackrel{d}{=} \{a^{-1} L(T_{ar}, az): r \geq 0, z \in \mathbb{R}\}$ .

*Proof.* For any  $a > 0$ ,  $X(t) = a^{-1/2} W(at)$  is also a Brownian motion. Let  $L^X(t, x)$  and  $L^W(t, x)$  denote the local times of  $X(t)$  and  $W(t)$ , respectively. Then a simple change of variable in (2.3) gives

$$L^X(t, x) = a^{-1/2} L^W(at, a^{1/2}x) \quad \text{a.s.} \tag{2.4}$$

which proves (a).

Let  $T_r^X = \inf\{t: L^X(t, 0) > r\}$  and  $T_r^W = \inf\{t: L^W(t, 0) > r\}$ . Then by (2.4)

$$T_r^X = a^{-1} T_{a^{1/2}r}^W \quad \text{a.s.} \tag{2.5}$$

Thus upon replacing  $a$  by  $a^2$  in (2.4) and (2.5) we obtain

$$L^X(T_r, z) = a^{-1} L^W(T_{ar}, az) \quad \text{a.s.}$$

which proves (b).  $\square$

If we now define

$$L^*(t) = \sup_{x \geq 0} L(t, x)$$

we have as an immediate consequence of the above Lemma that

$$L^*(t) \stackrel{d}{=} t^{1/2} L^*(1) \tag{2.6}$$

and

$$L^*(T_r) \stackrel{d}{=} r L^*(T_1). \tag{2.7}$$

*Remark.* Note that we are using  $L^*$  for the maximum over the positive  $x$ -axis rather than the entire real line.

In concluding this section we would like to point out that in the remainder of this paper the symbols  $c, c_1, c_2$  etc. will denote generic constants which may change from line to line. In addition we will freely interchange between the commonly used notations  $P(\cdot | X(0) = x)$  and  $P^x(\cdot)$ .

### 3. Transience and Lower Growth Rates

In this section we will prove that  $V(t)$  is a transient process and obtain estimates on how quickly it escapes to infinity. Recalling the path decomposition of a  $(\text{BES}(0))^2$  in Theorem 2.3, we begin by proving,

**Lemma 3.1.** *Let  $Y(t)$  be a diffusion with generator  $2x y'' + 4y'$  and  $Y(0)=1$ . Then for all  $m > 1, u > 0$*

$$P(\sup_{0 \leq t \leq u} Y(t) < m) \leq (2\pi u)^{-1/2}(m-1).$$

*Proof.* If  $X(t)$  is a diffusion with generator  $\frac{1}{2}y'' + \frac{3}{2x}y'$  and  $X(0)=1$  (so  $X$  is a  $\text{BES}(4)$ ), then (cf. [8]) the distribution of  $X(t)^2$  is the same as that of  $Y(t)$ . Moreover, there exists a Brownian motion  $W(t)$  started at  $W(0)=0$  such that

$$X(t) = 1 + W(t) + \frac{3}{2} \int_0^t X(s)^{-1} ds.$$

Since  $X(t) > 0$  for all  $t$ , we see that  $X(t) \geq 1 + W(t)$  and so

$$\begin{aligned} P(\sup_{0 \leq t \leq u} Y(t) < m) &\leq P(\sup_{0 \leq t \leq u} W(t) < m^{1/2} - 1) \\ &\leq P(\sup_{0 \leq t \leq u} W(t) < m - 1). \end{aligned}$$

By the reflection principle and scaling property

$$\begin{aligned} P(\sup_{0 \leq t \leq u} W(t) < m - 1) &= P(|W(u)| < m - 1) \\ &= P(|W(1)| < u^{-1/2}(m - 1)) \\ &\leq (2\pi u)^{-1/2}(m - 1). \quad \square \end{aligned}$$

Next we obtain the main estimate in the proof of transience.

**Lemma 3.2.** *For any  $\varepsilon > 0$ , there exists a constant  $c$  such that for all  $\lambda$  sufficiently small*

$$P(\inf_{1 \leq r \leq 2} (L^*(T_r) - r) < \lambda) \leq c \lambda^{\frac{1}{2} - \varepsilon}$$

*Proof.* By Lemma 2.4, for any  $n$

$$\begin{aligned} P(\inf_{1 \leq r \leq 2} (L^*(T_r) - r) < \lambda) &\leq \sum_{k=0}^{n-1} P(\inf_{1 + \frac{k}{n} \leq r \leq 1 + \frac{k+1}{n}} (L^*(T_r) - r) < \lambda) \\ &= \sum_{k=0}^{n-1} P\left(\inf_{1 \leq r \leq 1 + \frac{1}{n+k}} (L^*(T_r) - r) < \frac{\lambda n}{(n+k)}\right) \\ &\leq n P(\inf_{1 \leq r \leq 1 + \frac{1}{n}} (L^*(T_r) - r) < \lambda). \end{aligned}$$

Now for any  $x > 0$

$$\begin{aligned}
 &P\left(\inf_{1 \leq r \leq 1 + \frac{1}{n}} (L^*(T_r) - r) < \lambda\right) \\
 &\leq P\left(\inf_{1 \leq r \leq 1 + \frac{1}{n}} (L^*(T_r) - r) < \lambda, L^*(T_1) > 1 + 2\lambda, V(T_1) < x\right) \\
 &\quad + P\left(\inf_{1 \leq r \leq 1 + \frac{1}{n}} (L^*(T_r) - r) < \lambda, V(T_1) > x\right) \\
 &\quad + P(L^*(T_1) < 1 + 2\lambda) \\
 &= I + II + III.
 \end{aligned}$$

Observe that

$$I = \int_{z=0}^x P\left(\inf_{1 \leq r \leq 1 + \frac{1}{n}} (L^*(T_r) - r) < \lambda, L^*(T_1) > 1 + 2\lambda \mid V(T_1) = z\right) P(V(T_1) \in dz)$$

and that for  $0 \leq z \leq x$

$$\begin{aligned}
 &P\left(\inf_{1 \leq r \leq 1 + \frac{1}{n}} (L^*(T_r) - r) < \lambda, L^*(T_1) > 1 + 2\lambda \mid V(T_1) = z\right) \\
 &\leq P\left(\inf_{1 \leq r \leq 1 + \frac{1}{n}} (L(T_r, z) - r) < \lambda, L(T_1, z) > 1 + 2\lambda \mid V(T_1) = z\right) \\
 &\leq P\left(\inf_{0 \leq r \leq \frac{1}{n}} (L(T_r, z) - r) < -\lambda\right)
 \end{aligned}$$

since  $L(T_r, z) - r$  as a process in  $r$  has stationary independent increments. Furthermore since  $L(T_r, z) - r$  is a martingale, we have by Doob's inequality and Lemma 2.2(b)

$$\begin{aligned}
 P\left(\inf_{0 \leq r \leq \frac{1}{n}} (L(T_r, z) - r) < -\lambda\right) &= P\left(\sup_{0 \leq r \leq \frac{1}{n}} (r - L(T_r, z)) > \lambda\right) \\
 &\leq e^{-u\lambda} E \exp\left[u\left(\frac{1}{n} - L(T_1, z)\right)\right] \\
 &= \exp\left[-u\lambda + \frac{u}{n} - \frac{u}{n(1+2uz)}\right] \\
 &\leq \exp(-n\lambda^2/8z)
 \end{aligned}$$

if  $u = n\lambda/4z$ . Thus

$$I \leq \int_0^x \exp(-n\lambda^2/8z) P(V(T_1) \in dz) \leq \exp(-n\lambda^2/8x).$$

Next, since  $L^*(T_r)$  is increasing, if we assume that  $\lambda < n^{-1}$ , then

$$II \leq P(V(T_1) > x, L^*(T_1) < 1 + 2/n).$$

From this, Theorem 2.1, Theorem 2.3 and Lemma 3.1 we see that (using the notation of Theorem 2.3)

$$\begin{aligned} II &\leq Q(\rho > x, M < 1 + 2/n) \\ &= \int_1^{1+(2/n)} Q(\rho > x | M = m) Q(M \in dm) \\ &= \int_1^{1+(2/n)} Q(\sup_{0 \leq z \leq x} Y(z) < m | M = m) Q(M \in dm) \\ &= \int_1^{1+(2/n)} Q(\sup_{0 \leq z \leq x} Y(z) < m) m^{-2} dm \\ &\leq c x^{-1/2} \int_1^{1+(2/n)} (m-1) m^{-2} dm \\ &\leq c x^{-1/2} n^{-2}. \end{aligned}$$

Finally

$$III = 2\lambda(1 + 2\lambda)^{-1} \leq 2\lambda$$

since  $L(T_1, z)$  is a diffusion on natural scale. Thus for any  $n$  and any  $x > 0$

$$P(\inf_{1 \leq r \leq 2} (L^*(T_r) - r) < \lambda) = n \exp(-n\lambda^2/8x) + c x^{-1/2} n^{-1} + 2n\lambda.$$

Now given any  $\varepsilon > 0$  if we set  $x = \lambda^{\frac{6}{5} + 4\varepsilon}$  and  $n = \lambda^{-(\frac{4}{5} + \varepsilon)}$  the result follows.  $\square$

**Proposition 3.3.** For all  $\beta > 5$ ,

$$\liminf_{r \rightarrow \infty} \frac{L^*(T_r) - r}{r(\log r)^{-\beta}} = \infty \quad \text{a.s.}$$

*Proof.* Fix  $\beta > 5$  and let  $\lambda(r) = r(\log r)^{-\beta}$ . Then by Lemmas 2.4 and 3.2 for any  $\varepsilon > 0$ , if  $n$  is sufficiently large,

$$\begin{aligned} &P(L^*(T_r) - r < \lambda(r) \text{ for some } r \in [2^n, 2^{n+1})) \\ &\leq P(\inf_{1 \leq r \leq 2} (L^*(T_r) - r) < \lambda(2^{n+1}) 2^{-n}) \\ &\leq c n^{-\beta(\frac{1}{5} - \varepsilon)}. \end{aligned}$$

The result now follows from the Borel-Cantelli Lemma.  $\square$

We will now obtain estimates on the growth of  $I(T_r, h(r))$  where  $I(t, K)$  is defined by (1.1).

**Lemma 3.4.** There exists a constant  $c$  such that for all  $h$  and  $\lambda$  sufficiently small,

$$P(\sup_{1 \leq r \leq 2} \sup_{0 \leq z \leq h} (L(T_r, z) - r) > \lambda) \leq c \lambda h^{-1} \exp(-\lambda^2/32h).$$

*Proof.* Set  $Y(r) = \sup_{0 \leq z \leq h} (L(T_r, z) - r)$ . By Doob's inequality and Lemma 2.2(b), if  $u < (2h)^{-1}$

$$\begin{aligned}
 P(Y(r) > x) &\leq e^{-ux} E \exp(u(L(T_r, h) - r)) \\
 &= \exp(-ux - ur + ur(1 - 2uh)^{-1}).
 \end{aligned}$$

Setting  $u = x(4rh + 2xh)^{-1} < (2h)^{-1}$  gives

$$P(Y(r) > x) \leq \exp(-x^2(8rh + 4xh)^{-1}). \tag{3.1}$$

From this it follows immediately that  $E|Y(r)| < \infty$ , and consequently  $Y(r)$  is a submartingale, being the supremum of a collection of martingales. Thus by Doob's inequality and (3.1), if  $u < 1/h$

$$\begin{aligned}
 P(\sup_{1 \leq r \leq 2} Y(r) > \lambda) &\leq e^{-u\lambda} E \exp(uY(2)) \\
 &\leq e^{-u\lambda} \int_0^\infty u \exp(ux - x^2(16h + 4xh)^{-1}) dx.
 \end{aligned}$$

By splitting up the range of integration at  $x=4$  one can easily show that the integral is bounded above by

$$u(4 \exp(8u^2h) + 8h(1 - 8uh)^{-1} \exp(-4(1 - 8uh)(8h)^{-1}))$$

provided  $u < (8h)^{-1}$ . Thus if  $h$  is sufficiently small, then

$$P(\sup_{1 \leq r \leq 2} Y(r) > \lambda) \leq cu \exp(-u\lambda + 8u^2h).$$

The result now follows by setting  $u = \lambda/16h$ .  $\square$

**Proposition 3.5.** Assume that  $\lambda(r)$  and  $h(r)$  are increasing functions which satisfy

- i(i)  $\lambda(r)/r \rightarrow 0, h(r)/r \rightarrow 0$  as  $r \rightarrow \infty$ ,
- (ii) for some  $\gamma, \lambda(r)/h(2r)(\log r)^\gamma$  and  $rh(2r) \log \log r / \lambda^2(r)$  remain bounded as  $r \rightarrow \infty$ .

Then

$$\limsup_{r \rightarrow \infty} \frac{I(T_r, h(r)) - r}{\lambda(r)} < \infty \quad \text{a.s.}$$

*Proof.* By Lemmas 2.2 and 3.4, for any  $a > 0$

$$\begin{aligned}
 &P(I(T_r, h(r)) - r > a\lambda(r) \text{ for some } r \in [2^n, 2^{n+1})) \\
 &\leq P(I(T_r, h(2^{n+1})2^{-n}) - r > a\lambda(2^n)2^{-n} \text{ for some } r \in [1, 2)) \\
 &\leq c \frac{\lambda(2^n)}{h(2^{n+1})} \exp\left(-\frac{a^2 \lambda^2(2^n)}{32h(2^{n+1})2^n}\right).
 \end{aligned}$$

Since this gives rise to a convergent series if  $a$  is chosen large enough, the result follows from the Borel-Cantelli Lemma.  $\square$

We now put together Proposition 3.3 and 3.5 to prove the transience of  $V(t)$  and also to obtain estimates on its rate of escape.

**Theorem 3.6.** (a) For all  $\gamma > 10$

$$\liminf_{r \rightarrow \infty} \frac{V(T_r)}{r(\log r)^{-\gamma}} > 0 \quad \text{a.s.}$$

(b) For all  $\gamma > 11$

$$\liminf_{t \rightarrow \infty} \frac{V(t)}{t^{1/2}(\log t)^{-\gamma}} > 0 \quad \text{a.s.}$$

*Proof.* Fix  $\alpha > 5$ ,  $\beta \in (5, \alpha)$  and set  $h(r) = r(\log r)^{-2\alpha}$ . By Proposition 3.5 with  $\lambda(r) = r(\log \log r)^{1/2}(\log r)^{-\alpha}$ , for  $r$  sufficiently large (depending on  $\omega$ )

$$I(T_r, h(r)) \leq r + cr(\log \log r)^{1/2}(\log r)^{-\alpha}$$

while by Proposition 3.3 for  $r$  sufficiently large

$$L^*(T_r) \geq r + r(\log r)^{-\beta}.$$

Thus for all  $\alpha > 5$ ,  $\{L(T_r, z): z \geq 0\}$  takes its maximum on  $[h(r), \infty)$  if  $r$  is sufficiently large and so

$$\liminf_{r \rightarrow \infty} \frac{V(T_r)}{r(\log r)^{-2\alpha}} > 0.$$

Next, since  $L^*(t)$  is continuous and  $T_s \uparrow T_{r-}$  as  $s \uparrow r$ , we see that for  $r$  sufficiently large

$$L^*(T_{r-}) \geq r + r(\log r)^{-\beta}.$$

Now fix  $t$ ; then for some  $r$ ,  $T_{r-} \leq t \leq T_r$ . Thus if  $t$  is sufficiently large

$$L^*(t) \geq L^*(T_{r-}) \geq r + r(\log r)^{-\beta} \tag{3.2}$$

while for any increasing function  $k(t)$ ,

$$I(t, k(t)) \leq I(T_r, k(T_r)) \leq r + r(\log \log r)^{1/2}(\log r)^{-\alpha} \tag{3.3}$$

provided  $k(T_r) \leq h(r)$ . However by [6], for any  $\varepsilon > 0$ ,  $T_r/r^2(\log r)^{2+\varepsilon} \rightarrow 0$  as  $r \rightarrow \infty$ . Thus if we choose  $k(t) = t^{1/2}(\log t)^{-(2\alpha+1+\varepsilon)}$ , we have that for all  $t$  sufficiently large

$$L^*(t) > I(t, k(t)),$$

and so  $V(t) \geq k(t)$  by the argument given above. Since  $\varepsilon > 0$  was arbitrary, this completes the proof of (b).  $\square$

*Remark.* As pointed out earlier, it follows easily from this result that

$$\liminf_{t \rightarrow \infty} \frac{\bar{V}(t)}{t^{1/2}(\log t)^{-\gamma}} > 0$$

for all  $\gamma > 11$  where

$$\bar{V}(t) = \inf \{x \geq 0: L(t, x) \vee L(t, -x) = \sup_{x \in \mathbb{R}^1} L(t, x)\}, \tag{3.4}$$

i.e.,  $\bar{V}(t)$  is the absolute value of the most visited site on the entire  $x$ -axis.

In view of part (b) of the preceding theorem, it is natural to ask whether  $t^{1/2}(\log t)^{-\gamma}$  is a good approximation to the lower growth rate of  $V(t)$ . The following theorem shows that, up to a power of  $\log t$ , it is. Since it will be needed in the proof, we point out here that if  $Y(t)$  is a  $(\text{BES}(0))^2$  then from (2.1) and (2.2)

$$\liminf_{t \rightarrow 0} P^1(Y(t) < 1 - g(t)) > 0 \tag{3.5}$$

whenever  $g(t) = o(t^{1/2})$ .

**Theorem 3.7.** For all  $\gamma < 2$

$$\liminf_{t \rightarrow \infty} \frac{V(t)}{t^{1/2}(\log t)^{-\gamma}} < \infty \quad \text{a.s.}$$

*Proof.* Let  $p > 1$  and set  $r_n = \exp(n^p)$ . Given  $\gamma < 2$ , choose  $\beta \in (\gamma, 2)$  and let  $h(r) = r(\log r)^{-\beta}$ . Let  $h_n = h(r_n)$  and define

$$\hat{I}_n = \sup_{0 \leq z \leq h_{n+1}} (L(T_{r_{n+1}}, z) - L(T_{r_n}, z))$$

and

$$\hat{J}_n = \sup_{z > h_{n+1}} (L(T_{r_{n+1}}, z) - L(T_{r_n}, z)).$$

Recall that if  $Y(t)$  is a  $(\text{BES}(0))_2^2$ , then  $Y(t)$  is on natural scale, thus if  $\delta < 1$  and  $x \leq (1 - 3n^{-\delta})$

$$P^x(\sup_{t \geq 0} Y(t) < (1 - 2n^{-\delta})) = 1 - x(1 - 2n^{-\delta})^{-1} \geq n^{-\delta}.$$

Hence by the Markov property, Lemma 2.4 and (3.5)

$$\begin{aligned} P(\hat{J}_n < (1 - 2n^{-\delta})\hat{I}_n) &\geq P(\hat{J}_n < (1 - 2n^{-\delta})(r_{n+1} - r_n)) \\ &= P^{r_{n+1} - r_n}(\sup_{t \geq h_{n+1}} Y(t) < (1 - 2n^{-\delta})(r_{n+1} - r_n)) \\ &= P^1(\sup_{t \geq h_{n+1}(r_{n+1} - r_n)^{-1}} Y(t) < (1 - 2n^{-\delta})) \\ &\geq \int_{x=0}^{1 - 3n^{-\delta}} P^x(\sup_{t \geq 0} Y(t) < (1 - 2n^{-\delta})) \\ &\quad P^1(Y(h_{n+1}(r_{n+1} - r_n)^{-1}) \in dx) \\ &\geq n^{-\delta} P^1(Y(h_{n+1}(r_{n+1} - r_n)^{-1}) \leq 1 - 3n^{-\delta}) \\ &\geq cn^{-\delta} \end{aligned} \tag{3.6}$$

for  $n$  large provided  $\delta$  and  $p$  are chosen so that  $p\beta < 2\delta$ . Since the events  $(\hat{J}_n < (1 - 2n^{-\delta})\hat{I}_n)$ ,  $n = 1, 2, \dots$  are independent, the Borel-Cantelli Lemma gives

$$P(\hat{J}_n < (1 - 2n^{-\delta})\hat{I}_n \text{ i.o.}) = 1. \tag{3.7}$$

Next

$$\begin{aligned} P(L^*(T_{r_n}) > n^{-\delta}\hat{I}_n) &\leq P(L^*(T_{r_n}) > n^{-\delta}(r_{n+1} - r_n)) \\ &= \frac{r_n}{n^{-\delta}(r_{n+1} - r_n)}. \end{aligned}$$

Thus by Borel-Cantelli, with probability one

$$L^*(T_{r_n}) \leq n^{-\delta}\hat{I}_n \tag{3.8}$$

for  $n$  sufficiently large. Now set  $J(t, z) = \sup_{x > z} L(t, x)$ . Then by (3.7) and (3.8), infinitely often with probability one

$$\begin{aligned} J(T_{r_{n+1}}, h_{n+1}) &\leq \hat{J}_n + L^*(T_{r_n}) \\ &\leq (1 - n^{-\delta})\hat{I}_n \\ &\leq (1 - n^{-\delta})I(T_{r_{n+1}}, h_{n+1}). \end{aligned} \tag{3.9}$$

Thus by (3.9) and Theorem 1 of [10], infinitely often with probability one

$$\begin{aligned}
 V(T_{r_{n+1}}) &\leq h(r_{n+1}) \\
 &= h(L(T_{r_{n+1}}, 0)) \\
 &\leq h((3 T_{r_{n+1}} \log \log T_{r_{n+1}})^{1/2}) \\
 &\leq T_{r_{n+1}}^{1/2} (\log T_{r_{n+1}})^{-\gamma}
 \end{aligned}
 \tag{3.10}$$

which completes the proof.  $\square$

*Remark.* If

$$\hat{J}_n^- = \sup_{z < -h_{n+1}} (L(T_{r_{n+1}}, z) - L(T_{r_{n+1}}, z))$$

then following the proof of (3.6) we obtain

$$P(\hat{J}_n^- < (1 - 2n^{-\delta}) \hat{I}_n) \geq P(\hat{J}_n^- < (1 - 2n^{-\delta})(r_{n+1} - r_n)) \geq cn^{-\delta}.$$

Now  $\hat{J}_n$  depends only on the positive excursions and  $\hat{J}_n^-$  only on the negative excursions of a Brownian motion, so they are independent. Thus if  $\beta < 1$  we can choose  $\delta < \frac{1}{2}$  and  $p > 1$  to satisfy  $\beta p < 2\delta$ , and then

$$\begin{aligned}
 &\sum P(\hat{J}_n \vee \hat{J}_n^- < (1 - 2n^{-\delta}) \hat{I}_n) \\
 &\geq \sum P(\hat{J}_n \vee \hat{J}_n^- < (1 - 2n^{-\delta})(r_{n+1} - r_n)) \\
 &\geq \sum cn^{-2\delta}.
 \end{aligned}$$

Imitating the proof above we obtain

$$\liminf_{t \rightarrow \infty} \frac{\bar{V}(t)}{t^{1/2} (\log t)^{-\gamma}} < \infty \quad \text{a.s.}$$

for all  $\gamma < 1$ , where  $\bar{V}(t)$  is defined by (3.4).

#### 4. Law of the Iterated Logarithm

We will now investigate the upper growth rate of  $V(t)$ . In this case it is easier to deal directly with  $\bar{V}(t)$  as defined by (3.4). Thus correspondingly we introduce  $\bar{L}$ , the maximum value of  $L(t, x)$  on the entire axis, defined by

$$\bar{L}(t) = \sup_{x \in \mathbb{R}^1} L(t, x).$$

Observe that by Lemma 2.4, the analogue of (2.6) also holds for  $\bar{L}$ , i.e.

$$\bar{L}(t) \stackrel{d}{=} t^{1/2} \bar{L}(1).
 \tag{4.1}$$

The following notation will also prove useful: for the Brownian motion  $W(t)$  and any times  $0 \leq u \leq s$ , set

$$\begin{aligned}
 (W(s) - W(u))^* &= \sup \{|W(t) - W(u)| : u \leq t \leq s\} \\
 W^*(s) &= \sup \{|W(t)| : 0 \leq t \leq s\}.
 \end{aligned}$$

The proof of the L.I.L. for  $\bar{V}(t)$  is based on the intuitive idea given in the introduction and is similar to the proof of Theorem 1 in [7]. We will need the

following generalization of the Borel-Cantelli Lemma. It has been noted previously by Kesten [10] and follows from the proof of Corollary 1 on page 323 in Doob [5].

**Lemma 4.1.** *Let  $\mathcal{F}(s) = \sigma\{W(t): 0 \leq t \leq s\}$  and for  $k=0, 1, \dots$  assume that  $C_k \in \sigma\{W(t): s_k \leq t \leq s_{k+1}\}$  where  $0 \leq s_0 < s_1 < \dots$  and  $s_k \uparrow \infty$ . Then*

$$\sum_k P(C_k | \mathcal{F}(s_k)) = \infty \quad \text{a.s. implies } P(C_k \text{ i.o.}) = 1.$$

**Theorem 4.2.**

$$\limsup_{t \rightarrow \infty} \frac{\bar{V}(t)}{(2t \log \log t)^{1/2}} = 1 \quad \text{a.s.}$$

*Proof.* The upper bound is an immediate consequence of the Law of the Iterated Logarithm for Brownian motion.

To prove the lower bound assume that  $\alpha < 1$ . Now choose  $\delta > 0$  sufficiently small and  $\eta, K$  sufficiently large so that

$$P(\eta \leq W(1) \leq K\eta) \geq \exp(-\eta^2/2\alpha) + 2\delta. \tag{4.2}$$

This is possible since  $P(W(1) > \eta) \sim (2\pi\eta^2)^{-(1/2)} \exp(-\eta^2/2)$ . Next choose  $\lambda$  and  $\rho$  such that

$$P(\bar{L}(1) \leq \lambda) \geq 1 - \delta, \tag{4.3}$$

$$P(W^*(1) \leq \rho) \geq 1 - \delta. \tag{4.4}$$

Now let  $\zeta = 4(2\rho\eta^{-1} + K)\lambda$  and set  $b = P(\bar{L}(1) \geq \zeta)$ . Finally choose  $\bar{\rho}$  such that

$$P(W^*(1) \leq \bar{\rho}) \geq 1 - (b/2). \tag{4.5}$$

Fix  $t > 0$  and let  $t_k = kat / \log \log t$  where  $a = \eta^2/2\alpha^2$ . Define, for  $k = 0, 1, \dots, [(\log \log t)/a]$

$$E_k^1 = (\sup_{x \in \mathbb{R}^1} (L(t_{k+1}, x) - L(t_k, x)) \leq \lambda \gamma(t))$$

$$E_k^2 = ((W(t_{k+1}) - W(t_k))^* \leq \rho \gamma(t))$$

$$E_k^3 = ((k+1)\eta \gamma(t) \leq W(t_{k+1}) \leq (K+k)\eta \gamma(t))$$

where  $\gamma(t) = (at / \log \log t)^{1/2}$ . Now set

$$E_k = E_k^1 \cap E_k^2 \cap E_k^3 \quad \text{for } k=0, \dots, [(\log \log t)/a] - 2,$$

$$E_k = ((W(t_{k+1}) - W(t_k))^* \leq \bar{\rho} \gamma(t);$$

$$\sup_{x \in \mathbb{R}^1} (L(t_{k+1}, x) - L(t_k, x)) \geq \zeta \gamma(t)) \quad \text{for } k = [(\log \log t)/a] - 1,$$

$$E_k = E_k^2 \quad \text{for } k = [(\log \log t)/a].$$

Observe that for any  $x$ , by (4.1) and (4.3)

$$\begin{aligned} P(E_k^1 | W(t_k) = x) &= P(\bar{L}(t_1) \leq \lambda \gamma(t)) \\ &= P(\bar{L}(1) \leq \lambda) \\ &\geq 1 - \delta \end{aligned}$$

while by the scaling property of Brownian motion and (4.4)

$$\begin{aligned} P(E_k^2 | W(t_k) = x) &= P(W^*(t_1) \leq \rho \gamma(t)) \\ &= P(W^*(1) \leq \rho) \\ &\geq 1 - \delta. \end{aligned}$$

Next for  $k\eta\gamma(t) \leq x \leq (K+k-1)\eta\gamma(t)$ ,

$$\begin{aligned} P(E_k^3 | W(t_k) = x) &= P((k+1)\eta\gamma(t) \leq W(t_1) \leq (K+k)\eta\gamma(t) | W(0) = x) \\ &= P((k+1)\eta \leq W(1) \leq (K+k)\eta | W(0) = x/\gamma(t)) \\ &= P(\eta \leq W(1) \leq K\eta | W(0) = y) \end{aligned}$$

where  $y = x/\gamma(t) - k\eta$ . Observe that  $0 \leq y \leq (K-1)\eta$  and thus by the shape of the normal density we conclude that

$$\begin{aligned} P(E_k^3 | W(t_k) = x) &\geq P(\eta \leq W(1) \leq K\eta | W(0) = 0) \\ &\geq \exp(-\eta^2/2\alpha) + 2\delta. \end{aligned}$$

Thus for  $k=0, \dots, [(\log \log t)/a] - 2$

$$P(E_k | W(t_k) = x) \geq \exp(-\eta^2/2\alpha) \tag{4.6}$$

provided  $k\eta\gamma(t) \leq x \leq (K+k-1)\eta\gamma(t)$ , while for  $k = [(\log \log t)/a]$

$$P(E_k | W(t_k) = x) \geq 1 - \delta \tag{4.7}$$

independently of  $x$ . For  $k = [(\log \log t)/a] - 1$ , essentially the same calculations show that

$$P(E_k | W(t_k) = x) \geq b/2, \tag{4.8}$$

again provided that  $k\eta\gamma(t) \leq x \leq (K+k-1)\eta\gamma(t)$ . Let  $D_n = \bigcap_{k=0}^n E_k$  and  $\mathcal{F}(s) = \sigma\{W(r) : r \leq s\}$ . Then by the Markov property, for any  $n$

$$\begin{aligned} P(D_{n+1}) &= EE \left[ \prod_{k=0}^{n+1} 1_{E_k} \middle| \mathcal{F}(t_{n+1}) \right] \\ &= E \left[ \prod_{k=0}^n 1_{E_k} P(E_{n+1} | W(t_{n+1})) \right]. \end{aligned}$$

But  $(n+1)\eta\gamma(t) \leq W(t_{n+1}) \leq (K+n)\eta\gamma(t)$  on  $E_n$  if  $n \leq [(\log \log t)/a] - 3$ , thus by (4.6)

$$P(D_{n+1}) \geq \exp(-\eta^2/2\alpha) P(D_n) \quad \text{if } n \leq [(\log \log t)/a] - 3$$

while by (4.7) and (4.8)

$$\begin{aligned} P(D_{n+1}) &\geq (b/2) P(D_n) \quad \text{if } n = [(\log \log t)/a] - 2, \\ P(D_{n+1}) &\geq (1 - \delta) P(D_n) \quad \text{if } n = [(\log \log t)/a] - 1. \end{aligned}$$

Thus writing  $D = D_{[(\log \log t)/a]}$  we see that for some positive constant  $c$

$$P(D) \geq c \exp(-\eta^2 \log \log t / 2\alpha) = c \exp(-\alpha \log \log t). \tag{4.9}$$

Next observe that on  $D$

$$\bar{L}(t) \geq \zeta \gamma(t)$$

while for any  $k \leq \lceil (\log \log t)/a \rceil - 2$ , if  $s \in [kat/\log \log t, (k+1)at/\log \log t]$  then  $L(s, x)$  can only increase if

$$(k\eta - \rho)\gamma(t) \leq x \leq (K + k - 1 + \rho)\eta\gamma(t). \tag{4.10}$$

Now for a given  $x$ , since there are at most  $(2\rho\eta^{-1} + K)$  values of  $k$  for which  $k \leq \lceil (\log \log t)/a \rceil - 2$  and (4.10) holds, we see that on  $D$

$$L(t, x) \leq (2\rho\eta^{-1} + K)\lambda\gamma(t) = (\zeta/4)\gamma(t)$$

provided  $x \leq (\lceil (\log \log t)/a \rceil - 1)\eta\gamma(t) - (\bar{\rho} + \rho)\gamma(t)$ . (Observe that this restriction on  $x$  ensures that  $L(s, x)$  remains constant for

$$s \in [\lceil (\log \log t)/a \rceil - 1)at(\log \log t)^{-1}, t].$$

Thus for any  $\varepsilon > 0$ , if  $t$  is sufficiently large

$$D \subseteq \{\bar{L}(t) \geq \zeta\gamma(t)\}, \quad \sup_{x \leq \sqrt{2\alpha(1-\varepsilon)}\varphi(t)} L(t, x) \leq (\zeta/4)\gamma(t) \tag{4.11}$$

where  $\varphi(t) = (t \log \log t)^{1/2}$ . (Recall that  $\eta\alpha^{-1/2} = \sqrt{2}\alpha$ .)

We will now complete the proof by using the generalization of the Borel-Cantelli Lemma given in Lemma 4.1. Let  $p = \alpha^{-1}$  and set  $s_k = \exp(k^p)$ . Define

$$C_k = \left( \sup_{x \in \mathbb{R}^1} (L(s_{k+1}, x) - L(s_k, x)) \geq (\zeta/2)\gamma(s_{k+1}), \right. \\ \left. \sup_{x \leq \sqrt{2\alpha(1-\varepsilon)}\varphi(s_{k+1})} (L(s_{k+1}, x) - L(s_k, x)) \leq (\zeta/4)\gamma(s_{k+1}) \right).$$

Observe that if  $|y| \leq (3s_k \log \log s_k)^{1/2}$  then for sufficiently large  $k$ , by (4.9) and (4.11)

$$P(C_k | W(s_k) = y) = P\left(\sup_{x \in \mathbb{R}^1} (L(s_{k+1}, x) - L(s_k, x)) \geq (\zeta/2)\gamma(s_{k+1}), \right. \\ \left. \sup_{x \leq \sqrt{2\alpha(1-\varepsilon)}\varphi(s_{k+1}) - y} (L(s_{k+1}, x) - L(s_k, x)) \leq (\zeta/4)\gamma(s_{k+1}) \mid W(s_k) = 0\right) \\ \geq P(\bar{L}(s_{k+1} - s_k) \geq \zeta\gamma(s_{k+1} - s_k), \sup_{x \leq \sqrt{2\alpha(1-\varepsilon/2)}\varphi(s_{k+1} - s_k)} L(s_{k+1} - s_k, x) \\ \leq (\zeta/4)\gamma(s_{k+1} - s_k) \mid W(0) = 0) \geq ck^{-1}.$$

However by the L.I.L. for Brownian motion,  $|W(s_k)| \leq (3s_k \log \log s_k)^{1/2}$  eventually, and so for some (random)  $k_0$ , if  $k \geq k_0$ , then with probability one

$$P(C_k | \mathcal{F}(s_k)) = P(C_k | W(s_k)) \geq ck^{-1}.$$

Thus by Lemma 4.1,  $P(C_k \text{ i.o.}) = 1$ . Next observe that

$$\bar{L}(s_{k+1}) \geq \sup_{x \in \mathbb{R}^1} (L(s_{k+1}, x) - L(s_k, x))$$

while

$$\sup_{x \leq \sqrt{2\alpha(1-\varepsilon)}\varphi(s_{k+1})} L(s_{k+1}, x) \leq \bar{L}(s_k) + \sup_{x \leq \sqrt{2\alpha(1-\varepsilon)}\varphi(s_{k+1})} (L(s_{k+1}, x) - L(s_k, x)).$$

However by Theorem 1 of [10], since  $\varphi(s_k)/\gamma(s_{k+1}) \rightarrow 0$

$$\limsup_{k \rightarrow \infty} \frac{\bar{L}(s_k)}{\gamma(s_{k+1})} = 0.$$

Thus

$$P(\bar{L}(s_{k+1}) \geq (\zeta/2) \gamma(s_{k+1}), \sup_{x \leq \sqrt{2\alpha(1-\varepsilon)\varphi(s_{k+1})}} L(s_{k+1}, x) \leq (\zeta/3) \gamma(s_{k+1}) \text{ i.o.}) = 1,$$

i.e.,

$$\limsup_{k \rightarrow \infty} \frac{\bar{V}(s_{k+1})}{(s_{k+1} \log \log s_{k+1})^{1/2}} \geq \sqrt{2} \alpha(1-\varepsilon).$$

Since  $\varepsilon > 0$  and  $\alpha < 1$  were arbitrary, the proof is complete.  $\square$

*Remarks.* 1. A minor modification of the proof shows that for any  $K > 0$  and any  $\alpha < \sqrt{2}$

$$P(\bar{L}(t) > K \sup_{x \leq \alpha(t \log \log t)^{1/2}} L(t, x) \text{ i.o. as } t \uparrow \infty) = 1.$$

2. It would be interesting to obtain an analogue of Kolmogorov's test for  $\bar{V}(t)$ . In particular, do there exist deterministic functions  $\varphi(t)$  for which  $P(W(t) > \varphi(t) \text{ i.o.}) = 1$  and  $P(\bar{V}(t) > \varphi(t) \text{ i.o.}) = 0$ ? The answer to the analogous question for simple random walk is yes by Theorem 2 of Erdős and Révész [20].

3. It is an immediate consequence of the above theorem that  $\limsup_{t \rightarrow \infty} V(t)/(2t \log \log t)^{1/2} = 1$  a.s.

### 5. Random Walks

In this section,  $X_1, X_2, \dots$ , is a sequence of independent identically distributed random variables with  $P(X_i = 1) = P(X_i = -1) = 1/2$  and  $S_n = X_1 + \dots + X_n$ . Throughout this section  $k$  will always be restricted to the integers. Let

$$\begin{aligned} N(n, k) &= \sum_{j=0}^n 1_{\{k\}}(S_j), \\ N^*(n) &= \sup_{k \geq 0} N(n, k), \\ U(n) &= \inf \{k \geq 0: N(n, k) = N^*(n)\}. \end{aligned}$$

To prove the analogue of Theorems 3.6 and 3.7 for random walks, the basic tool is the invariance principle of Révész [15].

**Theorem 5.1.** *There exists a Brownian motion  $W(t)$  and a simple random walk  $S_n$  such that for all  $\varepsilon > 0$*

$$\sup_k |L(n, k) - N(n, k)| = o(n^{1/4+\varepsilon}) \text{ a.s.}$$

We begin with some estimates on Brownian local time. Let

$$\begin{aligned} D(t, x, y) &= L(t, x) - L(t, y), \\ M(t, x, y) &= \sup_{s \leq t} |D(s, x, y)|. \end{aligned}$$

**Lemma 5.2.** *There exist constants  $c_1$  and  $c_2$  such that for any integer  $k$ ,  $\delta \leq 1$  and  $t \geq 1$*

$$P\left(\sup_{\substack{|x-y| \leq \delta \\ x, y \in [k, k+1]}} M(t, x, y) > \lambda\right) \leq c_1 \delta^{-1} \exp(-\lambda/c_2 \delta^{1/2} t^{1/4}).$$

*Proof.* By Tanaka's formula, for any bounded stopping time  $T$ ,

$$L(T, x) = |W(T) - x| - |x| - \int_0^T \text{sgn}(W(s) - x) dW(s).$$

Now assume that  $x \leq y \leq x + 1$  and  $R \leq T$  are stopping times bounded by  $t$ . Then

$$\begin{aligned} E(|D(T, x, y) - D(R, x, y)| | \mathcal{F}(R)) &\leq (E(|D(T, x, y) - D(R, x, y)|^2 | \mathcal{F}(R)))^{1/2} \\ &\leq 2|x - y| + 2 \left( E \left( \left( \int_R^T 1_{[x, y]}(W(s)) dW(s) \right)^2 | \mathcal{F}(R) \right) \right)^{1/2} \\ &\leq 2|x - y| + 2 \left( E \left( \int_R^{R+t} 1_{[x, y]}(W(s)) ds | \mathcal{F}(R) \right) \right)^{1/2} \\ &= 2|x - y| + 2 \left( E^{W(R)} \int_0^t 1_{[x, y]}(W(s)) ds \right)^{1/2} \\ &= 2|x - y| + 2 \left( \int_0^t P^{W(R)}(W(s) \in [x, y]) ds \right)^{1/2} \\ &\leq ct^{1/4} |x - y|^{1/2}, \end{aligned}$$

provided  $t > 1$ . So by [3], p. 193,

$$E \exp(M(T, x, y)) / 8ct^{1/4} |x - y|^{1/2} \leq 2,$$

and consequently

$$P(M(t, x, y) > \lambda) \leq 2 \exp(-\lambda/8ct^{1/4} |x - y|^{1/2}).$$

Now fix  $k, \lambda, \delta$  and let  $\lambda_i = (1 - 2^{-1/4}) 2^{-i/4} \lambda$ , so that  $\sum_{i=0}^{\infty} \lambda_i = \lambda$ . Let

$$Q(\delta) = \{x \in \mathbb{R} : x = k + j\delta 2^{-i} \text{ for some } 0 \leq i < \infty, 0 \leq j \leq 2^i [\delta^{-1}] + 1\}.$$

Given  $x \in [k, k + 1]$  define  $x_i = \sup \{k + j\delta 2^{-i} \leq x : 0 \leq j \leq 2^i [\delta^{-1}] + 1\}$ . Now since  $L(t, x)$  is jointly continuous,

$$\sup_{\substack{|x-y| \leq \delta \\ x, y \in [k, k+1]}} M(t, x, y) > \lambda \quad \text{only if} \quad \sup_{\substack{|x-y| \leq \delta \\ x, y \in Q(\delta)}} M(t, x, y) > \lambda,$$

while for  $x, y \in Q(\delta)$  with  $|x - y| \leq \delta$

$$M(t, x, y) \leq M(t, x_0, y_0) + \sum_{i=1}^{\infty} (M(t, x_{i-1}, x_i) + M(t, y_{i-1}, y_i)).$$

(Observe that eventually each term in the infinite sum will be zero.) Thus if  $M(t, x, y) > \lambda$  then either  $M(t, x_0, y_0) > \lambda_0$  or for some  $i$ ,  $M(t, x_{i-1}, x_i)$

$\vee M(t, y_{i-1}, y_i) > \lambda_i/2$ . Hence for  $\lambda$  sufficiently large

$$\begin{aligned} &P\left(\sup_{\substack{|x-y| \leq \delta \\ x, y \in [k, k+1]}} M(t, x, y) > \lambda\right) \\ &\leq \sum_{i=0}^{\infty} 2(2^i[\delta^{-1}] + 1) \sup_{0 \leq j \leq 2^i[\delta^{-1}]} P(M(t, k + j\delta 2^{-i}, k + (j+1)\delta 2^{-i}) > \lambda_i/2) \\ &\leq c_1 \delta^{-1} \sum_{i=0}^{\infty} 2^{i+1} \exp(-\lambda_i/16c_2 t^{1/4} (\delta 2^{-i})^{1/2}) \\ &\leq c_1 \delta^{-1} \exp(-\lambda/c_2 t^{1/4} \delta^{1/2}). \quad \square \end{aligned}$$

**Lemma 5.3.** For every  $\varepsilon > 0$

$$\sup_k \sup_{\substack{s \leq t \\ z \in [k, k+1]}} |L(s, z) - L(s, k)| = o(t^{1/4+\varepsilon}) \quad \text{a.s.}$$

*Proof.* Let  $t_n = 2^n$ , then since  $L(t, z) = 0$  for  $z \geq \sup_{s \leq t} |W(s)|$ ,

$$\begin{aligned} &P\left(\sup_k \sup_{\substack{s \leq t_n \\ z \in [k, k+1]}} |L(s, z) - L(s, k)| \geq t_n^{1/4+\varepsilon/2}\right) \\ &\leq P(\sup_{s \leq t_n} |W(s)| \geq t_n) + 2^{n+2} \sup_{|k| \leq t_n} P\left(\sup_{\substack{s \leq t_n \\ z \in [k, k+1]}} |L(s, z) - L(s, k)| > t_n^{1/4+\varepsilon/2}\right) \\ &\leq 2 \exp(-t_n/2) + 2^{n+2} c_1 \exp(-t_n^{1/4+\varepsilon/2}/c_2 t_n^{1/4}). \end{aligned}$$

The result now follows from the Borel-Cantelli Lemma.  $\square$

**Theorem 5.4.** (a) For any  $\gamma > 11$

$$\liminf_{n \rightarrow \infty} \frac{U(n)}{n^{1/2}(\log n)^{-\gamma}} = \infty \quad \text{a.s.}$$

(b) For any  $\gamma < 2$

$$\liminf_{n \rightarrow \infty} \frac{U(n)}{n^{1/2}(\log n)^{-\gamma}} = 0 \quad \text{a.s.}$$

*Proof.* (a) From (3.2) and (3.3) we see that if  $r$  is sufficiently large,  $T_{r-} \leq t \leq T_r$  and  $\varepsilon > 0$ , then

$$L^*(t) \geq I(t, t^{1/2}(\log t)^{-(2\alpha+1+\varepsilon)}) + r(\log r)^{-\beta}$$

where  $5 < \beta < \alpha$ . Furthermore for each  $\eta > 0$ ,  $T_r \leq r^2(\log r)^{2+\eta}$  for large  $r$ , [6]; thus

$$L^*(t) \geq I(t, t^{1/2}(\log t)^{-(2\alpha+1+\varepsilon)}) + t^{1/2-\varepsilon} \tag{5.1}$$

eventually for any  $\varepsilon > 0$ . Now by Theorem 5.1 and Lemma 5.3

$$|L^*(n) - N^*(n)| = o(n^{1/4+\varepsilon}) \quad \text{a.s.,}$$

and

$$|I(n, n^{1/2}(\log n)^{-(2\alpha+1+\varepsilon)}) - \sup_{0 \leq k \leq n^{1/2}(\log n)^{-(2\alpha+1+\varepsilon)}} N(n, k)| = o(n^{1/4+\varepsilon}) \quad \text{a.s.}$$

Combining this with (5.1) gives (a).

(b) Recalling the notation of Theorem 3.7, we proved in (3.9) that infinitely often with probability one,

$$J(T_{r_{n+1}}, h_{n+1}) \leq (1 - n^{-\delta}) I(T_{r_{n+1}}, h_{n+1}).$$

Thus if  $j \leq T_{r_{n+1}} \leq j + 1$ ,

$$\begin{aligned} \sup_{0 \leq k \leq h_{n+1}} N(j+1, k) &\geq I(j+1, h_{n+1}) - o(j^{1/4+\varepsilon}) \\ &\geq I(T_{r_{n+1}}, h_{n+1}) - o(j^{1/4+\varepsilon}) \\ &\geq J(T_{r_{n+1}}, h_{n+1}) + n^{-\delta} I(T_{r_{n+1}}, h_{n+1}) - o(j^{1/4+\varepsilon}) \\ &\geq \sup_{k \geq h_{n+1}} N(j, k) + n^{-\delta} I(T_{r_{n+1}}, h_{n+1}) - o(j^{1/4+\varepsilon}). \end{aligned}$$

Now since

$$I(T_{r_{n+1}}, h_n) \geq r_{n+1},$$

and  $T_{r_{n+1}}/r_{n+1}^{2+\eta} \rightarrow 0$  for all  $\eta > 0$ , [6], we see that

$$j^{1/4+\varepsilon} = o(n^{-\delta} I(T_{r_{n+1}}, h_n)), \quad \text{a.s.}$$

if  $\varepsilon$  is sufficiently small. Thus for infinitely many  $j$ , with probability one

$$\sup_{0 \leq k \leq h_{n+1}} N(j+1, k) \geq \sup_{k \geq h_{n+1}} N(j, k) + \zeta(j)$$

where  $\zeta(j) \rightarrow \infty$  a.s. Since for each  $k$ ,  $N(j+1, k)$  and  $N(j, k)$  differ by at most 1, we have by the same argument as in (3.10) that infinitely often with probability one

$$\begin{aligned} U(j+1) &\leq h_{n+1} \\ &\leq T_{r_{n+1}}^{1/2} (\log T_{r_{n+1}})^{-\gamma} \\ &\leq (j+1)^{1/2} (\log(j+1))^{-\gamma}. \quad \square \end{aligned}$$

**Remarks.** 1. There is a two sided version of Theorem 5.4 which is proved similarly (see the remarks following Theorems 3.6 and 3.7).

2. Theorem 5.4 holds if  $S_n$  is a lattice-valued random walk with lattice size 1, where  $X_i$  is mean 0 and variance 1 with finite moments of all orders. This follows by replacing Theorem 5.1 with the results of Csáki and Révész [2].

3. There is an obvious random walk analogue of Theorem 4.2. We leave the statement and proof to the reader. See also Theorem 1 of [20].

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