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# JOINT CONTINUITY AND REPRESENTATIONS OF ADDITIVE FUNCTIONALS OF $d$-DIMENSIONAL BROWNIAN MOTION

Richard BASS

*Department of Mathematics, University of Washington, Seattle, WA 98195, U.S.A.*

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Conditions are given on a family of measures $\{\mu_a, 0 \leq a \leq 1\}$ so that the corresponding family $\{A_t^a, 0 \leq a \leq 1\}$ of additive functionals of $d$-dimensional Brownian motion will be jointly continuous in $a$ and $t$, a.s. This is then used to give a $d$-dimensional analogue to the representation $A_t = \int L_t^y \mu(\mathrm{d}y)$ that is valid for one-dimensional Brownian motion, where $L_t^y$ is local time at $y$. In place of local times at points, local times at hyperplanes are used.

additive functionals * Radon transform * local times * potentials * stochastic integrals

## 1. Introduction

It is well known that there is a one-to-one correspondence between additive functionals of $d$-dimensional Brownian motion and certain measures [8]. In this paper, we show that if a family of measures $\{\mu_a, 0 \leq a \leq 1\}$ satisfies certain conditions, then the family of corresponding additive functionals $\{A_t^a, 0 \leq a \leq 1\}$ will be jointly continuous in $a$ and $t$, a.s. We then use this result to give a $d$-dimensional analogue to the theorem of Ito and McKean [7] that states that any additive functional $A_t$ of one-dimensional Brownian motion can be represented as

$$A_t = \int L_t^y \mu(\mathrm{d}y) \tag{1.1}$$

where $L_t^y$ is the local time at $y$ for the one dimensional Brownian motion and $\mu$ is the measure corresponding to $A_t$. Of course, local times at points do not exist for dimensions greater than one; in their place we use local times at hyperplanes. The conditions required on the $\mu_a$ so that the $A_t^a$ will be jointly continuous, a.s., are that the $\mu_a$ satisfy a mild boundedness condition (3.6(i)) and that the $\mu_a$ be Hölder continuous in $a$ with respect to a certain metric for the weak topology on measures

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on $\mathbb{R}^d$: for some constants $k$ and $\eta>0$,

$$\sup_{\|h\|+\|\text{grad } h\|\le 1} \left| \int h(y)\mu_a(\mathrm{d}y) - \int h(y)\mu_b(\mathrm{d}y)) \right| \le K|b-a|^\eta$$

where $\|\cdot\|$ denotes sup norm. See (3.6).

To state the analogue of (1.1), we first need some notation. Let $W_t$ $(=W(t)=(W_t^1,\ldots,W_t^d))$ be a standard $d$-dimensional Brownian motion. If $\cdot$ denotes inner product, let $L_t(s,v)$ $(=L(t,s,v))$ denote local time of $W_t\cdot v$ at $s$. $L_t(s,v)$ may be thought of as local time for $W_t$ at the hyperplane $\{y: y\cdot v=s\}$. Let $A_t$ be an additive functional corresponding to a finite measure $\mu$ in the sense of McKean and Tanaka [8]. (This correspondence will be described in more detail in a moment.) Let $\mathcal{B}=\{v: |v|=1\}$, and let

$$A_t^b = \int\int_{\mathcal{B}}\int_{-\infty}^{\infty} I_b(s-y\cdot v)L_t(s,v)\,\mathrm{d}s\,\mathrm{d}v\,\mu(\mathrm{d}y), \tag{1.2}$$

where $I_b$ is a known deterministic function given by (4.2) that depends only on $b$ and the dimension $d$.

We prove that for each $u$,

$$\lim_{b\to 0}\sup_{t\le u}|A_t-A_t^b|=0 \quad \text{a.s.} \tag{1.3}$$

provided $\mu$ satisfies a very mild regularity condition (3.8). If $\mu$ is smooth enough, that is, if $\mu$ has a density that is sufficiently differentiable, we can give a representation of $A_t$ in terms of an integral of $L_t(s,v)$ that involves no limit whatsoever (see (4.6)).

To describe the correspondence between additive functionals and measures, we need a bit more notation. Let $S(r,x)=\{y: |y-x|<r\}$, $T_m=\inf\{t: |W_t|\ge m\}$, $u_m(x,y)$ the Green function of $d$-dimensional Brownian motion on $S(m,0)$ [2], and $U_m\mu(x)=\int u_m(x,y)\mu(\mathrm{d}y)$. Suppose $U_m\mu(x)<\infty$ for all $x$ and $m$. Then the additive functional associated with $\mu$ is defined to be the additive functional $A_t$ $(=A(t))$ such that $E^xA(T_m)=U_m\mu(x)$ for all $x$, for all $m$. Such a correspondence between additive functionals and measures exists for many processes besides Brownian motion (see [1]), but in the case of Brownian motion, the results of [3] and [11] make the correspondence much more explicit:

$$\begin{aligned} A(t\wedge T_m) &= -U_m\mu(W(t\wedge T_m))+U_m\mu(W(0)) \\ &\quad+\int_0^{t\wedge T_m}\text{grad}(U_m\mu)(W_s)\,\mathrm{d}W_s, \end{aligned} \tag{1.4}$$

where $\text{grad}(U_m\mu)$ is suitably interpreted.

In Section 2, we use the stochastic calculus to prove the joint continuity of $L(t,s,v)$ in $t$, $s$ and $v$. In particular, this guarantees that the integral in (1.2) makes sense. Our results in Section 2 have been extended by Yor [12] to a more general context. In Section 3 we investigate the joint continuity in $a$ and $t$ of a family of additive functionals $\{A_t^a, 0 \leq a \leq 1\}$. A subsequent paper [13] extends the results of Section 3 to other Markov processes. Finally, in Section 4 we use the methods of the Radon transform to derive our representation of $A_t$ as an integral of local times.

As a matter of notation, $c$ will denote constants, whose values, unless indicated otherwise, are not necessarily the same from one appearance to the next.

## 2. Joint continuity of local times

In this section we prove the joint continuity of local times at hyperplanes, as well as two propositions that will be needed later.

First, we prove the following, a variation of [9, VII, T59].

**Proposition 2.1.** *Let $A_t$ be a continuous additive functional of a Markov process $X(t)$ and suppose $E^y A_t \leq k$ for all $y$. Then $E^x A_t^n \leq n! k^n$.*

**Proof.** $A_t^n$ is equal to $n\int_0^t A_s^{n-1}\, dA_s$, to $\int_0^t A_t\, dA_s^{n-1}$, and to $\int_0^t A_s^{n-1}\, dA_s + \int_0^t A_s\, dA_s^{n-1}$, the first two by direct integration, the last by integration by parts. Therefore,

$$A_t^n = n\int_0^t (A_t - A_s)\, dA_s^{n-1},$$

and

$$\begin{aligned} E^x A_t^n &= nE^x \int_0^t E^{X(s)} A_{t-s}\, dA_s^{n-1} \\ &\leq nE^x \int_0^t E^{X(s)} A_t\, dA_s^{n-1} \leq nkE^x \int_0^t dA_s^{n-1}. \end{aligned}$$

The result follows by induction. □

The following is a generalization of a well-known theorem of Kolmogorov and is proved the same way.

**Proposition 2.2.** *Let $\{Y(t, a), 0 \leq t \leq u, a \in [0, 1]^n\}$ be a real-valued multiparameter process, $P^x$ a family of probabilities. If there exist real numbers $e(x)$, $c(x)$, $\gamma(x) > 0$, depending only on $x$, such that*

$$E^x[\sup_{t \leq u} |Y(t, a) - Y(t, b)|^{e(x)}] \leq c(x)|a - b|^{n+\gamma(x)}$$

*for each pair $a$, $b$, then there exists a subset $N$ of $\Omega$ and a version $Z$ of $Y$ such that $P^x(N) = 0$ for all $x$, and if $\omega \notin N$,*

$$\lim_{\delta \to 0} \sup_{|a-b| < \delta} \sup_{t \leq u} |Z(t, a) - Z(t, b)| = 0.$$

Recall that the way (2.2) is proved is to prove that $Y$ is uniformly continuous on a dense subset of $[0,1]^n$, a.s., and then to define $Z$ as a limit along this dense subset. Thus, the definition of $Z$ can be taken independent of $P^x$.

In the proof of (2.3) below, we apply (2.2) with $a \in [-m, m] \times \mathscr{B}$ instead of $[0,1]^n$ ($\mathscr{B} = \{v : |v| = 1\}$). The necessary modifications are trivial, however.

**Theorem 2.3.** *A version of* $\{L(t,s,v),\ s \in (-\infty, \infty)\ v \in \mathscr{B}\}$ *exists such that*

$$\lim_{\delta \to 0} \sup_{|v-w|<\delta} \sup_{|s-r|<\delta} \sup_{t \le u} |L(t,s,v) - L(t,r,w)| = 0, \quad a.s.$$

**Proof.** First, suppose we have shown the above theorem with $\sup_{t \le u}$ replaced by $\sup_{t \le u \wedge T_m}$, where $T_m = \inf\{t : |W_t| \ge m\}$. Since for almost all $\omega$ we can find an $m$ such that $u < T_m(\omega)$, we would have our theorem. So we will suppose we have a Brownian motion killed on leaving $S(m, 0)$, still denoted $W_t$, $L(t,s,v)$ the local time for the killed process. We will drop the subscript $m$ from $T_m$.

By Tanaka's formula,

$$L(t,s,v) = |W_{t \wedge T} \cdot v - s| - |W_0 \cdot v - s| - \int_0^{t \wedge T} \operatorname{sgn}(W_b \cdot v - s)\, \mathrm{d}(W_b \cdot v). \tag{2.4}$$

Fix $r, s, v, w$. Let $\beta = |s - r| + m|v - w|$. Suppose $|s-r|, |v-w| < \delta$. Then $\beta < (1+m)\delta$.

$$\begin{aligned} |W_{t\wedge T} \cdot v - s| - |W_{t \wedge T} \cdot w - r| &\le |s - r| + |W_{t \wedge T} \cdot (v - w)| \\ &\le |s - r| + |W_{t \wedge T}|\,|v - w| \le \beta. \end{aligned} \tag{2.5}$$

This holds, in particular, when $t = 0$.

On the other hand,

$$\int_0^{t \wedge T} \operatorname{sgn}(W_b \cdot v - s)\, \mathrm{d}(W_b \cdot v) - \int_0^{t \wedge T} \operatorname{sgn}(W_b \cdot w - r)\, \mathrm{d}(W_b \cdot w) = M_t + N_t,$$

where

$$M_t = \int_0^{t \wedge T} [\operatorname{sgn}(W_b \cdot v - s) - \operatorname{sgn}(W_b \cdot w - r)]\, \mathrm{d}(W_b \cdot v),$$

and

$$N_t = \sum_{i=1}^{d} \int_0^{t \wedge T} \operatorname{sgn}(W_b \cdot w - r)(v_i - w_i)\, \mathrm{d}W_b^i,$$

$$v = (v_1, \ldots, v_d), \quad w = (w_1, \ldots, w_d).$$

If $\langle N, N \rangle_t$ is the quadratic variation of $N$,

$$\langle N, N \rangle_t \le \int_0^{t \wedge T} \sum (v_i - w_i)^2\, \mathrm{d}b \le |v - w|^2 t \le t\delta^2.$$

We now estimate $\langle M, M\rangle_t$. Note that if $y \cdot v \geq s$, $y \cdot w \leq r$, $|y| \leq m$, then

$$y \cdot v = y \cdot w + y \cdot (v-w) \leq r + |y||v-w| \leq r + m|v-w|.$$

Similarly, if $y \cdot v \leq s$, $y \cdot w \geq r$, $|y| \leq m$,

$$y \cdot v \geq r - m|v-w|.$$

Thus,

$$|\text{sgn}\,(y \cdot v - s) - \text{sgn}(y \cdot w - r)| \leq 2(1_D(y \cdot v - r)),$$

where $D$ is the interval $[-\beta, \beta]$. We then get

$$\langle M, M\rangle_t \leq \int_0^{t \wedge T} 4(1_D(W_b \cdot v - r))\,\mathrm{d}b,$$

and $E^x\langle M, M\rangle_t \leq c\beta$, $c$ depending only on $t$.

Using the inequalities of Burkholder, Davis and Gundy [4],

$$E^x(\sup_{t \leq u}|M_t + N_t|^e) \leq cE^x(\langle M+N, M+N\rangle_u^{e/2})$$
$$\leq c(E^x\langle M, M\rangle_u^{e/2} + E^x\langle N, N\rangle_u^{e/2}).$$

Use (2.1) with $A_t = \langle M, M\rangle_t$, $X_t$ Brownian motion killed at time $T$, to get that this last expression is $\leq c\beta^{e/2}$. Combine this with (2.4) and (2.5) to get that

$$E^x(\sup_{t \leq u}|L(t, s, v) - L(t, r, w)|^e) \leq c\delta^{e/2}.$$

Finally, take $e$ sufficiently large and use (2.2) to complete the proof. □

## 3. Joint continuity of additive functionals

In this section we give conditions for a collection of additive functionals $\{A_t^a, 0 \leq a \leq 1\}$ to be a.s. jointly continuous in $t$ and $a$. At the end of the section we give weaker conditions for convergence in probability. Throughout this section $T_m = \inf\{t: |W_t| \geq m\}$, $u_m(x, y)$ is the Green function on $S(m, 0)$, and $U_m\nu(x) = \int u_m(x, y)\nu(\mathrm{d}y)$. Throughout we will assume $\mu$ (or $\mu_a$) is a finite measure satisfying $U_m\mu(x) < \infty$ for all $m$, for all $x$.

We need a lemma of Brosamler [3, 3.4]. This may be proved either as is done there or by approximating $U_m\mu$ from below by potentials $U_m\mu_n$ that are four times continuously differentiable and calculating $\Delta(U_m\mu_n)^2$.

**Lemma 3.1.** $\int |\text{grad}\, U_m\mu|^2(y)u_m(x, y)\,\mathrm{d}y + (U_m\mu)^2(x) = 2\int U_m\mu(y)u_m(x, y)\mu(\mathrm{d}y)$.

Suppose $D$ is a finely closed Borel set contained in $S(m, 0)$, $V = \inf\{t: W_t \in D^c\}$, and that $U_m\mu(x) \leq N$ for $x \in D$. Let $u_m(x, B) = E^x\int_0^{T_m} 1_B(W_s)\,\mathrm{d}s$ and let $u_D(x, B)$

denote $E^x \int_0^V 1_B(W_s)\,ds$. By the strong Markov property, $u_D(x,B) = u_m(x,B) - E^x u_m(W_V, B)$, and we conclude that $u_D$ has a density $u_D(x,y) = u_m(x,y) - E^x u_m(W_V, y)$.

The following lemma generalizes (3.1).

**Lemma 3.2.** *Let $D$, $V$, $U_m\mu$ be as above. Then*

$$E^x \int_0^V |\text{grad } U_m\mu|^2(W_s)\,ds = E^x(U_m\mu)^2(W_V) - (U_m\mu)^2(x) + 2\int U_m\mu(y)u_D(x,y)\mu(dy).$$

**Proof.** Recall that grad $U_m\mu$ exists as the a.e. limit of grad $U_m\mu_n$, where the $U_m\mu_n$'s are differentiable potentials increasing to $U_m\mu$ [2]. $U_m\mu \wedge \gamma N$ is superharmonic and dominated by the potential $U_m\mu$, hence it is equal to the potential $U_m\nu$ of some measure $\nu$. It follows easily, choosing $\gamma > 1$ appropriately, that grad $U_m\nu =$ grad $U_m\mu$ a.e. on $D$, and thus it suffices to prove the lemma when $U_m\mu$ is bounded.

$$E^x \int_0^V |\text{grad } U_m\mu|^2(W_s)\,ds = \int |\text{grad } U_m\mu|^2(y)u_D(x,y)\,dy$$

$$= \int |\text{grad } U_m\mu|^2(y)u_m(x,y)\,dy - E^x \int |\text{grad } U_m\mu|^2(y)u_m(W_V, y)\,dy,$$

and the result follows by using (3.1). □

**Proposition 3.3.** *Suppose $D$ and $V$ are as above and that $U_m\mu$ and $U_m\nu$ are bounded by $N$ on $D$. Let $\varepsilon = \sup_{x\in D}|U_m\mu(x) - U_m\nu(x)|$ and suppose $\varepsilon < 1$. Let*

$$M_t = \int_0^t (\text{grad } U_m\mu - \text{grad } U_m\nu)(W_s)\,dW_s.$$

*Then*

$$E^x(\sup_{t\le V}|M_t|^r) \le c\varepsilon^{r/2}.$$

**Proof.** Note that $u_D(x,y)$ is 0 for $y \in D^c$. Note also that $|U_m\mu(W_V) - U_m\nu(W_V)| \le \varepsilon$ by continuity of potentials on paths.

Apply (3.2) to $U_m\mu$, $U_m\nu$ and $U_m(\mu+\nu)$, and use the fact that $|\text{grad}(U_m\mu - U_m\nu)|^2 = 2|\text{grad } U_m\mu|^2 + 2|\text{grad } U_m\nu| - |\text{grad } U_m(\mu+\nu)|^2$ to get that if $x \in D$,

$$E^x \int_0^V |\text{grad } U_m\mu - \text{grad } U_m\nu|^2(W_s)\,ds$$
$$= E^x(U_m\mu - U_m\nu)^2(W_V) - (U_u\mu - U_m\nu)^2(x)$$

$$+2\int (U_m\mu - U_m\nu)(y)u_D(x,y)(\mu-\nu)(\mathrm{d}y)$$

$$\leq 2\varepsilon^2 + 2\varepsilon\left[\int u_D(x,y)\mu(\mathrm{d}y) + \int u_D(x,y)\nu(\mathrm{d}y)\right]$$

$$\leq 2\varepsilon^2 + 4\varepsilon N \leq c\varepsilon. \tag{3.4}$$

Thus $E^x\langle M, M\rangle_V \leq c\varepsilon$.

If $x \in D^c$, $V=0$ a.s. $P^x$ and $E^x\langle M, M\rangle_V = 0$. Then by the inequalities of Burkholder, Davis and Gundy,

$$E^x\left(\sup_{t\leq V}|M_t|^r\right) \leq cE^x\left(\langle M, M\rangle_V^{r/2}\right),$$

and by (2.1) applied to Brownian motion killed on leaving $D$, this is $\leq c\varepsilon^{r/2}$. □

**Theorem 3.5.** *Let $\{\mu_a, 0 \leq a \leq 1\}$ be a collection of measures, $U_m\mu_a$ the potentials, $A_t^a$ the associated additive functionals. Suppose that for each x, there exists an increasing sequence of finely closed sets $D_N$ (possibly depending on x), such that if $V_N = \inf\{t\colon W_t \in D_N^c\}$,*

(i) $V_N \to \infty$ a.s. $P^x$,

(ii) $\sup_{t\leq V_N}\sup_a U_m\mu_a(W_t) \leq N$, *and*

(iii) $\sup_{t\leq V_N}\sup_{|a-b|\leq\delta} |U_m\mu_a(W_t) - U_m\mu_b(W_t)| \leq c\delta^\gamma$ *for some $c, \gamma > 0$,*

*possibly depending on N.*

*Then there exist versions $B_t^a$ of $A_t^a$ such that for each u*

$$\lim_{\delta\to 0}\sup_{|a-b|\leq\delta}\sup_{t\leq u\wedge T_m} |B_t^a - B_t^b| = 0 \quad a.s.$$

**Proof.** Fix $x$.

$$A_t^b - A_t^a = (U_m\mu_a - U_m\mu_b)(W_t) - (U_m\mu_a - U_m\mu_b)(W_0)$$

$$-\int_0^t \mathrm{grad}(U_m\mu_a - U_m\mu_b)(W_s)\,\mathrm{d}W_s$$

for $t \leq T_m$ by (1.4). Then

$$E^x\left(\sup_{t\leq V_N\wedge T_m}|A_t^a - A_t^b|\right)^r \leq c\delta^{r\gamma}$$

$$+ cE^x\left(\sup_{t\leq V_N\wedge T_m}\left|\int_0^t |\mathrm{grad}(U_m\mu_a - U_m\mu_b)|^2(W_s)\,\mathrm{d}s\right|^r\right)$$

$$\leq c\delta^{r\gamma} + c\delta^{r\gamma/2},$$

using (3.3). The result follows from (2.2) by taking $r>2/\gamma$ and recalling (i) and the idea of the first paragraph of the proof of (2.3). □

The following theorem gives relatively simple conditions that can be used to check the hypotheses of (3.5).

**Theorem 3.6.** *Let x be fixed. Suppose* $\{\mu_a, 0\le a\le 1\}$ *is a collection of measures such that*

(i) $\mu_a(S(\delta, x))\le k_1\delta^{d-2+\gamma}$ *for some* $k_1$, $\gamma>0$ *depending only on* $x$;

(ii) $$\left|\int h(y)\mu_a(\mathrm{d}y)-\int h(y)\mu_b(\mathrm{d}y)\right|\le k_2|b-a|^\eta(\|h\|+\|\mathrm{grad}\ h\|)$$

*for some* $k_2$, $\eta>0$. *Then*

(a) $U_m\mu_a(x)\le k_3$, $k_3$ *depending only on* $k_1$ *and* $\gamma$;

(b) $|U_m\mu_a(x)-U_m\mu_b(x)|\le k_4|b-a|^\tau$, $k_4$ *and* $\tau$ *depending only on* $k_1$, $k_2$, $\gamma$ *and* $\eta$;

(c) *If* $A_t^a$ *is the additive functional associated to* $\mu_a$, *then there exist versions* $B_t^a$ *of* $A_t^a$ *such that for each* $\mu$,

$$\lim_{\delta\to 0}\sup_{|a-b|\le\delta}\sup_{t\le u}|B_t^a-B_t^b|=0\quad \text{a.s.}$$

Here $\|\cdot\|$ is sup norm. Condition (ii) says that the $\mu_a$'s are Hölder continuous with respect to a certain metric for the weak topology.

An example that motivates condition (i) is the following:

Suppose for $0\le a\le 1$, $f_a:[0,1]^{d-1}\to S(m,0)$. Let $\mu_a(D)=$ Lebesgue measure in $\mathbb{R}^{d-1}$ of $\{y: f_a(y)\in D\}$. Thus, $f_a([0,1]^{d-1})$ is a family of hypersurfaces lying in $\mathbb{R}^d$. (For example, if $d=2$, $f_a$ is a family of curves.) Each $A_t^a$ may be thought of as a 'local time' for the hypersurface $f_a([0,1]^{d-1})$. If $f_a$ is reasonably smooth, condition (i) is easily seen to be satisfied with $\gamma=1$.

Notice that if $\{\mu_a\}$ has uniformly bounded densities, (i) is satisfied with $\gamma=2$.

**Proof of (3.6).** The proof of (a) follows easily from the nature of the singularity of $u_m(x,y)$ at $y=x$, which is on the order of $|\log|y-x\||$ if $d=2$, $|y-x|^{2-d}$ if $d>2$ [2].

To prove (b), let $g$ be a continuously differentiable function on $[0,\infty)$ such that $0\le g\le 1$, $g=1$ on $[1,\infty)$, $g=0$ on $[0,\frac{1}{2}]$, and $\|g'\|\le 4$. Fix $\delta<1$ for the moment, and define $h_\delta(y)=g(|y-x|/\delta)$. $h_\delta$ is 1 on $S(m,0)-S(\delta,x)$, 0 on $S(\frac{1}{2}\delta,x)$, and $\|\mathrm{grad}\ h_\delta\|\le 4/\delta$.

$$|U_m\mu_a(x)-U_m\mu_b(x)|\le\left|\int u_m(x,y)h_\delta(y)\mu_a(\mathrm{d}y)-\int u_m(x,y)h_\delta(y)\mu_b(\mathrm{d}y)\right|$$
$$+\int u_m(x,y)(1-h_\delta(y))\mu_a(\mathrm{d}y)+\int u_m(x,y)(1-h_\delta(y))\mu_b(\mathrm{d}y). \tag{3.7}$$

$u_m(x,y)(1-h_\delta(y))$ is nonzero only on $S(\delta,x)$. By the nature of the singularity of $u_m(x,y)$ at $y=x$ and (i), it is easy to check that the sum of the last two terms on the right hand side of (3.7) is $\leq c\delta^\gamma$, $c$ depending only on $k_1$ and $\gamma$.

On the other hand, $u_m(x,y)h_\delta(y)$ is bounded and differentiable with

$$\|u_m(x,\cdot)h_\delta(\cdot)\|+\|\mathrm{grad}(u_m(x,\cdot)h_\delta(\cdot))\|\leq c\delta^{-r},\quad r>0,$$

$c$, $r$ depending only on the dimension $d$.

Therefore,

$$|U_m\mu_a(x)-U_m\mu_b(x)|\leq c(\delta^\gamma+|b-a|^\eta\delta^{-r}),\quad \gamma,c,r>0,$$

where $c$ depends only on $k_1$, $\gamma$ and $k_2$.

As a function of $\delta$, the right hand side attains its minimum value at $\delta_0=(|b-a|^\eta r/\gamma)^{1/(\gamma+r)}$, which is $<1$ if $|b-a|$ is sufficiently small. Now, choosing $\delta=\delta_0$, we conclude that

$$|U_m\mu_a(x)-U_m\mu_b(x)|\leq c|b-a|^{\gamma\eta/(\gamma+r)}((r/\gamma)^{\gamma/(\gamma+r)}+(r/\gamma)^{-r/(\gamma+r)}).$$

Finally, set $\tau=\gamma\eta/(\gamma+r)$, and observe that $\tau>0$.

The proof of (c) follows from (a), (b) and (3.5). □

In the statement of some of the results, it will be convenient to refer to

**Hypothesis 3.8.** There exist constants $c$ and $\gamma>0$, independent of $x$ and $\delta$, such that $\mu(S(\delta,x))\leq c\delta^{d-2+\gamma}$ for all $x$ and $\delta$.

As in the remarks following (3.6), (3.8) is satisfied if $\mu$ has a bounded density ($\gamma=2$) or if $\mu$ is a measure supported on a hypersurface in an appropriate way ($\gamma=1$). Since we must have $\mu(S(\delta,x))=o(\delta^{d-2})$ if $U_m(x)$ is to be finite, (3.8) is a reasonably mild condition.

The theorem that will be needed in Section 4 is

**Theorem 3.9.** *Suppose* $K:\mathbb{R}^d\to[0,\infty)$ *with* (i) $\int K(y)\,dy=1$, (*ii*) $\int|y|K(y)\,dy<\infty$, *and* (iii) $K(\alpha y)=K(y)$ *whenever* $|\alpha|=1$. *Let* $K_a(y)=a^{-d}K(y/a)$. *Let* $\mu_a(dy)=\int K_a(z)\mu(dy-z)\,dz$. *Let* $A_t$ *be the additive functional associated with* $\mu$, $A_t^a$ *the additive functional associated with* $\mu_a$.

*If* $\mu$ *satisfies* (3.8), *then there exist versions* $B_t^a$ *of* $A_t^a$ *such that*

$$\limsup_{a\to 0\ t\leq u}|B_t^a-A_t|=0\quad a.s.$$

**Proof.** As in the first paragraph of (2.3), it suffices to prove

$$\lim_{a\to 0}\ \sup_{t\leq u\wedge T_m}|B_t^a-A_t|=0\quad \text{a.s.}$$

Let $\mu_0=\mu$. We will prove $\{\mu_a, 0\leq a\leq 1\}$ satisfies (i) and (ii) of (3.6) with $k_1$ and $\gamma$ independent of $x$. Then (a) and (b) of (3.6) together with (3.5) will prove the theorem.

First of all,

$$\mu_a(S(\delta,x)) = \int K_a(z)\mu(S(\delta,x)-z)\,\mathrm{d}z = \int K_a(z)\mu(S(\delta,x-z))\,\mathrm{d}z$$
$$\leq c\delta^{d-2+\gamma}\int K_a(z)\,\mathrm{d}z.$$

Now suppose $h$ is continuously differentiable. Observe that

$$h * K_a(y) = \int h(y-z)K_a(z)\,\mathrm{d}z = \int h(y-z)a^{-d}K(z/a)\,\mathrm{d}z$$
$$= \int h(y-az)K(z)\,\mathrm{d}z.$$

It follows, then, that since $|h(y-az)-h(y-bz)|\leq\|\text{grad } h\|\,|a-b|\,|z|$, we have

$$|h * K_a(y) - h * K_b(y)| \leq \|\text{grad } h\|\,|a-b|\int |z|K(z)\,\mathrm{d}z.$$

Since $\int h(y)\mu_a(\mathrm{d}y) = \int h * K_a(z)\mu(\mathrm{d}z)$,

$$\left|\int h(y)\mu_a(\mathrm{d}y) - \int h(y)\mu_b(\mathrm{d}y)\right| \leq \int \sup_z |h * K_a(z) - h * K_b(z)|\mu(\mathrm{d}z)$$

$$\leq |a-b|\,\|\text{grad } h\|\int |z|K(z)\,\mathrm{d}z\,\mu(\mathbb{R}^d). \tag{3.10}$$

If $a$ or $b=0$, minor modifications to the above proof show that (3.10) still holds. □

Note $\mu_a(dy) = [\int K_a(z-y)\mu(dz)]\,\mathrm{d}y$, or $\mu_a$ has a density $f_a$. Then one version of $A_t^a$ is $\int_0^t f_a(W_s)\,\mathrm{d}s$. If $K$ is continuous and bounded, $K_a$ will be continuous in $a$, and hence $A_t^a$ will be continuous in $a$. In this case, there is no need to take versions of $A_t^a$.

**Corollary 3.11.** *Suppose $\mu$ satisfies (3.8). Let*

$$A_t^a = \sigma^{-1}\int\int_0^t a^{-d}1_{(S(a,x))}(W_s)\,\mathrm{d}s\mu(\mathrm{d}x),$$

*where $\sigma$ is the volume of the unit ball. Then versions $B_t^a$ of $A_t^a$ exist so that*

$$\lim_{a\to 0}\sup_{t\leq u}|A_t - B_t^a| = 0 \quad \text{a.s.}$$

**Proof.** Let $K(y)=\sigma^{-1}1_{(S(1,0))}(y)$. $\mu_a(\mathrm{d}y)$ has a density

$$\int K_a(z-y)\mu(\mathrm{d}z)=\sigma^{-1}a^{-d}1_{(S(a,z))}(y)\mu(\mathrm{d}z).$$

The result follows by Fubini and the remark above. □

As the proof shows, any $K$ satisfying (3.9) may be substituted for the indicator of the unit ball.

If $\mu$ is a finite measure with $U_m\mu(x)<\infty$ for all $x$ and $m$, but perhaps not satisfying (3.8), we can still get the analogue of (3.9) if we replace a.s. convergence by convergence in probability.

**Theorem 3.12.** *Suppose $K:\mathbb{R}^d\to[0,\infty)$ with* (i) $\int K(y)\,\mathrm{d}y=1$, (ii) $K(\alpha y)=K(y)$ *if* $|\alpha|=1$, (iii) $K(|y|)$ *is decreasing in* $|y|$, *and* (iv) *for all* $r>0$, $a^{-d}K(y/a)\to 0$ *uniformly as* $a\to 0$ *for* $y\in S^c(r,0)$. *Let* $K_a$, $\mu_a$, $A^a$ *be defined as in* (3.9). *Then* $\sup_{t\le u}|A_t^a-A_t|\to 0$ *in* $P^x$*-probability as* $a\to 0$ *for all* $x$.

**Proof.** First of all, suppose $\mu$ and $\nu$ are two measures such that

$$U_m\mu(W_t),\ U_m\nu(W_t)\le N,\qquad |U_m\mu(W_t)-U_m\nu(W_t)|\le\varepsilon \tag{3.13}$$

for $t\le V\wedge T_m$, $V$ the hitting time to a finely open set. By (1.4) and (3.3),

$$E^x(\sup_{t\le V\wedge T_m}|A_t^\mu-A_t^\nu|)^2\le c\varepsilon. \tag{3.14}$$

where $A^\mu$, $A^\nu$ are the additive functional corresponding to $\mu$, $\nu$, respectively.

Secondly, if $V$ is a stopping time,

$$P^x(\sup_{t\le u}|A_t^\mu-A_t^\nu|>\eta)\le P^x(\sup_{t\le V}|A_t^\mu-A_t^\nu|>\eta)+P^x(u>V). \tag{3.15}$$

Fix $x$. Take $m>4|x|$. Let $J_a(y)=K_a(y)1_{(S^c(m/2,0))}(y)$. $J_a(y)\to 0$ uniformly as $a\to 0$, so $\mu*J_a$ has a density tending to 0 uniformly, hence the additive functional associated with $\mu*J_a$ tends to 0 uniformly on $[0,T_m]$.

Let $\nu_a=\mu*(K_a(\cdot)1_{(S(m/2,0))})$, $B_t^a$ the corresponding additive functional. Since the Green function $u_m(x,y)$ is superharmonic on $S(m,0)$, $U_m\nu_a\uparrow U_m\mu$ on $S(m/2,0)$ by (i), (ii) and (iii).

Let $V_0(N)=\inf\{t:\ U_m\mu(W_t)>N\}$.

$$\infty>U_m\mu(x)\ge E^xU_m\mu(W(V_0(N)))\ge NP^x(V_0(N)<\infty),$$

and hence $P^x(V_0(N)<\infty)\to 0$ as $N\to\infty$. Since $V_0(N)$ is nondecreasing in $N$, we have $V_0(N)\to\infty$ a.s. $P^x$ as $N\to\infty$.

Let

$$V(a,\varepsilon)=\inf\{t:\ U_m\mu(W(t\wedge V_0(N))-U_m\nu_a(W(t\wedge V_0(N)))>\varepsilon\}.$$

Since $U_m \nu_a(W(t \wedge V_0(N)))$ is an increasing family of potentials, by [9, VII, T36], $\lim_{a\to\infty} P^x(V(a,\varepsilon)<\infty)=0$. Note that since potentials are finely continuous, $V_0(N)$ and $V(a,\varepsilon)$ are hitting times to finely open sets.

Finally, given $\eta$, $\delta$ and $u$, let $\varepsilon \le \delta\eta^2$. Pick $m$ large enough, then $N$ large enough, and then $a$ small enough so that $P^x(u>T_{m/2} \wedge V_0(N) \wedge V(a,\varepsilon))<\delta$. By right continuity of potentials, (3.13) holds for $\mu$ and $\nu_a$ on $0\le t\le T_{m/2}\wedge V_0(N)\wedge V(a,\varepsilon)$. Applying Chebyshev to (3.14) and then using (3.15),

$$P^x(\sup_{t\le u}|A_t-B_t^a|>\eta)\le c\varepsilon/\eta^2+\delta\le(c+1)\delta.$$

Also,

$$P^x(\sup_{t\le u}|A_t^a-B_t^a|>\eta)\le P^x(\sup_{t\le T_m}|A_t^a-B_t^a|>\eta)+P^x(u>T_m),$$

which does it if $a$ is small enough. □

## 4. Integral representations

In this section we apply the Radon transform to the potentials of a measure $\mu$ to derive (1.3). See [6] for more details about the Radon transform. $S(m,0)$, $T_m$, $u_m$ and $U_m$ are as in Sections 2 and 3, and $\mu$ is a finite measure such that $U_m\mu(x)<\infty$ for all $x$ and $m$.

First a lemma. Let $G_m(x,s,v)=\int_{\{y\cdot v=s\}} u_m(x,y)\,dy$, where $dy$ is Lebesgue measure on the $d-1$ dimensional hyperplane $\{y: y\cdot v=s\}$ and $|v|=1$.

**Lemma 4.1.** (i) *$G_m(x,\cdot,v)$ has support in $[-m,m]$*; (ii) *$G_m(x,s,v)$ is uniformly bounded for fixed $m$*; (iii) $G_m(x,s,v)=E^xL(T_m,s,v)$.

**Proof.** (i) follows from the fact that $u_m$ is supported on $S(m,0)$.

Fix $m$. Fix $v$. Since Brownian motion is rotationally symmetric, we may assume without loss of generality tht $v$ is the unit vector in the first coordinate direction. Let $B=(-2m,2m)\times\mathbb{R}^{d-1}$, $V=\inf\{t: W_t\in B^c\}$. Let $U_B(x,dy)$ be the potential for $B$. By the strong Markov property,

$$U_B(x,dy)=u_m(x,y)\,dy+\int P^x(W(T_m)\in dz)\,U_B(z,dy).$$

If $h_0:\mathbb{R}\to[0,\infty)$, $y=(y_1,\dots,y_d)$, and $h(y)$ is defined to be $h_0(y_1)$, using the independence of $W_s^1$ from $(W_s^2,\dots,W_s^d)$, it follows easily that $\int h(y)\,U_B(x,dy)=U_0h_0(x)$, where $U_0$ is the potential for a one-dimensional Brownian motion killed on leaving $(-2m,2m)$.

Take a sequence of functions $h_0^\varepsilon=(2\varepsilon)^{-1}1_{(s-\varepsilon,s+\varepsilon)}$. Since $U_0(x,dy)$ has a continuous density $u_0(x,y)$ and $h_0^\varepsilon(y_1)\,dy_1$ is a sequence of measures converging weakly to the point mass at $s$, $U_0 h_0^\varepsilon(x) \to u_0(x, s)$, which we know to be $E^x L(V, s, v) < \infty$. It is not difficult to check that

$$\int h^\varepsilon(y) u_m(x, y)\, \mathrm{d}y \to G_m(x, s, v).$$

We then have

$$E^x L(V, s, v) = u_0(x, s) = G_m(x, s, v) + \int P^x(W(T_m) \in \mathrm{d}z) u_0(z, s)$$
$$= G_m(x, s, v) + E^x E^{W(T_m)} L(V, s, v).$$

Since $u_0(x, s)$ is bounded by a constant depending only on $m$ and not $x$ or $s$, (ii) follows. Since $L_t$ is an additive functional, using the strong Markov property at $T_m$ gives (iii). □

Define $I_b(y)$ to be

$$I_b(y) = (2\pi)^{-d} \int_0^\infty \cos(qy) q^{d-1}\, \mathrm{e}^{-bq^2/2}\, \mathrm{d}q. \tag{4.2}$$

Looking in a table of Fourier cosine transforms [5], we get

$$I_b(y) = (2\pi)^{-d}(-1)^{(d-1)/2}\pi^{1/2}(2b)^{-d/2} \exp(-y^2/2b) H_{d-1}((2b)^{-1/2} y)$$

if $d$ is odd, where $H_{d-1}$ is the $(d-1)$st Hermite polynomial;

$$I_b(y) = (2\pi)^{-d}\tfrac{1}{2}(b/2)^{-d/2}\Gamma(d/2) F(d/2; \tfrac{1}{2}; -y^2/2b)$$

if $d$ is even, where $F$ is the confluent hypergeometric function:

$$F(c; d; z) = \sum_{k=0}^\infty (c)_k z^k / ((d)_k k!)$$

with $(c)_k = \Gamma(c+k)/\Gamma(c) = (c)(c+1) \cdots (c+k-1)$.

Here is our main theorem:

**Theorem 4.3.** *Let $\mathcal{B} = \{v: |v| = 1\}$, let $A_t$ be the additive functional associated with $\mu$, and let*

$$A_t^b = \int \int_{\mathcal{B}} \int_{-\infty}^\infty I_b(s - y \cdot v) L_t(s, v)\, \mathrm{d}s\, \mathrm{d}v\, \mu(\mathrm{d}y).$$

*Then* (i) *for all $x$ and $u$, $\sup_{t \le u} |A_t^b - A_t| \to 0$ in $P^x$-probability as $b \to 0$;* (ii) *if, in addition, $\mu$ satisfies* (3.8),

$$\sup_{t \le u} |A_t^b - A_t| \to 0 \ \textit{a.s.} \quad \textit{as } b \to 0.$$

**Proof.** Let $\hat{}$ denote the Fourier transform. If $h_b(y) = (2\pi b)^{-d/2}\, \mathrm{e}^{-|y|^2/2b}$, recall that $\hat{h}_b(r) = \mathrm{e}^{-|r|^2 b/2}$. Suppose $f: \mathbb{R}^d \to \mathbb{R}$ is bounded with compact support. Recall also, then, that the Fourier transform of $f * h_b$ is $\hat{f}\hat{h}_b$, and by the Fourier inversion formula,

$$f * h_b(y) = (2\pi)^{-d} \int e^{-iy\cdot r}\hat{f}(r)\hat{h}_b(r)\,dr.$$

Substitute $r = qv$, where $v$ is a unit vector, and we have

$$f * h_b(y) = (2\pi)^{-d} \int_{\mathscr{B}} \int_0^\infty e^{-iqy\cdot v} q^{d-1}\hat{f}(qv)\hat{h}_b(qv)\,dq\,dv.$$

Since $v$ is a unit vector, $\hat{h}_b(qv) = e^{-bq^2/2}$, while

$$\hat{f}(qv) = \int e^{iqv\cdot z} f(z)\,dz = \int_{-\infty}^{\infty} \int_{(v\cdot z = s)} e^{iqs} f(z)\,dz\,ds.$$

We thus get the Radon transform formula

$$f * h_b(y) = (2\pi)^{-d} \int_{\mathscr{B}} \int_0^\infty \int_{-\infty}^{\infty} e^{iq(s-y\cdot v)} q^{d-1} \times e^{-bq^2/2}\left(\int_{(v\cdot z=s)} f(z)\,dz\right) ds\,dq\,dy. \tag{4.4}$$

Fix $m$. Apply (4.4) to $f(y) = u_m(x, y) \wedge N$. Letting $N\uparrow\infty$, the left side of (4.4) converges to $u_m(x,\cdot) * h_b$ by monotone convergence, while the right side converges, by dominated convergence and (4.1) to

$$(2\pi)^{-d} \int_{\mathscr{B}} \int_0^\infty \int_{-\infty}^{\infty} e^{iq(s-y\cdot v)} q^{d-1}\, e^{-bq^2/2} G_m(x, s, v)\,ds\,dq\,dv.$$

Finally, integrate with respect to $\mu(dy)$. If we let $\mu_b = \mu * h_b$, we get

$$U_m\mu_b(x) = (2\pi)^{-d} \int\int_{\mathscr{B}} \int_0^\infty \int_{-\infty}^{\infty} e^{iq(s-y\cdot v)} q^{d-1} \times e^{-bq^2/2} G_m(x, s, v)\,ds\,dq\,dv\,\mu(dy).$$

Because of the boundedness of $G_m(x, s, v)$ and the presence of the $e^{-bq^2/2}$ term, we may interchange the order of integration as we please. So

$$\begin{aligned} U_m\mu_b(x) &= \operatorname{Re} U_m\mu_b(x) \\ &= E^x \int\int_{\mathscr{B}} \int_{-\infty}^{\infty} (2\pi)^{-d} \int_0^\infty \operatorname{Re} e^{iq(s-y\cdot v)} q^{d-1} \\ &\qquad \times e^{-bq^2/2}\,dq\, L(T_m, s, v)\,ds\,dv\,\mu(dy) \\ &= E^x \int\int_{\mathscr{B}} \int_{-\infty}^{\infty} I_b(s - y\cdot v) L(T_m, s, v)\,ds\,dv\,\mu(dy) = E^x A^b(T_m). \end{aligned} \tag{4.5}$$

Since this holds for all $x$, we must have that $A_t^b$ is the additive functional associated to $\mu_b$, at least up to time $T_m$. (The difference beween $A_t^b$ and the additive functional associated with $\mu_b$ is a continuous martingale with paths of bounded variation, hence 0. In particular, $A_t^b$ is nondecreasing and nonnegative.)

(3.9) or (3.12) gives the desired result for $\sup_{t\le u\wedge T_m}|A_t^b - A_t|$, and since $m$ was arbitrary, the theorem follows. □

Suppose $d$, the dimension, is odd, and that $\mu$ has a density $f$ that has a sufficiently smooth derivative. More exactly, define $g_v(s) = \int_{(y\cdot v=s)} f(y)\,dy$, and suppose that $D_{d-1}g_v(s) = \partial^{d-1}g_v(s)/\partial s^{d-1}$ is in $L_1(-\infty,\infty)$, $\mu$ still a finite measure.

**Theorem 4.6.**

$$A_t = \tfrac{1}{2}(-1)^{(d-1)/2}\int_{\mathscr{B}}\int_{-\infty}^{\infty} D_{d-1}g_v(s)L_t(s,v)\,ds\,dv \quad a.s.$$

**Proof.** First note

$$\hat{g}_v(q) = \int_{-\infty}^{\infty} e^{iqr}g_v(r)\,dr = \int_{-\infty}^{\infty} e^{iqr}\int_{(y\cdot v=r)} f(y)\,dy\,dr = \hat{f}(qv).$$

Then $(D_{d-1}q_v)\hat{}\,(q) = (-1)^{(d-1)/2}q^{d-1}\hat{f}(qv)$.

Since $d-1$ is even and $L_t(s,v) = L_t(-s,-v)$, we get from (4.5) that

$$U_m\mu_b(x) = \tfrac{1}{2}(2\pi)^{-d}E^x\int_{\mathscr{B}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int e^{iq(y\cdot v-s)}q^{d-1}$$

$$\times e^{-bq^2/2}f(y)L(T_m,s,v)\,dy\,dq\,ds\,dv$$

$$= \tfrac{1}{2}(2\pi)^{-d}E^x\int_{\mathscr{B}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iqs}q^{d-1}e^{-bq^2/2}\hat{f}(qv)L(T_m,s,v)\,dq\,ds\,dy$$

$$= \tfrac{1}{2}(-1)^{(d-1)/2}(2\pi)^{-d}E^x\int_{\mathscr{B}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-iqs}\widehat{D_{d-1}g_v}(q)$$

$$\times e^{-bq^2/2}\,dq\,L(T_m,s,v)\,ds\,dv$$

$$= \tfrac{1}{2}(-1)^{(d-1)/2}E^x\int_{\mathscr{B}}\int_{-\infty}^{\infty} D_{d-1}g_v * h_b(s)L(T_m,s,v)\,ds\,dv$$

$$= \tfrac{1}{2}(-1)^{(d-1)/2}E^x\int_{\mathscr{B}}\int_{-\infty}^{\infty} D_{d-1}g_v(s)L(T_m,s,v)\,ds\,dv$$

$$+ \tfrac{1}{2}(-1)^{(d-1)/2}\int_{\mathscr{B}}\int_{-\infty}^{\infty}(D_{d-1}g_v * h_b(s) - D_{d-1}g_v(s))G_m(x,s,v)\,ds\,dv.$$

The second term on the right of this last line $\to 0$ as $b\to 0$ since $G_m$ is bounded and $D_{d-1}g_v(\cdot)\in L_1$. By writing $h_b = h_b\,1_{(S(\delta,0))} + h_b 1_{(S^c(\delta,0))}$ and arguing as in the proof of (3.12), $U_m\mu_b(x) = U_m\mu * h_b(x) \to U_m(x)$ pointwise as $b\to 0$.

Letting $b\to 0$, we have for all $x$,

$$U_m\mu(x)=\tfrac{1}{2}(-1)^{(d-1)/2}E^x\int_{\mathscr{B}}\int_{-\infty}^{\infty}D_{d-1}g_v(s)L(T_m,s,v)\,\mathrm{d}s\,\mathrm{d}v.$$

Since

$$A(t\wedge T_m)-\tfrac{1}{2}(-1)^{(d-1)/2}\int_{\mathscr{B}}\int_{-\infty}^{\infty}D_{d-1}g_v(s)L(t\wedge T_m,s,v)\,\mathrm{d}s\,\mathrm{d}v$$

is a continuous martingale with paths of bounded variation, hence 0, a.s., and $m$ was arbitrary, the theorem follows. □.

If $d=1$, this reduces to the usual representation $A_t^\mu=\int L_t^y\mu(\mathrm{d}y)$.

If $d$ is even, we get an analogous result with $D_{d-1}g_v$ replaced by $\mathscr{H}D_{d-1}g_v$, where $\mathscr{H}$ is the Hilbert transform:

$$\mathscr{H}h(x)=\lim_{\varepsilon\to 0}\int_{(|y|>\varepsilon)}y^{-1}h(y-x)\,\mathrm{d}y,$$

for $h$ locally integrable [10].

The reason for the appearance of the Hilbert transform is that here $d-1$ is odd and $q^{d-1}=(\operatorname{sgn}q)|q|^{d-1}$ for $q<0$. Thus when we change the limits of integration of $q$ from $[0,\infty)$ to $(-\infty,\infty)$, we must include a $\operatorname{sgn}q$ term. This gives rise to the Hilbert transform since $\widehat{\mathscr{H}h}(q)=-\mathrm{i}\operatorname{sgn}q\hat{h}(q)$.

If $d$ is even, it would probably be more convenient to define the measure $\mu'$ on $\mathbb{R}^d\times\mathbb{R}$ with density $f'(x,y)=f(x)$, $(x,y)\in\mathbb{R}^d\times[-m,m]$. Define $W_s'=(W_s,X_s)$, where $X_s$ is a Brownian motion independent of $W_s$.

$$A_t^\mu=\int_0^t f(W_s)\,\mathrm{d}s=\int_0^t f'(W_s')\,\mathrm{d}s$$

for $t$ less than the exit time of $S(m,0)$, and we are now in the situation where $d$ is odd.

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