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MARKOV PROCESSES WITH LIPSCHITZ SEMIGROUPS

BY

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ABSTRACT. For f a function on a metric space, let

$$\operatorname{Lip} f = \sup_{x \neq y} |f(x) - f(y)|/d(x, y),$$

and say that a semigroup P_t is Lipschitz if $\operatorname{Lip}(P_t f) \leqslant e^{Kt} \operatorname{Lip} f$ for all f, t, where K is a constant. If one has two Lipschitz semigroups, then, with some additional assumptions, the sum of their infinitesimal generators will also generate a Lipschitz semigroup. Furthermore a sequence of uniformly Lipschitz semigroups has a subsequence which converges in the strong operator topology.

Examples of Markov processes with Lipschitz semigroups include all diffusions on the real line which are on natural scale whose speed measures satisfy mild conditions, as well as some jump processes. One thus gets Markov processes whose generators are certain integro-differential operators. One can also interpret the results as giving some smoothness conditions for the solutions of certain parabolic partial differential equations.

1. Introduction. If f is a function on a metric space, let

$$\operatorname{Lip} f = \sup_{x \neq y} |f(x) - f(y)|/d(x, y).$$

Say that a semigroup P_t is Lipschitz if there is a constant K such that $\operatorname{Lip}(P_t f) \leqslant e^{Kt} \operatorname{Lip} f$ for all t and f. It is not too surprising that the Markov processes that arise out of Ito's theory of stochastic differential equations have semigroups that are Lipschitz. What may be more surprising is that any diffusion on the reals that is on natural scale and whose speed measure satisfies certain mild conditions also has Lipschitz semigroups and, in fact, $\operatorname{Lip}(P_t f) \leqslant \operatorname{Lip} f$.

One consequence of this fact is that it can be shown that if

$$Ag(x) = a(x)g''(x) + b(x)g'(x) + \int \left[g(y) - g(x) - g'(x)(y - x) \right] m(x, dy)$$

when g is twice continuously differentiable, then A will be the generator of a strong Markov process that has Lipschitz semigroups, provided a is nonnegative, continuous, and bounded, b is Lipschitz, and m satisfies a Lipschitz-like condition (see §7). This is an extension of results of Skorokhod **[16]** and Lepeltier and Marchal **[14]** in that here a need not be Lipschitz and the conditions on m are weaker than those that they use. Komatsu **[12]** and Stroock **[17]** have results that can be applied here, but only in the case that a is strictly positive; we allow a to be zero.

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The fact that diffusions on the real line have Lipschitz semigroups can be given an interpretation in terms of partial differential equations. If P_t is the semigroup of a diffusion with infinitesimal generator $A = (d/dm)(d/dx)$, where m is the speed measure, then $u(t, x) = P_t f(x)$ is the solution of the parabolic partial differential equation $\partial u/\partial t = Au$ with initial condition $u(0, x) = f(x)$. If A is written as $a(x)(d^2/dx^2)$, our a need not be continuous nor strictly positive. The assertion that P_t is Lipschitz is then an assertion about the smoothness of the solution u in the spatial variable in terms of the smoothness of the initial condition f. We thus generalize some of the results of Fichera [7], Freidlin [8], Kostin [13], and Oleinik [15]. See also [5] and [11].

In §2 we prove that if one has two Lipschitz semigroups, then the sum of their infinitesimal generators will also generate a Lipschitz semigroup. We also prove that given a sequence of uniformly Lipschitz semigroups, there will be a subsequence that converges in the strong operator topology. In §§3 and 4 we prove that any diffusion on the real line that is on natural scale and whose speed measure satisfies certain mild conditions will have Lipschitz semigroups. In §5 we consider the drift term, and in §§6 and 7 we extend the class of jump processes considered by Skorokhod and Lepeltier and Marchal in two different ways. In §7 we also give an example of a semigroup that is not Lipschitz.

2. Semigroups. Let E be r-dimensional Euclidean space. Let C_0 be the continuous functions that vanish at infinity. Let C_0^2 be the set of functions in C_0 whose first and second order partial derivatives are also in C_0. Let d be the Euclidean metric, and let $\|f\| = \sup_{x\in E}|f(x)|$.

Call a nonnegative contraction semigroup, P_t, Feller if P_t: $C_0 \to C_0$ and if $\|P_t f - f\| \to 0$ as $t \to 0$ whenever $f \in C_0$. A standard construction gives a correspondence between strong Markov Processes (Z_t, P^x) and Feller semigroups P_t [3].

If $f \in C_0$, define

$$\operatorname{Lip} f = \inf\{K: |f(y) - f(x)| \leqslant Kd(y, x) \text{ for all } y, x \in E\}.$$

Let $L = \{f \in C_0: \operatorname{Lip} f < \infty\}$. If P_t is a Feller semigroup, let Lip-exp P_t (for Lipschitz exponent) be defined by

$$\text{Lip-exp } P_t = \inf\left\{K: \operatorname{Lip} P_t f \leqslant e^{Kt} \operatorname{Lip} f \text{ for all } f \in L, \text{ for all } t > 0\right\}.$$

Call a semigroup Lipschitz if Lip-exp $P_t < \infty$.

After proving a lemma, we show that if a semigroup is a combination of two Lipschitz semigroups, it will also be Lipschitz.

Let $D_i f$ be the ith partial of f.

LEMMA 2.1. *Suppose P_t is a contraction semigroup such that if $f \in C_0^2$,*

$$\|P_t f - f\| \leqslant Kt\left(\sum_{i=1}^{r} \|D_i f\| + \sum_{i,j=1}^{r} \|D_i D_j f\|\right).$$

If P_t maps C_0 into bounded continuous functions, then P_t maps C_0 into C_0.

PROOF. Under the hypotheses given, for each x, there is a probability $P_t(x, \cdot)$ such that $P_t f(x) = \int f(y) P_t(x, dy)$ for all $f \in C_0$. Fix x, N. Let $g \in C_0^2$ such that $g(y) = 4N^2 - (d(x, y))^2$ if $d(x, y) \leqslant N$, $0 \leqslant g(y) \leqslant 3N^2$ if $d(x, y) > N$,

$\|D_i g\| \leqslant 4N$ for each i, and $\|D_i D_j g\| \leqslant 4$ for each pair i, j. $|P_t g(x) - g(x)| \leqslant Kt(4rN + 4r^2)$. So $P_t(x, S_N^c(x)) \leqslant 4r^2 Kt(1 + N)/N^2$, where $S_N(x)$ is the ball of radius N about x.

If $f \in C_0$, $\varepsilon > 0$, take N large so that $4r^2Kt(1 + N)/N^2 \leqslant \varepsilon/\|f\|$. Then if $d(x, \{y: |f(y)| > \varepsilon\}) \geqslant N$,

$$|P_t f(x)| \leqslant \varepsilon P_t(x, S_N(x)) + \|f\| P_t(x, S_N^c(x)) \leqslant 2\varepsilon,$$

which shows $P_t f \in C_0$. □

For the remainder of this section we will suppose that all infinitesimal generators have domains that contain C_0^2, and that all generators satisfy

$$\|Af\| \leqslant K_A\Big(\sum_i \|D_i f\| + \sum_{i,j} \|D_i D_j f\|\Big) \quad \textit{if } f \in C_0^2, \textit{for some constant } K_A. \tag{2.2}$$

By Dynkin's formula, if A generates P_t, P_t will satisfy the hypotheses of 2.1.

THEOREM 2.3. *Let P_t, Q_t be two Lipschitz semigroups with strong infinitesimal generators A, B, respectively. Suppose C is the closure of the restriction of $A + B$ to C_0^2 and that C is the strong infinitesimal generator of a contraction semigroup R_t. Then R_t will be Lipschitz, and* Lip-exp $R_t \leqslant$ Lip-exp P_t + Lip-exp Q_t.

PROOF. By the Trotter product formula **[6]**, if $f \in C_0^2$, $(P_{t/n}Q_{t/n})^n f \to R_t f$. Since P_s, Q_s both map C_0 into C_0 for all s, R_t maps C_0^2 into continuous functions and hence, by 2.1, C_0 into C_0. Since C_0^2 is contained in the domain of C, $R_t f \to f$ uniformly as $t \to 0$ if $f \in C_0^2$, hence if $f \in C_0$. Thus R_t is Feller.

If $f \in C_0^2$, and K_1 = Lip-exp P_t, K_2 = Lip-exp Q_t, then

$$\mathrm{Lip}(P_{t/n}Q_{t/n})^n f \leqslant (e^{K_1 t/n} e^{K_2 t/n})^n \,\mathrm{Lip}\, f = e^{(K_1 + K_2)t}\,\mathrm{Lip}\, f.$$

It follows readily that R_t is Lipschitz, and Lip-exp $R_t \leqslant K_1 + K_2$. □

Suppose P_t is as above.

(2.4) *Suppose $m(x, dy)$ is a nonnegative kernel that is uniformly bounded, weakly continuous in x, and there exists M such that $m(x, S_M^c(x)) = 0$ for all x, where $S_M(x)$ is the ball of radius M about x.*

If B is the operator given by $Bg(x) = \int[g(y) - g(x)]m(x, dy)$, it is known **[2]** that B will generate a nonnegative contraction semigroup Q_t. By the conditions on B, B: $C_0 \to C_0$. Since B is bounded, one can write an explicit formula for Q_t **[10]** and see that Q_t maps C_0 into continuous functions. Note that

$$\|Bg\| \leqslant \sum_i \|D_i g\| \int (y_i - x_i) m(x, dy),$$

where x_i is the ith coordinate of x, and so by 2.1, Q_t is Feller.

It is also known (**[2]** or **[10]**) that $A + B$ restricted to the domain of A generates a nonnegative contraction semigroup R_t on C_0. Thus, in this special case we do not have to suppose the existence of R_t, and we have

COROLLARY 2.5. *Suppose P_t is a Lipschitz semigroup with generator A. Let m be a kernel satisfying* (2.4) *and let B be defined by $Bg(x) = \int[g(y) - g(x)]m(x, dy)$. Then $A + B$ restricted to C_0^2 has an extension that generates a Feller semigroup R_t and* Lip-exp $R_t \leqslant$ Lip-exp P_t + Lip-exp Q_t, *where Q_t is the semigroup generated by B.*

PROOF. The only comment required is that if Q_t is not Lipschitz, then Lip-exp $Q_t = \infty$, and the inequality is trivial. □

The next result says that if one has a sequence of uniformly Lipschitz semigroups, one can find a subsequence that converges.

THEOREM 2.6. *Suppose P_t^n is a sequence of Lipschitz semigroups with generators A_n. Suppose $K = \sup_n$ Lip-exp $P_t^n < \infty$, and $\sup_n K_{A_n} < \infty$, where K_{A_n} is as in* (2.2). *Then there is a subsequence $P_t^{n_j}$ and a Lipschitz semigroup P_t such that $P_t^{n_j} f \to P_t f$ uniformly whenever $f \in C_0$, and* Lip-exp $P_t \leq K$.

Furthermore, if $\lim_{n\to\infty} A_n f = Af$ for $f \in C_0^2$ for some operator A, then A restricted to C_0^2 has an extension which generates P_t.

PROOF. If $f \in C_0^2$, then $P_t^n f$ is Lipschitz in t with Lipschitz constant $c \sup_n \|A_n f\|$, where c is independent of n. On the other hand, $P_t^n f$ is Lipschitz in x, and Lip $P_t^n f \leq e^{Kt}$ Lip f. Thus $P_t^n f$ is equicontinuous in x and t, and by Ascoli-Arzela, there is a subsequence that converges uniformly on compact sets in $[0, \infty) \times E$. The limit, call it F, will be continuous. By the argument of 2.1, if $d(x, \{y: |f(y)| > \varepsilon\}) \geq N$,

$$|P_t^n f(x)| \leq \varepsilon + 4r^2\|f\| \sup_n K_{A_n} t(1 + N)/N^2,$$

which, taking a limit along the subsequence, shows that $F \in C_0$. Also, F will be Lipschitz in x with Lipschitz constant no bigger than e^{Kt} Lip f.

Pick a countable dense subset of C_0^2. By a diagonalization process, one can find a subsequence $P_t^{n_j}$ that converges for each f in this countable subset. Call the limit $P_t f$. Since this countable subset is dense in C_0^2, hence in C_0, extend P_t to all of C_0, by continuity. Since each P_t^n is a contraction, it is easy to check that P_t is a nonnegative contraction semigroup that is Feller and that Lip-exp $P_t \leq K$.

Now suppose $A_n f \to Af$ uniformly if $f \in C_0^2$. $A_n f = \lim_{t\to 0}(P_t^n f - f)/t$ is continuous, and so Af is continuous and bounded. By the hypotheses and the argument of 2.1, for fixed x and t the family of probabilities $P_t^n(x, \cdot)$ is tight. Since $P_t^{n_j} f(x) \to P_t f(x)$ whenever $f \in C_0$, $P_t^{n_j} Af(x) \to P_t Af(x)$. This holds for each x and t. Similarly, for fixed x, the family $P_t(x, \cdot)$ is tight, and so $P_t Af(x) \to Af(x)$ as $t \to 0$.

Then if $f \in C_0^2$,

$$\begin{aligned} P_t f - f &= \lim_{j\to\infty} (P_t^{n_j} f - f) = \lim_j \int_0^t P_s^{n_j} A_{n_j} f\, ds \\ &= \lim_j \int_0^t P_s^{n_j}(A_{n_j} f - Af)\, ds + \lim_j \int_0^t P_s^{n_j} Af\, ds = \int_0^t P_s Af\, ds, \end{aligned}$$

the limit being a pointwise limit, and using the facts that $P_s^{n_j}$ is a contraction and $A_{n_j} f \to Af$ uniformly. Dividing both sides by t and letting $t \to 0$ shows that A restricted to C_0^2 is the restriction of the generator of P_t. □

Let us say that a Lipschitz semigroup Q_t is approximable if (i) there exist kernels m_n, each satisfying (2.4); (ii) if Q_t^n is the semigroup generated by B_n where $B_n g(x) = \int [g(y) - g(x)] m_n(x, dy)$, then Lip-exp $Q_t^n \leq$ Lip-exp Q_t; and (iii) $Q_t^n f \to Q_t f$ uniformly as $n \to \infty$ whenever $f \in C_0$.

Putting 2.6 and 2.5 together, we have

THEOREM 2.7. *Suppose P_t is a Lipschitz semigroup with generator A. Suppose Q_t is a Lipschitz semigroup that is approximable such that if B_n is the operator defined in* (ii) *of the definition of "approximable", $B_n f \to Bf$ whenever $f \in C_0^2$, where B is the infinitesimal generator of Q_t. Then $A + B$ restricted to C_0^2 has an extension that generates a Lipschitz semigroup R_t, and* Lip-exp $R_t \leqslant$ Lip-exp P_t + Lip-exp Q_t.

PROOF. Let R_t^n be the semigroup generated by $A + B_n$. By 2.6, there is a subsequence that converges to a semigroup; call it R_t.

$$\text{Lip-exp } R_t \leqslant \sup_n \text{ Lip-exp } R_t^n \leqslant \text{Lip-exp } P_t + \sup_n \text{ Lip-exp } Q_t^n$$
$$\leqslant \text{Lip-exp } P_t + \text{Lip-exp } Q_t. \quad \square$$

We need one final result.

THEOREM 2.8. *Suppose for each k, Q_t^k is approximable,* $\sup_k$ Lip-exp $Q_t^k = M < \infty$, *and $Q_t^k \to Q_t f$ for all $f \in C_0$. Then Q_t is approximable.*

PROOF. Let the semigroups that approximate Q_t^k be denoted by $Q_t^{k_n}$. Let $\{f_i\}$ be a countable dense subset of C_0. Let $\{(g_j, g_j^k, g_j^{kn})\}$ be an ordering of $\{(Q_r f_i, Q_r^k f_i, Q_r^{k_n} f_i)$: r rational$\}$.

There exists a K_1 such that if $k \geqslant K_1$, $\|g_1 - g_1^k\| \leqslant \frac{1}{2}$. For each $k \geqslant K_1$, there exists N_1^k such that if $n \geqslant N_1^k$, $\|g_1^{kn} - g_1^k\| \leqslant \frac{1}{2}$. There exists $K_2 > K_1$ so that if $k \geqslant K_2$, $\|g_1 - g_1^k\| \leqslant \frac{1}{4}$, $\|g_2 - g_2^k\| \leqslant \frac{1}{4}$. There exists N_2^k such that if $k \geqslant K_2$, $n \geqslant N_2^k$, $\|g_1^{kn} - g_1^k\| \leqslant \frac{1}{4}$, $\|g_2^{kn} - g_2^k\| \leqslant \frac{1}{4}$. Continue, defining K_p so that $\|g_j - g_j^k\| \leqslant 2^{-p}, j = 1, 2, \ldots, p$, if $k \geqslant K_p$, and so on.

The sequence $Q_r^{K_p, N_p} f_i$ $(= g_j^{K_p, N_p})$ will converge to $Q_r f_i$ $(= g_j)$ for each i and rational r. Since each Q_t^{kn} is a contraction semigroup, we have $Q_t^{K_p, N_p} f$ converges to $Q_t f$ for all $t \geqslant 0$, all $f \in C_0$. $\square$

3. Diffusions with Lipschitz coefficients. We now consider Markov processes on the real line (E = reals). The processes that we will consider will all contain C_0^2 in the domains of their infinitesimal generators. If Z_t is the Markov process with generator $Af(x) = a^2(x)f''(x)$, where a is a Lipschitz function, then using Ito's theory of stochastic differential equations (see **[1]**) it is relatively easy to show that the semigroup of Z_t is Lipschitz.

THEOREM 3.1. *Suppose P_t is the semigroup generated by A, where $Af(x) = a^2(x)f''(x)$, $\|a\| < \infty$, and α =* Lip $a < \infty$. *Then P_t is Lipschitz.*

PROOF. If $f \in C_0^2$, $\|P_t f - f\| = \|\int_0^t P_s Af\, ds\| \leqslant \|a\|^2 t \|f''\|$. By 2.1, P_t is Feller.

To show P_t is Lipschitz, let P be a probability under which W_t is a standard Brownian motion, and let X_t satisfy $X_t = x + \int_0^t a(X_s)\, dW_s$ and Y_t satisfy $Y_t = y + \int_0^t a(Y_s)\, dW_s$.

$$E(X_t - Y_t)^2 = (x-y)^2 + 2(x-y)E\int_0^t [a(X_s) - a(Y_s)]\, dW_s$$
$$+ E\Big[\int_0^t [a(X_s) - a(Y_s)]\, dW_s\Big]^2$$
$$= (x-y)^2 + E\int_0^t [a(X_s) - a(Y_s)]^2\, ds$$
$$\leqslant (x-y)^2 + \alpha^2 \int_0^t E(X_s - Y_s)^2\, ds.$$

By Gronwall's inequality, $E(X_t - Y_t)^2 \leqslant (x-y)^2 e^{\alpha^2 t}$. If $f \in L$,

$$|Ef(X_t) - Ef(Y_t)| \leqslant (\operatorname{Lip} f) E|X_t - Y_t| \leqslant (\operatorname{Lip} f)\big(E(X_t - Y_t)^2\big)^{1/2}$$
$$\leqslant (\operatorname{Lip} f)|x-y| e^{\alpha^2 t/2}.$$

If Z_t is the process with A as its generator, then, as is well known, the distribution of Z_t under P^x is the same as X_t under P and the distribution of Z_t under P^y is the same as Y_t under P. So,

$$|P_t f(x) - P_t f(y)| = |E^x f(Z_t) - E^y f(Z_t)| = |Ef(X_t) - Ef(Y_t)|$$
$$\leqslant \operatorname{Lip} f |x-y| e^{\alpha^2 t/2},$$

or Lip-exp $P_t = \alpha^2/2$. □

By a more careful analysis, however, we can do much better and show Lip-exp $P_t = 0$.

THEOREM 3.2. *Under the same hypotheses as* 3.1, Lip-exp $P_t = 0$.

PROOF. Take h so that $\alpha h < 1$; let $\lambda = 2/h^2$. Consider the Markov process X_t^h, with semigroup P_t^h, constructed as follows: starting at x, the process waits at x a length of time that is exponential with parameter λ. At that time it jumps either to $x + a(x)h$ or $x - a(x)h$, both with probability $\frac{1}{2}$ ($a(x)$ is possibly 0). It waits at the new point a length of time that is exponential with parameter λ, and so on. It is easy to see that the generator of P_t^h is given by

$$A^h f(x) = [f(x + a(x)h) + f(x - a(x)h) - 2f(x)]/h^2.$$

By 2.6, since $A^h f \to Af$ if $f \in C_0^2$, Lip-exp P_t will be 0 if we show Lip-exp $P_t^h = 0$. There is no question of subsequences here, since there is only one semigroup that has generator A.

Let Q be the transition probability for the Markov chain that, starting at x, jumps either to $x + a(x)h$ or $x - a(x)h$, both with probability $\frac{1}{2}$. Let Q^n be the iterates of Q. Since X_t^h is a pure jump process with waiting times that are identically distributed,

$$P_t^h f = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(Q^k f)(\lambda t)^k}{k!}.$$

If we show Lip $Q^k f \leqslant$ Lip f, we would then have Lip $P_t^h f \leqslant$ Lip f, our desired result.

$$|Q^{k+1} f(y) - Q^{k+1} f(x)| = |Q(Q^k f)(y) - Q(Q^k f)(x)|.$$

If we show Lip $Qf \leqslant$ Lip f for all f, then since the case $k = 0$ is trivial, we would do a simple induction and be done.

Suppose $y > x$.

$$\begin{aligned}|Qf(y) - Qf(x)| &= \left|\tfrac{1}{2}f(y + a(y)h) + \tfrac{1}{2}f(y - a(y)h)\right.\\ &\qquad \left.-\tfrac{1}{2}f(x + a(x)h) - \tfrac{1}{2}f(x - a(x)h)\right|\\ &\leqslant \tfrac{1}{2}\mathrm{Lip}\, f|(y - x) + (a(y) - a(x))h|\\ &\qquad + \tfrac{1}{2}\mathrm{Lip}\, f|(y - x) - (a(y) - a(x))h|.\end{aligned}$$

Now $|(a(y) - a(x))h| \leqslant \alpha h|y - x| < |y - x|$. Therefore,

$$|(y - x) + (a(y) - a(x))h| = (y - x) + (a(y) - a(x))h$$

and $|(y - x) - (a(y) - a(x))h| = (y - x) - (a(y) - a(x))h$. We thus have $|Qf(y) - Qf(x)| \leqslant \mathrm{Lip}\, f(y - x)$, which completes the proof. □

4. Diffusions on the real line. Since Lip-exp $P_t = 0$ under the conditions of 3.2, regardless of a, any process that can be approximated by diffusions with Lipschitz coefficients will also have Lipschitz exponent 0. We exploit this to show that a large class of diffusions on the line have Lipschitz exponent 0. We consider those diffusions on R which are on natural scale and whose speed measures s satisfy

(4.1)(i) *s is finite on finite intervals and*

(ii) *there is a constant $\beta > 0$ such that $s(x, y] \geqslant \beta(y - x)$ for all $x < y$.*

The first condition insures that there are no boundary points, the second that the process does not move too fast.

Let W_t be a standard Brownian motion and let L_t^y be local time for W at y. Recall that L_t^y is jointly continuous in t and y, increasing in t, that $\int_A L_t^y\, dy = \int_0^t 1_A(W_s)\, ds$, and $E^x(L_t^y)^2$ as a function of y is bounded on compact intervals. Recall that a diffusion on the real line which is on natural scale is characterized by its speed measure s. If $H_t = \int L_t^y s(dy)$, $T_t = \inf\{u > 0\colon H_u \geqslant t\}$, then W_{T_t} will have the same distribution under P^x as X_t, for all x. (See **[4]** or **[9]**.)

Note that by (4.1)(ii), $H_t = \int L_t^y\, s(dy) \geqslant \beta \int L_t^y\, dy = \beta t$. So $T_t \leqslant t/\beta$, and if $f \in C_0^2$, by Dynkin's identity,

$$E^x f(X_t) - f(x) = E^x f(W_{T_t}) - f(x) = E^x \textstyle\int_0^{T_t} \tfrac{1}{2} f''(W_s)\, ds \leqslant (t/2\beta)\|f''\|.$$

It is known that P_t maps C_0 into continuous functions, and by 2.1, $P_t\colon \bar{C}_0 \to \bar{C}_0$. Furthermore, by the continuity of L_t^y in t, H_t is continuous.

Our method is to approximate s by speed measures s_n such that the corresponding processes X_t^n with semigroups P_t^n have generators $a_n^2 f''$, where a_n is Lipschitz, and to show that P_t^n converges to P_t, the semigroup of X_t.

THEOREM 4.2. *Suppose X_t with semigroup P_t is a diffusion on R that is on natural scale and that its speed measure s satisfies* (4.1). *Then* Lip-exp $P_t = 0$ *and P_t is approximable.*

PROOF. First, suppose there exists N such that s on $[-N, N]^c$ is Lebesgue measure. We may assume $\beta \leqslant 1$ in (4.1)(ii). Define $S(x) = s(-N, x]$ for $-N \leqslant x \leqslant N$, and let S_n be a sequence of increasing C^2 functions on $[-N, N]$ that tend pointwise to S, except possibly at points of discontinuity of S, such that $S_n'(x) \geqslant \beta$ for all x, $S_n(-N) = 0$, and $S_n(N) = S(N)$. This is possible since S is increasing and bounded. Let s_n be the measure on $[-N, N]$ induced by S_n, and define s_n on $[-N, N]^c$ to be Lebesgue measure on $[-N, N]^c$. Let X_t^n be the diffusion which has s_n as its speed measure, P_t^n the corresponding semigroup, and H_t^n and T_t^n defined analogously to H_t and T_t.

It is not hard to check that the generator of P_t^n is of the form $a_n^2 f''$, where a_n is differentiable, bounded above by $\beta^{-1/2}$, and bounded below away from 0 (since S_n' is bounded above). By 3.2, Lip-exp $P_t^n = 0$. Furthermore, the measures s_n are uniformly bounded on finite intervals and if g is continuous,

$$\int_{[-N,N]} g(y) s_n(dy) \to \int_{[-N,N]} g(y) s(dy).$$

As before, $H_u^n - H_t^n \geqslant \beta(u - t)$, $T_t^n \leqslant t/\beta$, and $P_t^n f(x) - f(x) \leqslant (t/2\beta)\|f''\|$.

Suppose $f \in C_0^2$. Then $\|P_u^n f - P_t^n f\| \leqslant \|P_t^n\| \, \|P_{u-t}^n f - f\| \leqslant (t - u)\|f''\|/2\beta$, or P_t^n is Lipschitz in t.

Suppose now $f \in L$. Since X_t has the same distribution as W_{T_t} and X_t^n has the same distribution as $W_{T_t^n}$,

$$\begin{aligned}|P_t f(x) - P_t^n f(x)| &= |E^x f(W_{T_t}) - E^x f(W_{T_t^n})| \leqslant (\mathrm{Lip}\, f) E^x |W_{T_t} - W_{T_t^n}| \\ &\leqslant (\mathrm{Lip}\, f)\big(E^x(W_{T_t} - W_{T_t^n})^2\big)^{1/2}.\end{aligned}$$

Since $W_t^2 - t$ is a martingale and $(W_{T_t} - W_{T_t^n})^2 = (W_{T_t \vee T_t^n} - W_{T_t \wedge T_t^n})^2$,

$$E^x(W_{T_t} - W_{T_t^n})^2 = E^x|T_t - T_t^n|.$$

Fix ω. Suppose $T_t = u$. Then $H_u = t$. If $H_u^n = t - \varepsilon$, $H_{u+\varepsilon/\beta}^n \geqslant H_u^n + \beta(\varepsilon/\beta) = t$. So $T_t^u \leqslant u + \varepsilon/\beta$. If $H_u^n = t + \varepsilon$, $H_{u-\varepsilon/\beta}^n \leqslant H_u^n - \varepsilon = t$, and $T_t^n \geqslant u - \varepsilon/\beta$. Thus $|T_t - T_t^n| \leqslant |H_{T_t} - H_{T_t}^n|/\beta$. We have then

$$E^x|T_t - T_t^n| \leqslant E^x|H_{T_t} - H_{T_t^n}|/\beta.$$

Let $h(y) = 1$ if $|y| \leqslant n$, $N + 1 - |y|$ if $N \leqslant |y| \leqslant N + 1$, and 0 otherwise. By the construction of s_n and s,

$$G_n = |H_{T_t} - H_{T_t}^n| = \left|\int h(y) L_{T_t}^y \, s(dy) - \int h(y) L_{T_t}^y \, s_n(dy)\right|.$$

For each ω, $h(y) L_{T_t}^y(\omega)$ as a function of y is continuous with compact support, and so $G_n(\omega) \to 0$. On the other hand,

$$\begin{aligned}\sup_n E^x\Big(\int h(y) L_{T_t}^y \, s_n(dy)\Big)^2 &\leqslant \sup_n \int h^2(y) E^x(L_{T_t}^y)^2 \, s_n(dy) \\ &\leqslant \sup_n \int h^2(y) E^x(L_{t/\beta}^y)^2 \, s_n(dy) \\ &\leqslant \sup_{|y| \leqslant N+1} E^x(L_{t/\beta}^y)^2 \sup_n s_n[-N - 1, N + 1] < \infty.\end{aligned}$$

A similar equation holds for $E^x(\int h(y)L_{T_t}^y\, s(dy))^2$; hence $\sup_n E_x^x G_n^2 < \infty$, or the G_n are uniformly integrable, and so $E^x G_n \to 0$.

Thus, $P_t^n f(x) \to P_t f(x)$ for each x if $f \in L$. Since $P_t^n f$ is equicontinuous if $f \in L$, this, 2.6, and 2.1, gives uniform convergence. Hence Lip-exp $P_t = 0$.

Finally, suppose s satisfies (4.1). Define s_N to be equal to s on $[-N, N]$ and equal to Lebesgue measure on $[-N, N]^c$, and let P_t, P_t^N, X_t, X_t^N be the corresponding semigroups and processes.

If x is fixed and $T_N = \inf\{t > 0: |X_t - x| \geqslant N\}$, X_t and X_t^N may be constructed as time changes of the same Brownian motion so that they have the same paths up to and including time T_N. If g is the function constructed in the proof of 2.1,

$$\left|E^x g\left(X_{T_N \wedge t}^N\right) - g(x)\right| \leqslant \left|E^x \int_0^{T_N \wedge t} \frac{4}{\beta}\, ds\right| \leqslant 4\frac{t}{\beta},$$

or $P^x(t \geqslant T_N) \leqslant 4t/(\beta N^2)$. The same holds for X_t, and so if $f \in C_0$,

$$|P_t f(x) - P_t^N f(x)| \leqslant 2\|f\| P^x(t \geqslant T_N) \leqslant 8t\|f\|/(\beta N^2) \to 0$$

as $N \to \infty$. Thus if $f \in L$, $P_t^N f \to P_t f$ pointwise, hence uniformly on intervals since $P_t^N f$ is equicontinuous, hence uniformly by 2.1. Therefore Lip-exp $P_t = 0$ and P_t is approximable. □.

5. Drift. So far we have considered diffusions on the line that are on natural scale, that is, they have no drift. Now we consider adding drift terms.

THEOREM 5.1. *If* Lip $b < \infty$, *A is the operator defined by $Af(x) = b(x)f'(x)$ for $f \in C_0^2$, and P_t is the semigroup generated by A, then P_t is approximable and* Lip-exp $P_t \leqslant$ Lip b.

PROOF. Let X_t^h be the process that starting at x, waits at x a length of time that is exponential with parameter $1/h$, then jumps to $x + b(x)h$, waits there a length of time that is exponential with parameter $1/h$, and so on. Let Q be the transition probability that at each step jumps from x to $x + b(x)h$, Q^n the iterates. Then exactly as in the proof of 3.2,

$$\text{Lip } P_t^h f \leqslant e^{-t/h} \sum_{k=0}^{\infty} \frac{(\text{Lip } Q^k f)(t/h)^k}{k!},$$

where P_t^h is the semigroup for X_t^h.

$$|Qf(y) - Qf(x)| = |y + b(y)h - (x + b(x)h)| \leqslant |y - x|(1 + (\text{Lip } b)h).$$

By induction, Lip $Q^k f \leqslant (1 + h \text{ Lip } b)^k$, and so

$$\text{Lip } P_t^h \leqslant e^{-t/h} e^{t/h(1 + h \text{ Lip } b)} = e^{t \text{ Lip } b}.$$

Now let $h \to 0$ and use 2.6. □

6. Jumps. Ito's method of solving stochastic differential equations has been extended by Skorokhod [**16**] and Lepeltier and Marchal [**14**] to include the case where jumps are allowed. Thus, they consider solutions to the equation $X_t = x + \int_0^t \int F(X_s, u)\, q(ds, du)$ where $p(ds, du)$ is the random measure associated to a certain process with stationary, independent increments, $m(du) = E(\int_0^1 p(ds, du))$, $q(ds, du) = p(ds, du) - m(du)ds$, and F satisfies $\int[F(y, u) - F(x, u)]^2 m(du) \leq K|y - x|^2$. In a way very similar to 3.1, we can show that the semigroup P_t corresponding to X_t is Lipschitz and Lip-exp P_t is some function of K. Again, a more careful analysis shows that in some cases Lip-exp $P_t = 0$.

PROPOSITION 6.1. *Let $m(x, dy)$ be a kernel on the real line such that*:

(i) *m is weakly continuous in x*;

(ii) *$m(x, \cdot)$ puts positive mass a_i on each of n points $x + h_i(x)$, where $h_i(x)$ may possibly be* 0;

(iii) $\sum_{i=1}^n a_i h_i(x) = 0$ *for all x*;

(iv) $\sup_i \|h_i\| < \infty$ *and* Lip $h_i < 1$ *for all i*.

Define A by

$$Ag(x) = \int [g(y) - g(x)] m(x, dy) \tag{6.2}$$

and let X_t, P_t be the process and semigroup generated by A. Then P_t is Lipschitz and Lip-exp $P_t = 0$.

PROOF. Note that by (iii), $\int(y - x)m(x, dy) = 0$, and so if $g \in C_0^2$,

$$Ag(x) = \int [g(y) - g(x) - g'(x)(y - x)] m(x, dy). \tag{6.3}$$

It follows that whenever A is given by (6.3) $\|Ag\| \leq \|g''\| \, \|\int(y - x)^2 m(x, dy)\|$ and, using 2.1, P_t is Feller.

Let $M = \sum_{i=1}^n a_i$. Let Q be the Markov chain that moves from x to $x + h_i(x)$ with probability a_i/M. It is not hard to check that X_t is the process that waits at x a length of time that is exponential with parameter M, moves according to Q, then waits again, etc. Suppose $y > x$.

$$\begin{aligned} |Qf(y) - Qf(x)| &= \left|\sum a_i M^{-1} f(y + h_i(y)) - \sum a_i M^{-1} f(x + h_i(x))\right| \\ &\leq (\mathrm{Lip}\, f) M^{-1} \sum a_i |y + h_i(y) - (x + h_i(x))|. \end{aligned}$$

Since Lip $h_i < 1$, $|h_i(y) - h_i(x)| < |y - x|$, so

$$|y + h_i(y) - x - h_i(x)| = y - x + h_i(y) - h_i(x).$$

Then

$$\begin{aligned} \|Qf(y) - Qf(x)| &\leq (\mathrm{Lip}\, f) M^{-1} \sum a_i (y - x) \\ &\quad + (\mathrm{Lip}\, f) M^{-1}\left(\sum a_i h_i(y) - \sum a_i h_i(x)\right) \\ &= (\mathrm{Lip}\, f)|y - x|. \end{aligned}$$

Thus Lip $Qf \leq$ Lip f.

As in 3.2, Lip $Q^k f \leq$ Lip f,

$$\mathrm{Lip}\, P_t f \leq e^{-Mt} \sum_{k=0}^{\infty} \frac{\mathrm{Lip}\, Q^k f (Mt)^k}{k!} \leq \mathrm{Lip}\, f$$

or Lip-exp $P_t = 0$. □

PROPOSITION 6.4. *Suppose m_n is a sequence of kernels such that* (i) $\sup_n \|m_n(x, E)\| < \infty$, (ii) $\sup_n \|\int |y - x| m_n(x, dy)\| < \infty$, *and* (iii) *if A_n is defined in terms of m_n by* (6.2) *and P_t^n is the semigroup generated by A_n, then each P_t^n is Lipschitz and* $\sup_n$ Lip-exp $P_n^n < \infty$. *Suppose $m_n(x, dy)$ converges weakly to $m(x, dy)$ as $n \to \infty$ for each x, A is defined in terms of m by* (6.2), *and P_t is the semigroup generated by A. Then P_t is approximable, Lipschitz, and* Lip-exp $P_t \leqslant \sup_n$ Lip-exp P_t^n.

PROOF. If $f \in C_0^2$, note that

$$\|P_t^n f - f\| \leqslant t\|Af\| \leqslant t\|f'\| \int |y - x| m(x, dy)$$
$$\leqslant t\|f'\| \sup_n \left\| \int |y - x| m_n(x, dy) \right\|.$$

By 2.6, there is a subsequence n_j and a semigroup Q_t such that $P_t^{n_j} f \to Q_t f$ uniformly whenever $f \in C_0^2$.

If $f \in C_0^2$,

$$Q_t f - f = \lim_{j=\infty} (P_t^{n_j} f - f) = \lim \int_0^t A_{n_j} P_r^{n_j} f \, dr$$
$$= \lim \int_0^t A_{n_j} (P_r^{n_j} f - Q_r f) \, dr + \lim \int_0^t A_{n_j} Q_r f \, dr.$$

The first limit is 0 since the A_n's are uniformly bounded. The second limit is $\int_0^t A Q_r f(x) \, dr$ for each x, since $Q_r f$ is bounded and continuous and m_n converges weakly to m. Since A is bounded, dividing both sides by t gives $\lim_{t \to 0} (Q_t f - f)/t = Af$, uniformly in x. Therefore

$$Q_t f - P_t f = \int_0^t A(Q_r f - P_r f) \, dr \leqslant \|A\| \int_0^t (Q_r f - P_r f) \, dr,$$

or $Q_t f = P_t f$ if $f \in C_0^2$, hence for all $f \in C_0$.

Since any convergent subsequence of P_t^n converges to P_t, P_t^n converges to P_t. □

We use 6.1 and 6.4 to get

THEOREM 6.5. *Let μ be a continuous measure with $\int u^2 \mu(du) < \infty$. Let F satisfy $\|\int |F(x, u)|^2 \mu(du)\| < \infty$, $\|\int_{|u| \leqslant \varepsilon} |F(x, u)|^2 \mu(du)\| \to 0$ as $\varepsilon \to 0$, and $|F(x, u) - F(y, u)| \leqslant |x - y|$ for all x, y. Let m be the kernel defined by*

$$m(x, B) = \int 1_{B-x}(F(x, u)) \mu(du). \tag{6.6}$$

Let A be defined by (6.3). *Then A will be the generator of an approximable semigroup with Lipschitz exponent* 0.

Comment. Since $m(du)$ is on the order of $o(u^{-2})du$, the condition of **[9]**, $\int [F(x, u) - F(y, u)]^2 m(du) \leqslant K|x - y|^2$, translates, very roughly, to $|F(x, u) - F(y, u)|$ is on the order of $u|x - y|$, a stronger requirement on Lip $F(\cdot, u)$ than in 6.5.

PROOF. If $\mu_\varepsilon(du) = 1_{[\varepsilon,\infty)}(|u|)\mu(du)$, m_ε and A_ε are defined analogously to m and A, and $g \in C_0^2$,

$$Ag - A_\varepsilon g \leqslant \|g''\| \left\| \int (y-x)^2(m(x,dy) - m_\varepsilon(x,dy)) \right\|$$

$$= \|g''\| \left\| \int_{|u|<\varepsilon} |F(x,u)|^2 \mu(du) \right\| \to 0.$$

By 2.6, if the semigroup generated by A_ε has Lipschitz exponent 0, we will be done. Thus it suffices to assume μ is a bounded measure.

Next we approximate the drift by jumps. Let $h(x) = \int (y-x)m(x,dy)$. Let u_0 be fixed, let $\varepsilon_x(dy)$ denote point mass at x, and let $\mu_n = \mu + n\varepsilon_{u_0}$. Redefine $F(x,u_0)$ to be $h(x)/n$. If n is sufficiently large, $|F(x,u_0) - F(y,u_0)| \leqslant |x-y|$. Define m_n in terms of μ_n by (6.6) and define A_n in terms of m_n by (6.2). One can now check that $A_n g \to Ag$ if $g \in C_0^2$, and so by 2.6, it suffices to consider the case when μ is bounded and A is given by (6.2).

Finally, let μ_k be a sequence of uniformly bounded measures that each put mass on at most k points so that $\mu_k(\{u_0\}) = \mu(\{\mu_0\})$ for each k, $\mu_k \to \mu$ weakly, and $\sup_k \|\int |F(x,u)|^2 \mu_k(du)\| < \infty$. If the m_k are defined by (6.6) the $m_k(x,dy)$ will then converge weakly to $m(x,dy)$ for each x. Note that

$$\int |y - x| m_k(x,dy) = \int |F(x,u)| \mu_k(du)$$

$$\leqslant \sup_k \left(\left\| \int |F(x,u)|^2 \mu_k(du) \right\| + \mu_k(E) \right).$$

Using 6.4 it suffices to consider the case where A is given by (6.2) and m satisfies (6.1)(i)–(iv). But then 6.1 gives the desired result. □

7. Other jump processes. There are a number of jump processes that have Lipschitz semigroups, but which do not fit into the Skorokhod framework. For example, let A be given by $Ag(x) = a(x)[g(x+1) - g(x)]$, where Lip $a < \infty$. This is the process that waits a length of time that is exponential with parameter $1/a(x)$, then jumps 1. Under the Skorokhod framework, the F could not even be continuous, let alone Lipschitz.

PROPOSITION 7.1. *Let A be given by $Ag(x) = a(x)[g(x+h) - g(x)]$, $\|a\| < \infty$. If P_t is the semigroup generated by A,* Lip-exp $P_t \leqslant h$ Lip a.

PROOF. If $\|a\| = 0$ or Lip $a = \infty$, the result is trivial. So assume $\|a\| > 0$ and Lip $a < \infty$. Let $M = \|a\|$, $b(x) = a(x)/M$.

Let Q be the transition operator for the chain that goes from x to $x + h$ with probability $b(x)$, and stays at x with probability $1 - b(x)$. Let X_t be the process that jumps according to Q and in between waits lengths of time that are exponential with parameter M.

It is not hard to check that X_t has generator A.

$$\begin{aligned}
|Qf(y) - Qf(x)| &\\
&= |b(y)f(y+h) + (1-b(y))f(y) - b(x)f(x+h) - (1-b(x))f(x)| \\
&\leqslant b(x)|f(y+h) - f(x+h)| + [1-b(x)]|f(y) - f(x)| \\
&\quad + |b(y) - b(x)|\,|f(y+h) - f(y)| \\
&\leqslant \operatorname{Lip} f|y-x| + (\operatorname{Lip} b)|y-x|(\operatorname{Lip} f)h \\
&= \operatorname{Lip} f|y-x|(1 + h \operatorname{Lip} b).
\end{aligned}$$

If P_t is the semigroup corresponding to X_t, $P_t = e^{-Mt}\sum Q^k(Mt)^k/k!$, or Lip $P_t \leqslant \exp(Mth \operatorname{Lip} b)$, or Lip-exp $P_t \leqslant Mh \operatorname{Lip} b = h \operatorname{Lip} a$. □

We can then get the following theorem:

THEOREM 7.2. *Suppose A is given by* (6.3), *where m is any kernel that satisfies* (i) $\int (y-x)^2 m(x, dy)$ *is uniformly bounded*; (ii) $\int_{|y-x|>\varepsilon}(y-x)m(x, dy) \to 0$ *uniformly*; *and* (iii) *there is a number $M > 0$ such that for any interval $[\alpha, \beta]$ that does not contain* 0, $\operatorname{Lip}(\int_{\alpha+x}^{\beta+x}(y-x)m(x, dy)) \leqslant M$. *Then the semigroup generated by A will be Lipschitz.*

PROOF. If we define $m_\varepsilon(x, dy) = 1_{S_\varepsilon^c(x)}(y)m(x, dy)$, and define A_ε in terms of m_ε by (6.3), then just as in the proof of 6.5, we can show that we need only consider the case where m is uniformly bounded.

If we let $h(x) = \int (y-x)m(x, dy)$, and $a(x) = h(x)/\|h\|$, define $m_n(x, dy) = m(x, dy) + nh(x)\varepsilon_{x+1/n}(dy)$ if h is positive, $m_n(x, dy) = m(x, dy) + nh(x)\varepsilon_{x-1/n}(dy)$ if h is negative. If we define A_n in terms of m_n by (6.2), then $A_n g \to Ag$ whenever $g \in C_0^2$, and so by 2.6, we need only consider the case where m is uniformly bounded, and A is given by (6.2).

Finally, approximate m by m_k where

$$m_k(x, dy) = \sum_{j=-k^2}^{k^2} \varepsilon_{j/k}(dy)m(x, [j/k, (j+1)/k)).$$

By 6.4, if the semigroups generated by A_k, where the A_k are defined by (6.2), have a uniformly bounded Lipschitz exponent, we will have our result. That they do follows from hypothesis (iii), 7.1, and 2.7. □

Consider operators that are of the form described in either 4.2, 5.1, 6.5, or 7.2. If A is the sum of such operators, then by 2.7 A restricted to C_0^2 will have an extension that generates a Lipschitz semigroup.

We give an example to show that not every semigroup is Lipschitz. Let P_t be the semigroup whose infinitesimal generator A is given by $Af(x) = f(x+1) - f(x)$ if $x \geqslant 0$, $f(x-1) - f(x)$ if $x < 0$. The Markov process corresponding to P_t behaves like the standard Poisson process starting at x if $x \geqslant 0$ and behaves like the negative of the Poisson process starting at x if $x < 0$. If $f(x) = x/(1 + |x|^2)$ and $t > 0$, it is easy to check that $\inf_{x\geqslant 0} P_t f(x) > 0$ and $\sup_{x<0} P_t f(x) < 0$. Thus $P_t f$ is not even continuous at 0, let alone Lipschitz there.

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