

# Law of the Iterated Logarithm for Set-indexed Partial Sum Processes with Finite Variance

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## 1. Introduction

Let  $X_j$ ,  $j \in \{1, 2, \dots\}^d$  be independent, identically distributed random variables that are mean 0 with variance 1. If  $\mathcal{A}$  is a collection of Borel subsets of  $[0, 1]^d$ , define the partial sum process  $\{S_n(B); B \in \mathcal{A}\}$  by

$$S_n(B) = \sum_{j \in nB} X_j.$$

The purpose of this paper is to prove a functional law of the iterated logarithm (LIL) for a suitably smoothed and normalized version of  $S_n$ .

The first functional LIL is due to Strassen [8]. If  $B_t$  is one-dimensional Brownian motion, he showed that the set of functions  $\beta_t(\cdot)$  defined by

$$\beta_t(s) = B_{st}/(2t \log \log t)^{\frac{1}{2}}, \quad 0 \leq s \leq 1,$$

is relatively compact in the uniform topology and the set of limit points is the set of functions which are absolutely continuous and whose derivatives have  $L_2$  norm less than or equal to 1. Using Skorokhod embedding, Strassen showed that random walks also obeyed a functional LIL and that the usual LIL, as well as many other interesting results, could be obtained as a consequence.

The extension of Strassen's results to Brownian motion indexed by points in  $\mathbb{R}_+^d$  and to partial sum processes indexed by points in  $\mathbb{R}_+^d$  was done by Pyke [7] and Wichura [9], respectively. For a Brownian motion indexed by a large collection  $\mathcal{A}$  of sets, the functional law was proved by Bass and Pyke [2], and then a type of Skorokhod embedding was used to extend this for partial sum processes. However, if  $H(\delta)$  is the log-entropy of  $\mathcal{A}$  (defined in Sect. 2), it was necessary that  $H(\delta) \leq \delta^{-r}$  for some  $r < 1$  and that the  $X_j$ 's have at least  $(2 + \varepsilon)/(1 - r)$  moments for some  $\varepsilon > 0$ . Moreover, the sets in  $\mathcal{A}$  had to have uniformly smooth boundaries. Morrow and Philipp [6] were able to wea-

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\* Research partially supported by NSF Grant MCS 83-00581

ken the smooth boundary condition somewhat, but required the  $X_j$ 's to have at least  $(2 + \varepsilon)(1 + r)/(1 - r)$  moments.

In this paper we prove a functional LIL that requires only finite variance for the  $X_j$ 's. Our restriction on  $\mathcal{A}$  is that its log-entropy with inclusion,  $H$ , satisfy

$$(1.1) \quad \int_0^1 (H(x)/x)^{\frac{1}{2}} dx < \infty.$$

That is the weakest one could expect, since (1.1) (using entropy without inclusion) is the condition usually used to show that the Brownian motion indexed by  $\mathcal{A}$  has sample continuous paths [5]. Furthermore, no smoothness whatsoever is required on the boundaries of the sets in  $\mathcal{A}$ . When we take  $d = 1$ ,  $\mathcal{A} = \{[0, t]: 0 \leq t \leq 1\}$ , we get Strassen's LIL, while  $d \geq 2$ ,  $\mathcal{A} = \{[0, t]: t \in [0, 1]^d\}$  yields Wichura's results for partial sums of iid random variables.

The key to our method is a modification of the metric entropy technique. Instead of truncating the  $X_j$ 's once at the beginning and then doing the chaining argument, we perform a truncation at each stage of the chaining. This gives us an exponential estimate, and then more or less standard techniques are used to get our functional LIL. Some simplifications due to de Acosta [4] are used in this latter stage of the argument.

Our methods can also be used to prove the central limit theorem for set-indexed partial sum processes with finite variance. This gives a new proof of the recent result of Alexander and Pyke [1], which was done by completely different methods.

In Sect. 2 we introduce notation and state our assumptions and results. Section 3 contains the entropy argument. Section 4 gives the proof of relative compactness, while Sect. 5 identifies the limit points. A sketch of the proof of the central limit theorem is given in Sect. 6.

## 2. Statement of Results

Let  $I^d = [0, 1]^d$ . If  $A$  is any Borel set, let  $|A|$  denote the Lebesgue measure of  $A$ , and let  $rA = \{rt: t \in A\}$ . For  $j = (j_1, \dots, j_d)$ , let

$$R_j = (j_1 - 1, j_1] \times \dots \times (j_d - 1, j_d],$$

the cube with upper corner at  $j$ . Suppose  $X_j$ ,  $j \in \{1, 2, \dots\}^d$  are iid random variables with mean 0 and variance 1. Our smoothed and normalized version of  $S_n$  will be  $Z_n$ , given by

$$(2.1) \quad Z_n(B) = n^{-d/2} \sum_j |nB \cap R_j| X_j.$$

In the case where  $d = 1$  and we identify the point  $t$  with the set  $[0, t]$ ,  $n^{d/2} Z_n(t)$  is just the usual process obtained by linearly interpolating between the points  $\{(i, S_i): i \leq n\}$ . As a function of  $B$ ,  $Z_n$  is well defined for all Borel sets  $B \subseteq I^d$  and for each  $n$  is uniformly continuous with respect to the metric  $d(A, B) = |A \Delta B|$ .

Let

$$LL\ n = \begin{cases} \log \log n & \text{if } n \geq 16 \\ 1 & \text{if } n \leq 15. \end{cases}$$

Let  $\varphi_n = (2LL\ n)^{1/2}$ .

We will suppose that  $\mathcal{A}$  is a collection of Borel subsets of  $I^d$  that is totally bounded with inclusion:

(2.2) for each  $\delta$ , there exists a finite subcollection  $\mathcal{A}(\delta)$  of  $\mathcal{A}$  with the following property:

if  $A \in \mathcal{A}$ , there exists  $B, B^+$  (depending on  $A$ ) in  $\mathcal{A}(\delta)$  such that  $B \subseteq A \subseteq B^+$  and  $|B^+ - B| \leq \delta$ .

For each  $\delta$ , let  $H(\delta)$ , the log-entropy, be the log of the cardinality of the smallest such subcollection  $\mathcal{A}(\delta)$ .

We suppose  $\mathcal{A}$  is entropy-integrable:

(2.3) 
$$\int_0^1 (H(x)/x)^{\frac{1}{2}} dx < \infty.$$

For any  $F: \mathcal{A} \rightarrow \mathbb{R}$ , let

$$\|F\|_{\mathcal{A}} = \sup_{B \in \mathcal{A}} |F(B)|.$$

Let  $\mathcal{K}$  be the subset of the functions from  $\mathcal{A}$  to  $\mathbb{R}$  given by

(2.4) 
$$\mathcal{K} = \{F: \text{for some } f: I^d \rightarrow \mathbb{R} \text{ with } \int_{I^d} f^2(t) dt \leq 1, F(B) = \int_B f(t) dt \text{ for all } B \in \mathcal{A}\}.$$

Given  $F \in \mathcal{K}$ , we extend the domain of  $F$  to all Borel subsets  $B$  of  $I^d$  by defining

$$F(B) = \int_B f(t) dt.$$

We can now state our theorem.

**Theorem 2.1.**  $\{Z_n/\varphi_n\}$  is relatively compact with respect to the metric  $\|\cdot\|_{\mathcal{A}}$ , and the set of limit points is exactly  $\mathcal{K}$ , a.s.

Before proceeding to the proof of Theorem 2.1, we introduce a bit more notation and make some observations that will be useful later.

Let  $\mathcal{K}^\varepsilon$  be the subset of the functions from  $\mathcal{A}$  to  $\mathbb{R}$  given by

(2.5) 
$$\mathcal{K}^\varepsilon = \{F: \text{for some } F_0 \in \mathcal{K}, \|F - F_0\|_{\mathcal{A}} \leq \varepsilon\}.$$

By Cauchy-Schwartz, if  $F \in \mathcal{K}$ ,  $B$  Borel,

(2.6) 
$$|F(B)| = \left| \int_B f(t) dt \right| \leq \left( \int_{I^d} 1_B(t) dt \right)^{\frac{1}{2}} \left( \int_{I^d} f^2(t) dt \right)^{\frac{1}{2}} \leq |B|^{\frac{1}{2}},$$

and so if  $F \in \mathcal{K}^\varepsilon$ ,  $B \in \mathcal{A}$ ,

(2.7) 
$$|F(B)| \leq \varepsilon + |B|^{\frac{1}{2}}.$$

Throughout,  $c$ , with or without subscripts, will denote a constant whose value may change from place to place.

For the rest of the paper, it will be technically convenient to assume  $\mathcal{A}$  is contraction closed:

$$(2.8) \quad \text{if } B \in \mathcal{A}, 0 \leq r \leq 1, \text{ then } rB \in \mathcal{A}.$$

There is no loss of generality in making this assumption, because if  $\mathcal{A}^* = \{rB : B \in \mathcal{A}, 0 \leq r < 1\}$  has log-entropy  $H^*(\delta)$ , then replacing  $\mathcal{A}$  in Theorem 2.1 by the larger  $\mathcal{A}^*$  only creates a better theorem, while

**Lemma 2.2.**  $\int_0^1 (H^*(x)/x)^{\frac{1}{2}} dx < \infty.$

*Proof.* Let  $\mathcal{A}^*(2\delta) = \{sB : s/\delta \text{ is an integer, } 0 \leq s \leq 1, B \in \mathcal{A}(\delta)\}$ . It is easy to check that if  $rA \in \mathcal{A}^*$ , there exist  $B, B^+ \in \mathcal{A}^*(2\delta)$  with  $B \subseteq rA \subseteq B^+$ ,  $|B^+ - B| \leq 2\delta$  and that  $H^*(2\delta) \leq 2 \log \delta^{-1} + H(\delta)$ . The result follows.  $\square$

The following lemma will make working with (2.3) easier.

**Lemma 2.3.** *Suppose  $G(x)$  is a strictly decreasing continuous function with  $G(x) \geq x^{-\frac{1}{2}}$  and  $\int_0^1 (G(x)/x)^{\frac{1}{2}} dx < \infty$ . Suppose  $x_0 < \frac{1}{2}$ , and define  $x_i$  to satisfy*

$$G(x_i)/x_i = (G(x_0)/x_0) 2^i, \quad i = 1, 2, \dots$$

Then

- (a)  $x_i/x_{i+1} \leq 2$ ;
- (b)  $G(x_{i+1})/G(x_i) \leq 2$ ;
- (c)  $\sum_{i=0}^{\infty} (x_i G(x_i))^{\frac{1}{2}} \leq 5 \int_0^1 (G(x)/x)^{\frac{1}{2}} dx$ ; and
- (d) if  $\zeta \leq \frac{1}{2}$ ,  $\sum_{i=0}^{\infty} \zeta^{+x_i^{-\frac{1}{2}}} \leq 15 \zeta \int_0^1 (G(x)/x)^{\frac{1}{2}} dx$ .

*Proof.* Since  $G(x)/x$  is strictly decreasing and  $G(x) \geq x^{-\frac{1}{2}}$ ,  $x_i$  decreases to 0. Then

$$x_i/x_{i+1} = \frac{x_i}{G(x_i)} \frac{G(x_{i+1})}{x_{i+1}} \frac{G(x_i)}{G(x_{i+1})} \leq 2,$$

and

$$G(x_{i+1})/G(x_i) = \frac{G(x_{i+1})}{x_{i+1}} \frac{x_i}{G(x_i)} \frac{x_{i+1}}{x_i} \leq 2.$$

Observe that

$$(2.9) \quad (x_n G(x_n))^{\frac{1}{2}} = x_n (G(x_n)/x_n)^{\frac{1}{2}} \leq \int_0^{x_n} (G(x)/x)^{\frac{1}{2}} dx \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and

$$(2.10) \quad (G(x_i)/x_i)^{\frac{1}{2}} - (G(x_{i-1})/x_{i-1})^{\frac{1}{2}} = (G(x_0)/x_0)^{\frac{1}{2}} (2^{i/2} - 2^{(i-1)/2}) \geq (G(x_i)/x_i)^{\frac{1}{2}}/5.$$

Using (2.10) and summing by parts, we have

$$\begin{aligned} \int_0^1 (G(x)/x)^{\frac{1}{2}} &\geq \sum_{i=0}^{\infty} \int_{x_{i+1}}^{x_i} (G(x)/x)^{\frac{1}{2}} dx \geq \sum_{i=0}^n (x_i - x_{i+1}) (G(x_i)/x_i)^{\frac{1}{2}} \\ &= x_0 (G(x_0)/x_0)^{\frac{1}{2}} + \sum_{i=1}^n x_i [(G(x_i)/x_i)^{\frac{1}{2}} - (G(x_{i-1})/x_{i-1})^{\frac{1}{2}}] - x_{n+1} (G(x_n)/x_n)^{\frac{1}{2}} \\ &\geq \sum_{i=0}^n (x_i G(x_i))^{\frac{1}{2}} / 5 - (x_n G(x_n))^{\frac{1}{2}}. \end{aligned}$$

This with (2.9) gives (c).

Since  $(x^{-\frac{1}{2}} - 1) |\log \zeta| \geq x^{-\frac{1}{2}} / 10 \geq |\log x| / 4 - \log 3$  for  $\zeta, x \leq \frac{1}{2}$ ,

$$\sum_{i=0}^{\infty} \zeta^{x_i^{-\frac{1}{2}}} \leq 3 \zeta \sum_{i=0}^{\infty} x_i^{\frac{1}{2}} \leq 3 \zeta \sum_{i=0}^{\infty} (x_i G(x_i))^{\frac{1}{2}},$$

and (d) follows.  $\square$

A comment about measurability: since by (2.2)  $\mathcal{A}$  is totally bounded, then  $\mathcal{A}$  is separable. Since for each  $n, Z_n$  is uniformly continuous with respect to the metric  $d(A, B) = |A \Delta B|$ , it follows that  $Z_n^{-1}(\mathcal{B}) \subseteq \sigma(X_j : j \in \{1, 2, \dots\}^d)$ , where  $\mathcal{B}$  is the Borel  $\sigma$ -field on  $\mathcal{C}(\mathcal{A})$ , the set of uniformly continuous functions on  $\mathcal{A}$  with norm  $\|\cdot\|_{\mathcal{A}}$ . A similar comment applies to each of the other processes we will define.

### 3. Entropy

This section gives an exponential bound for partial sums of truncated random variables. For  $0 \leq a \leq b \leq \infty$ , define

$$(3.1) \quad X_j(n, a, b) = X_j 1_{(an^{d/2}/\varphi_n \leq |X_j| < bn^{d/2}/\varphi_n)},$$

$$(3.2) \quad Z_n(A, a, b) = n^{-d/2} \sum_{j \in \{1, 2, \dots\}^d} |nA \cap R_j| (X_j(n, a, b) - EX_j(n, a, b)),$$

and

$$(3.3) \quad U_n(A, a, b) = n^{-d/2} \sum_{j \in \{1, 2, \dots\}^d} |nA \cap R_j| |X_j(n, a, b)|.$$

Note  $Z_n(A, a, b)$  is mean 0, and if  $A$  and  $B$  are any two sets,

$$\begin{aligned} (3.4) \quad \text{Var}(Z_n(A, a, b) - Z_n(B, a, b)) &\leq n^{-d} \sum_j (|nA \cap R_j| - |nB \cap R_j|)^2 \text{Var} X_j \\ &\leq n^{-d} \sum_j |nA \Delta B \cap R_j| = |A \Delta B|, \end{aligned}$$

since  $||nA \cap R_j| - |nB \cap R_j|| \leq |n(A \Delta B) \cap R_j| \leq 1$ . Note also that  $Z_n(A, a, b) - Z_n(B, a, b)$  is the sum of independent random variables, each of which is bounded by  $2b/\varphi_n$ .

A similar calculation gives

$$(3.5) \quad \text{Var} U_n(A, a, b) \leq |A|,$$

while

$$E|X_{\mathcal{J}}(n, a, b)| \leq EX_{\mathcal{J}}^2/(an^{d/2}/\varphi_n),$$

and hence

$$(3.6) \quad EU_n(A, a, b) \leq n^{-d/2} \sum_{\mathcal{J}} |nA \cap R_{\mathcal{J}}| \varphi_n/a n^{d/2} = \varphi_n|A|/a.$$

Our estimate is the following.

**Theorem 3.1.** *Given  $\lambda, \delta,$  and  $\varepsilon > 0,$  there exists constants  $c, \alpha,$  and  $n_0$  (depending on  $\lambda, \delta,$  and  $\varepsilon$ ) such that if  $a \leq \alpha$  and  $n \geq n_0,$*

$$P\left(\sup_{B, C \in \mathcal{A}, |B \Delta C| \leq \delta} |Z_n(B, 0, a) - Z_n(C, 0, a)| > \lambda \varphi_n\right) \leq c \exp\left(\frac{-\lambda^2 \varphi_n^2}{(2 + \varepsilon) \delta}\right).$$

*Proof.* We will select numbers  $\bar{\lambda}, \lambda_i, \delta_i, a_i, \alpha$  later so that  $\delta_i, a_i, \lambda_i \downarrow 0, \alpha < a_0$  and

$$(3.7) \quad \lambda \geq \bar{\lambda} + 4 \sum_{i=0}^{\infty} \lambda_i.$$

If  $B$  and  $C$  are any sets in  $\mathcal{A},$  there exists sets  $B_i, B_i^+, C_i, C_i^+$  in  $\mathcal{A}(\delta_i)$  such that  $B_i \subseteq B \subseteq B_i^+$  and  $|B_i^+ - B_i| \leq \delta_i$  and similarly for  $C, C_i, C_i^+,$  for each  $i.$  Since for each  $a \leq \alpha$  and each  $n, Z_n(A, 0, a)$  is a continuous function of  $A$  with respect to  $d(A, B) = |A \Delta B|,$  we can write

$$\begin{aligned} Z_n(B, 0, a) &= Z_n(B_0, 0, a) + \sum_{i=0}^{\infty} [Z_n(B_{i+1}, 0, a_i \wedge a) - Z_n(B_i, 0, a_i \wedge a)] \\ &\quad + \sum_{i=1}^{\infty} [Z_n(B, a_i \wedge a, a_{i-1} \wedge a) - Z_n(B_i, a_i \wedge a, a_{i-1} \wedge a)], \end{aligned}$$

and similarly for  $Z_n(C, 0, a),$  provided  $\alpha < a_0.$  (It will be apparent from the remainder of the proof that both series are absolutely convergent, a.s.) Then the only way  $Z_n(C, 0, a) - Z_n(B, 0, a)$  can be larger than  $\lambda \varphi_n$  for some  $B$  and  $C$  in  $\mathcal{A}$  with  $|C \Delta B| \leq \delta$  is if at least one of the following holds:

(i) for some  $B_0, C_0 \in \mathcal{A}(\delta_0)$  with  $B_0 \subseteq B, C_0 \subseteq C, |B - B_0| \leq \delta_0, |C - C_0| \leq \delta_0,$

$$|Z_n(C_0, 0, a) - Z_n(B_0, 0, a)| \geq \bar{\lambda} \varphi_n;$$

(ii) for some  $i,$  for some  $B_i \in \mathcal{A}(\delta_i), B_{i+1} \in \mathcal{A}(\delta_{i+1}), |B_{i+1} \Delta B_i| \leq 2\delta_i,$

$$|Z_n(B_{i+1}, 0, a_i \wedge a) - Z_n(B_i, 0, a_i \wedge a)| \geq \lambda_i \varphi_n;$$

(iii) same as (ii) with  $B$ 's replaced by  $C$ 's;

(iv) for some  $i,$  for some  $B_i, B_i^+ \in \mathcal{A}(\delta_i), B \in \mathcal{A}, B_i \subseteq B \subseteq B_i^+, |B_i^+ - B_i| \leq \delta_i,$

$$|Z_n(B, a_i \wedge a, a_{i-1} \wedge a) - Z_n(B_i, a_i \wedge a, a_{i-1} \wedge a)| \geq \lambda_i \varphi_n;$$

or (v) same as (iv) with  $B$ 's replaced by  $C$ 's.

The number of pairs  $B_i, B_i^+$  in  $\mathcal{A}(\delta_i)$  is  $\leq \exp(2H(\delta_i)),$  while the number of pairs  $B_i \in \mathcal{A}(\delta_i), B_{i+1} \in \mathcal{A}(\delta_{i+1})$  is  $\leq \exp(2H(\delta_{i+1})),$  since  $H(x)$  increases as  $x$  decreases.

We then have

$$(3.8) \quad P\left(\sup_{B, C \in \mathcal{A}, |B \Delta C| \leq \delta} |Z_n(C, 0, a) - Z_n(B, 0, a)| > \lambda \varphi_n\right) \leq p_0 + \sum_{i=0}^{\infty} r_i + \sum_{i=1}^{\infty} s_i,$$

where

$$p_0 = \exp(2H(\delta_0)) \max_{B_0, C_0 \in \mathcal{A}(\delta_0), |C_0 \Delta B_0| \leq \delta + 2\delta_0} P(|Z_n(C_0, 0, a) - Z_n(B_0, 0, a)| \geq \bar{\lambda} \varphi_n),$$

$$r_i = 2 \exp(2H(\delta_{i+1})) \max_{\substack{B_i \in \mathcal{A}(\delta_i), B_{i+1} \in \mathcal{A}(\delta_{i+1}) \\ |B_i \Delta B_{i+1}| \leq 2\delta_i}} P(|Z_n(B_{i+1}, 0, a_i \wedge a) - Z_n(B_i, 0, a_i \wedge a)| \geq \lambda_i \varphi_n),$$

and

$$s_i \leq 2 \exp(2H(\delta_i)) \max_{B_i, B_i^+ \in \mathcal{A}(\delta_i), |B_i^+ - B_i| \leq \delta_i} P\left(\sup_{B_i \subseteq B \subseteq B_i^+} |Z_n(B, a_i \wedge a, a_{i-1} \wedge a) - Z_n(B_i, a_i \wedge a, a_{i-1} \wedge a)| \geq \lambda_i \varphi_n\right).$$

Here  $p_0$  bounds the terms in (i),  $r_i$  the terms in (ii) and (iii), and  $s_i$  the terms in (iv) and (v).

We now proceed to estimate  $p_0$ ,  $r_i$ , and  $s_i$ . Recalling (3.4) and the fact that  $Z_n(A, 0, a) - Z_n(B, 0, a)$  is the sum of terms bounded by  $2a/\varphi_n$ , we have by Bernstein's inequality (see [3]),

$$(3.9) \quad p_0 \leq 2 \exp(2H(\delta_0)) \exp\left(\frac{-\bar{\lambda}^2 \varphi_n^2}{2(\delta + 2\delta_0) + 2(2a/\varphi_n)(\bar{\lambda} \varphi_n/3)}\right) \\ = 2 \exp(2H(\delta_0)) \exp(-\bar{\lambda}^2 \varphi_n^2 / (2\delta + 4\delta_0 + 4a\bar{\lambda}/3)).$$

Similarly,

$$(3.10) \quad r_i \leq 4 \exp(2H(\delta_{i+1})) \exp(-\lambda_i^2 \varphi_n^2 / (4\delta_i + 4a_i \lambda_i/3)).$$

To estimate  $s_i$ , note that

$$\sup_{B_i \subseteq B \subseteq B_i^+} |Z_n(B, a_i \wedge a, a_{i-1} \wedge a) - Z_n(B_i, a_i \wedge a, a_{i-1} \wedge a)| \\ \leq U_n(B_i^+ - B_i, a_i, a_{i-1}) + EU_n(B_i^+ - B_i, a_i, a_{i-1}).$$

By the definition of  $U_n$ ,  $U_n(B_i^+ - B_i, a_i, a_{i-1})$  is the sum of independent random variables each of which is bounded by  $a_{i-1}/\varphi_n$ , and by (3.6), if  $|B_i^+ - B_i| \leq \delta_i$ ,

$$EU_n(B_i^+ - B_i, a_i, a_{i-1}) \leq \varphi_n \delta_i / a_i,$$

Then, provided

$$(3.11) \quad \lambda_i \varphi_n \geq 3 \varphi_n \delta_i / a_i,$$

by Bernstein's inequality,

$$(3.12) \quad s_i \leq 2 \exp(2H(\delta_i)) \max_{B_i, B_i^+ \in \mathcal{A}(\delta_i), |B_i^+ - B_i| \leq \delta_i} P(U_n(B_i^+ - B_i, a_i, a_{i-1}) - EU_n(B_i^+ - B_i, a_i, a_{i-1}) \geq \lambda_i \varphi_n/3) \\ \leq 2 \exp(2H(\delta_i)) \exp\left(\frac{-\lambda_i^2 \varphi_n^2/9}{2\delta_i + 2(a_{i-1}/\varphi_n)(\lambda_i \varphi_n/3)/3}\right) \\ = 2 \exp(2H(\delta_i)) \exp(-\lambda_i^2 \varphi_n^2 / (18\delta_i + 2a_{i-1} \lambda_i)).$$

We now choose  $\delta_i, \lambda_i, a_i, \bar{\lambda}$ , and  $\alpha$ . Let  $G(x)$  be a continuous, strictly decreasing function so that  $G(x) \geq x^{-\frac{1}{2}}$ ,  $G(x) \geq H(x)$ , but

$$\gamma = \int_0^1 (G(x)/x)^{\frac{1}{2}} dx < \infty.$$

For  $x \leq 1$ , note that  $G(x) \geq x^{-\frac{1}{2}}$ . Take  $\bar{\lambda}$  less than but sufficiently close to  $\lambda$  so that  $\bar{\lambda}^2 \geq \lambda^2/(1 + \varepsilon/4)$ . Let  $\beta = (\lambda - \bar{\lambda})/20\gamma$ . Choose  $\delta_0 < 1$  small enough so that  $\delta_0 < \varepsilon\delta/8$  and  $\delta_0^{-\frac{1}{2}}\beta^2/60 \geq \lambda^2/2\delta$ . Let  $\delta_i$  be defined by  $G(\delta_i)/\delta_i = (G(\delta_0)/\delta_0)2^i$ , and let

$$\lambda_i = (\delta_i G(\delta_i))^{\frac{1}{2}} \beta, \quad i = 0, 1, \dots$$

Let  $a_i = 3\delta_i/\lambda_i, i = 0, 1, \dots$ . Take  $\alpha = \min(a_0/2, \varepsilon\delta/4(1 + \lambda))$ .

By Lemma 2.3(c), (3.7) is satisfied. By the definition of  $a_i$ , (3.11) is satisfied. From our choices of  $\bar{\lambda}, \alpha$ , and  $\delta_0$  and (3.9) we get

$$(3.13) \quad p_0 \leq c \exp(-\lambda^2 \varphi_n^2 / (2 + \varepsilon) \delta)$$

provided  $a \leq \alpha$ . By Lemma 2.3(b),  $H(\delta_{i+1}) \leq G(\delta_{i+1}) \leq 2G(\delta_i)$ . Since  $\lambda_i^2 \varphi_n^2 / \delta_i = G(\delta_i) \beta^2 \varphi_n^2 \geq 64G(\delta_i)$  for  $n$  sufficiently large, we have

$$\begin{aligned} r_i &\leq 4 \exp(-\lambda_i^2 \varphi_n^2 / 8\delta_i + 4G(\delta_i)) \\ &\leq 4 \exp(-G(\delta_i) \beta^2 \varphi_n^2 / 16) \\ &\leq 4 \exp(-\delta_i^{-\frac{1}{2}} \lambda^2 \varphi_n^2 / 2\delta) \end{aligned}$$

for  $n$  sufficiently large, uniformly in  $i$ . By Lemma 2.3(d) with  $\zeta = \exp(-\lambda^2 \varphi_n^2 / 2\delta)$ , we have

$$(3.14) \quad \sum_{i=0}^{\infty} r_i \leq c \exp(-\lambda^2 \varphi_n^2 / 2\delta).$$

By our choice of  $\delta_i$  and  $\lambda_i, a_{i-1}/a_i = 2^{\frac{1}{2}}$ , so

$$\begin{aligned} s_i &\leq 2 \exp(-\lambda_i^2 \varphi_n^2 / 30\delta_i + 2G(\delta_i)) \\ &\leq 2 \exp(-G(\delta_i) \beta^2 \varphi_n^2 / 60) \\ &\leq 2 \exp(-\delta_i^{-\frac{1}{2}} \lambda^2 \varphi_n^2 / 2\delta) \end{aligned}$$

for  $n$  sufficiently large, uniformly in  $i$ . By Lemma 2.3(d) again,

$$(3.15) \quad \sum_{i=0}^{\infty} s_i \leq c \exp(-\lambda^2 \varphi_n^2 / 2\delta).$$

Substituting (3.13), (3.14), and (3.15) in (3.8) proves the theorem.  $\square$

#### 4. Relative Compactness

We now proceed to prove part of Theorem 2.1, that  $Z_n/\varphi_n$  has limit points, all contained in  $\mathcal{X}$ . We separate out some preliminary propositions for convenience. First we have

**Proposition 4.1.** *For each  $a$ ,*

$$\|Z_n(\cdot, a, \infty)/\varphi_n\|_{\mathcal{A}} \rightarrow 0 \text{ a.s. as } n \rightarrow \infty.$$

*Proof.* Since for each  $A$ ,

$$|Z_n(A, a, \infty)| \leq U_n(I^d, a, \infty) + EU_n(I^d, a, \infty),$$

it suffices to show

$$(4.1) \quad U_n(I^d, a, \infty)/\varphi_n \rightarrow 0, \text{ a.s.}$$

and

$$(4.2) \quad EU_n(I^d, a, \infty)/\varphi_n \rightarrow 0.$$

If  $\mathcal{j} = (j_1, \dots, j_d)$ , let  $\mu(\mathcal{j}) = \max(j_1, \dots, j_d)$ . Given  $i$  and  $a$ , let

$$\psi(i, a) = \sup\{k \text{ integer: } a k^{d/2}/\varphi_k < i + 1\}.$$

Since the number of  $\mathcal{j}$ 's in  $\{1, 2, \dots\}^d$  with  $\mu(\mathcal{j}) = k$  is less than or equal to  $c k^{d-1}$  for a constant  $c$ , we have

$$(4.3) \quad \begin{aligned} \sum_{\mathcal{j}} 1_{(i+1 > a\mu(\mathcal{j})^{d/2}/\varphi_{\mu(\mathcal{j})})} (\mu(\mathcal{j})^{d/2} \varphi_{\mu(\mathcal{j})})^{-1} &= \sum_k \sum_{\mathcal{j}: \mu(\mathcal{j})=k} 1_{(i+1 > ak^{d/2}/\varphi_k)} (k^{d/2} \varphi_k)^{-1} \\ &\leq c \sum_k 1_{(i+1 > ak^{d/2}/\varphi_k)} (k^{d/2-1}/\varphi_k) \\ &\leq c \psi(i, a)^{d/2} / \varphi_{\psi(i, a)} \\ &\leq c(i+1)/a. \end{aligned}$$

We see from this that

$$(4.4) \quad \begin{aligned} \sum_{\mathcal{j}} E|X_{\mathcal{j}}(\mu(\mathcal{j}), a, \infty)|/\mu(\mathcal{j})^{d/2} \varphi_{\mu(\mathcal{j})} &\leq \sum_{\mathcal{j}} \sum_{i+1 > a\mu(\mathcal{j})^{d/2}/\varphi_{\mu(\mathcal{j})}} (i+1) P(i \leq |X_{\mathcal{j}}| \leq i+1) (\mu(\mathcal{j})^{d/2} \varphi_{\mu(\mathcal{j})})^{-1} \\ &\leq \sum_{i=0}^{\infty} [\sum_{\mathcal{j}} 1_{(i+1 > a\mu(\mathcal{j})^{d/2}/\varphi_{\mu(\mathcal{j})})} (\mu(\mathcal{j})^{d/2} \varphi_{\mu(\mathcal{j})})^{-1}] (i+1) P(i \leq |X_1| \leq i+1) \\ &\leq (c/a) \sum_{i=0}^{\infty} (i+1)^2 P(i \leq |X_1| \leq i+1) \leq (c/a) E(|X_1| + 1)^2 < \infty, \end{aligned}$$

where we use the fact that the  $X_{\mathcal{j}}$ 's are iid,  $X_{\mathcal{j}}(\mu(\mathcal{j}), a, \infty)$  is defined by (3.1), and  $1 = (1, 1, \dots, 1)$ .

An immediate consequence of (4.4) is that

$$(4.5) \quad \sum_{\mathcal{j}} |X_{\mathcal{j}}(\mu(\mathcal{j}), a, \infty)|/\mu(\mathcal{j})^{d/2} \varphi_{\mu(\mathcal{j})} < \infty, \text{ a.s.}$$

Given  $\varepsilon$ , pick  $n_1(\omega)$  such that

$$\sum_{\mu(\mathcal{j}) > n_1} |X_{\mathcal{j}}(\mu(\mathcal{j}), a, \infty)|/\mu(\mathcal{j})^{d/2} \varphi_{\mu(\mathcal{j})} < \varepsilon.$$

Then

$$U_n(I^d, a, \infty)/\varphi_n \leq \sum_{\mu(\mathcal{J}) \leq n_1} |X_{\mathcal{J}}(n, a, \infty)|/n^{d/2} \varphi_n + \sum_{n \geq \mu(\mathcal{J}) > n_1} |X_{\mathcal{J}}(\mu(\mathcal{J}), a, \infty)|/\mu(\mathcal{J})^{d/2} \varphi_{\mu(\mathcal{J})},$$

or  $\limsup_{n \rightarrow \infty} U_n(I^d, a, \infty)/\varphi_n \leq 2\varepsilon$ . Since  $\varepsilon$  is arbitrary, this proves (4.1).

Just as (4.1) follows from (4.5), we see that (4.2) follows from (4.4), and the proof is complete.  $\square$

For notational convenience, given  $m$  and  $\ell \in \{1, \dots, m\}^d$ , let

$$(4.6) \quad Q_{\ell m} = m^{-1} R_{\ell}.$$

If  $n$  is a multiple of  $m$ , let

$$(4.7) \quad Y_{nm}(A, a, b) = \sum_{\ell \in \{1, \dots, m\}^d} \frac{|A \cap Q_{\ell m}|}{|Q_{\ell m}|} Z_n(Q_{\ell m}, a, b).$$

**Proposition 4.2.** *Given  $1/8 > \varepsilon > 0$ ,  $m$  fixed, and  $n$  a multiple of  $m$ , there exists  $\alpha$  such that if  $a < \alpha$  and  $n$  is sufficiently large,*

$$P(Y_{nm}(\cdot, 0, a)/\varphi_n \notin \mathcal{K}^{4\varepsilon}) \leq c \exp(-(1 + \varepsilon) \varphi_n^2/2).$$

*Proof.* If we let

$$y_{nm}(\ell) = \sum_{\ell \in \{1, \dots, m\}^d} |Q_{\ell m}|^{-1} Z_n(Q_{\ell m}, 0, a) 1_{Q_{\ell m}}(\ell),$$

we see that for all  $A$

$$Y_{nm}(A, 0, a) = \int_A y_{nm}(\ell) d\ell.$$

So to show  $Y_{nm}(\cdot, 0, a)/\varphi_n \in \mathcal{K}^{4\varepsilon}$ , we need to consider

$$(4.8) \quad \varphi_n^{-2} \int_{I^d} (y_{nm}(\ell))^2 d\ell = \varphi_n^{-2} \sum_{\ell \in \{1, \dots, m\}^d} |Z_n(Q_{\ell m}, 0, a)|^2 / |Q_{\ell m}|.$$

Let

$$T_{n\ell} = |Z_n(Q_{\ell m}, 0, a)|^2 / |Q_{\ell m}|,$$

and let

$$\bar{T}_{n\ell} = \begin{cases} T_{n\ell} & \text{if } T_{n\ell} \leq (1 + 2\varepsilon) \varphi_n^2 \\ 0 & \text{otherwise.} \end{cases}$$

By Bernstein's inequality,

$$(4.9) \quad \begin{aligned} P(\bar{T}_{n\ell} \neq T_{n\ell} \text{ for some } \ell) &\leq m^d P(T_{n1} \geq (1 + 2\varepsilon) \varphi_n^2) \\ &= m^d P(|Z_n(Q_{1m}, 0, a)| \geq (1 + 2\varepsilon)^{\frac{1}{2}} \varphi_n |Q_{1m}|^{\frac{1}{2}}) \\ &\leq 2m^d \exp\left(-\frac{(1 + 2\varepsilon) \varphi_n^2 |Q_{1m}|}{2|Q_{1m}| + 2(2a/\varphi_n) ((1 + 2\varepsilon)^{\frac{1}{2}} \varphi_n |Q_{1m}|^{\frac{1}{2}})/3}\right) \\ &\leq 2m^d \exp(- (1 + \varepsilon) \varphi_n^2/2) \end{aligned}$$

if  $a$  is sufficiently small,  $1 = (1, \dots, 1)$ .

By the same argument, if  $x \leq 2\varphi_n^2$  and  $a$  is sufficiently small,

$$P(\bar{T}_{n\ell} > x) \leq P(T_{n\ell} > x) \leq 2 \exp\left(\frac{-x|Q_{1m}|}{2|Q_{1m}| + 2(2a/\varphi_n)x^{\frac{1}{2}}|Q_{1m}|^{\frac{1}{2}}/3}\right) \leq 2 \exp(-x(1-\varepsilon)/2),$$

and so

$$(4.10) \quad \begin{aligned} E e^{u\bar{T}_{n\ell}} &= u \int_0^{(1+2\varepsilon)\varphi_n^2} e^{ux} P(\bar{T}_{n\ell} > x) dx \\ &\leq 2u \int_0^{(1+2\varepsilon)\varphi_n^2} e^{ux} e^{-x(1-\varepsilon)/2} dx \\ &\leq 2/\varepsilon < \infty \end{aligned}$$

if  $u = (1 - 2\varepsilon)/2$ .

Since the  $\bar{T}_{n\ell}$  are independent (recall  $n$  is a multiple of  $m$ ), from (4.10) we get

$$(4.11) \quad \begin{aligned} P\left(\sum_{\ell \in \{1, \dots, m\}^d} \bar{T}_{n\ell} \geq (1+4\varepsilon)\varphi_n^2\right) &\leq \exp(-u(1+4\varepsilon)\varphi_n^2) E \exp(u \sum_{\ell} \bar{T}_{n\ell}) \\ &= \exp(-u(1+4\varepsilon)\varphi_n^2) \prod_{\ell} E \exp(u \bar{T}_{n\ell}) \\ &\leq c \exp(-(1+\varepsilon)\varphi_n^2/2), \end{aligned}$$

and hence

$$(4.12) \quad \begin{aligned} P\left(\sum_{\ell \in \{1, \dots, m\}^d} T_{n\ell} \geq (1+4\varepsilon)\varphi_n^2\right) &\leq P(T_{n\ell} \neq \bar{T}_{n\ell} \text{ for some } \ell) \\ &\quad + P\left(\sum_{\ell} \bar{T}_{n\ell} \geq (1+4\varepsilon)\varphi_n^2\right) \\ &\leq c \exp(-(1+\varepsilon)\varphi_n^2/2). \end{aligned}$$

By (2.6),  $rF \in \mathcal{K}$  implies  $\|F - rF\|_{\mathcal{A}} = (1-r)\|rF\|_{\mathcal{A}}/r \leq (1-r)/r$ , which implies  $F \in \mathcal{K}^{(1-r)/r}$ . By (4.8), if  $\sum_{\ell} T_{n\ell}(\omega) \leq (1+4\varepsilon)\varphi_n^2$ , then  $(1+4\varepsilon)^{-\frac{1}{2}} Y_{nm}(\cdot, 0, a)(\omega)/\varphi_n \in \mathcal{K}^{4\varepsilon}$ , and so, letting  $r = (1+4\varepsilon)^{-\frac{1}{2}}$ ,  $Y_{nm}(\cdot, 0, a)(\omega)/\varphi_n \in \mathcal{K}^{4\varepsilon}$ . This with (4.12) proves the proposition.  $\square$

We need one more preliminary proposition.

**Proposition 4.3.** *Suppose  $A$  is fixed. Given  $\eta > 0$ , there exists  $m_0$  such that if  $m \geq m_0$ ,  $n$  is a multiple of  $m$ ,  $a > 0$ , then*

$$\text{Var}(Y_{nm}(A, 0, a) - Z_n(A, 0, a)) \leq \eta.$$

Moreover,  $Y_{nm}(A, 0, a) - Z_n(A, 0, a)$  is the sum of independent mean 0 random variables, each bounded by  $2a/\varphi_n$ .

*Proof.* By the definition of  $Y_{nm}$  (see (4.7)),

$$\begin{aligned}
 (4.13) \quad Y_{nm}(A, 0, a) - Z_n(A, 0, a) &= \sum_{\ell \in \{1, \dots, m\}^d} \left\{ \frac{|A \cap Q_{\ell m}|}{|Q_{\ell m}|} Z_n(Q_{\ell m}, 0, a) - Z_n(A \cap Q_{\ell m}, 0, a) \right\} \\
 &= n^{-d/2} \sum_{\ell} \sum_{j \in nQ_{\ell m}} \left( \frac{|A \cap Q_{\ell m}|}{|Q_{\ell m}|} - |n(A \cap Q_{\ell m}) \cap R_j| \right) \\
 &\quad \cdot (X_j(n, 0, a) - EX_j(n, 0, a)).
 \end{aligned}$$

Since  $||A \cap Q_{\ell m}|/|Q_{\ell m}| - |n(A \cap Q_{\ell m}) \cap R_j|| \leq 1$ , the second part of the proposition is clear. From (4.13),

$$\begin{aligned}
 (4.14) \quad \text{Var}(Y_{nm}(A, 0, a) - Z_n(A, 0, a)) &\leq n^{-d} \sum_{\ell \in \{1, \dots, m\}^d} \sum_{j \in nQ_{\ell m}} (|A \cap Q_{\ell m}|/|Q_{\ell m}| - |n(A \cap Q_{\ell m}) \cap R_j|)^2 \\
 &= n^{-d} \sum_{\ell} \sum_{j \in nQ_{\ell m}} (|n(A \cap Q_{\ell m}) \cap R_j|^2 - (|A \cap Q_{\ell m}|/|Q_{\ell m}|)^2) \\
 &\leq \sum_{\ell} (|A \cap Q_{\ell m}| - |A \cap Q_{\ell m}|^2/|Q_{\ell m}|) \\
 &= \sum_{\ell} |A \cap Q_{\ell m}| |A^c \cap Q_{\ell m}|/|Q_{\ell m}|,
 \end{aligned}$$

since  $\sum_{j \in nQ_{\ell m}} |n(A \cap Q_{\ell m}) \cap R_j| = n^d |A \cap Q_{\ell m}|$ .

If we let

$$h_m(B, \ell) = \sum_{\ell \in \{1, \dots, m\}^d} \frac{|B \cap Q_{\ell m}|}{|Q_{\ell m}|} 1_{Q_{\ell m}}(\ell),$$

by the Lebesgue density theorem,  $h_m(B, \ell) \rightarrow 1_B(\ell)$  for almost all  $\ell$  as  $m \rightarrow \infty$ . So by dominated convergence and (4.14),

$$\begin{aligned}
 \text{Var}(Y_{nm}(A, 0, a) - Z_n(A, 0, a)) &\leq \int_{I^d} h_m(A, \ell) h_m(A^c, \ell) d\ell \\
 &\rightarrow \int_{I^d} 1_A(\ell) 1_{A^c}(\ell) d\ell = 0.
 \end{aligned}$$

Hence if  $m$  is sufficiently large,  $\text{Var}(Y_{nm}(A, 0, a) - Z_n(A, 0, a)) \leq \eta$ .  $\square$

We now prove

**Theorem 4.4.** For each  $\varepsilon$ ,

$$P(Z_n(\cdot)/\phi_n \notin \mathcal{K}^{22\varepsilon} \text{ i.o.}) = 0.$$

*Proof.* Let  $1/16 > \varepsilon > 0$ . Pick  $\delta < \varepsilon^2/2$ . Pick  $m$  large enough so that if  $n$  is a multiple of  $m$ ,

$$\text{Var}(Y_{nm}(B, 0, a) - Z_n(B, 0, a)) \leq \varepsilon^2/4$$

for each  $B \in \mathcal{A}(\delta)$ ; this is possible by Proposition 4.3. Then if  $n$  is a multiple of  $m$ ,

$$\begin{aligned}
 (4.15) \quad & P(Z_n(\cdot, 0, a)/\varphi_n \notin \mathcal{K}^{15\varepsilon}) \leq P(\sup_{B \in \mathcal{A}(\delta)} |Z_n(B, 0, a) - Y_{nm}(B, 0, a)| > \varepsilon \varphi_n) \\
 & + P(Y_{nm}(\cdot, 0, a)/\varphi_n \notin \mathcal{K}^{4\varepsilon}) \\
 & + P(\sup_{\substack{A \in \mathcal{A}, B \in \mathcal{A}(\delta), \\ |A \Delta B| < \delta}} |Y_{nm}(A, 0, a) \\
 & - Y_{nm}(B, 0, a)| > 9\varepsilon \varphi_n, Y_{nm}(\cdot, 0, a)/\varphi_n \in \mathcal{K}^{4\varepsilon}) \\
 & + P(\sup_{\substack{A \in \mathcal{A}, B \in \mathcal{A}(\delta) \\ |A \Delta B| \leq \delta}} |Z_n(A, 0, a) - Z_n(B, 0, a)| > \varepsilon \varphi_n) \\
 & = I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

By Proposition 4.3 and Bernstein's inequality,

$$\begin{aligned}
 I_1 & \leq \exp(H(\delta)) \sup_{B \in \mathcal{A}(\delta)} P(|Z_n(B, 0, a) - Y_{nm}(B, 0, a)| > \varepsilon \varphi_n) \\
 & \leq c \exp(-\varepsilon^2 \varphi_n^2 / (2(\varepsilon^2/4) + 2(2a/\varphi_n)(\varepsilon \varphi_n)/3)) \\
 & \leq c(\log n)^{-\rho}
 \end{aligned}$$

for some  $\rho > 1$ , provided  $a$  is chosen small enough.

By Proposition 4.2,

$$I_2 \leq c \exp(-(1 + \varepsilon) \varphi_n^2 / 2) \leq c(\log n)^{-\rho}$$

for some  $\rho > 1$ , provided  $a$  is chosen small enough.

By (2.7), if  $Y_{nm}(\cdot, 0, a)/\varphi_n \in \mathcal{K}^{4\varepsilon}$  and  $|A \Delta B| < \delta$ ,

$$|Y_{nm}(A, 0, a) - Y_{nm}(B, 0, a)|/\varphi_n \leq 8\varepsilon + \delta^{\frac{1}{2}} < 9\varepsilon.$$

So  $I_3 = 0$ .

Finally, provided  $a$  is sufficiently small,

$$I_4 \leq c \exp(-\varepsilon^2 \varphi_n^2 / (2 + \varepsilon) \delta) \leq c(\log n)^{-\rho}$$

for some  $\rho > 1$  by Theorem 3.1.

Choose  $q$  sufficiently small so that  $1 \leq q^d \leq 1 + \varepsilon$ . If we take  $a$  sufficiently small and  $n_k = m[q^k]$ , where  $[q^k]$  is the integer part of  $q^k$ , then substituting our bounds for  $I_1, I_2, I_3$ , and  $I_4$  in (4.15) gives

$$\sum_k P(Z_{n_k}(\cdot, 0, a)/\varphi_{n_k} \notin \mathcal{K}^{15\varepsilon}) \leq c \sum_k k^{-\rho} < \infty.$$

By Borel-Cantelli,

$$P(Z_{n_k}(\cdot, 0, a)/\varphi_{n_k} \notin \mathcal{K}^{15\varepsilon} \text{ i.o.}) = 0.$$

Since by Proposition 4.1,  $\|Z_n(\cdot, a, \infty)/\varphi_n\|_{\mathcal{A}} \rightarrow 0$  a.s., it follows that  $P(Z_{n_k}(\cdot)/\varphi_{n_k} \notin \mathcal{K}^{16\varepsilon} \text{ i.o.}) = 0$ .

Given  $F \in \mathcal{K}$  with  $F(A) = \int_A f(t) dt$ , define  $F_r, 0 \leq r \leq 1$ , by

$$(4.16) \quad F_r(A) = \int_A f_r(t) dt,$$

where

$$(4.17) \quad f_r(t) = r^d f(rt).$$

Note

$$F_r(A) = \int_A r^d f(r \ell) d\ell = \int_{rA} f(\ell) d\ell = F(rA)$$

and

$$\int_{I^d} f_r^2(\ell) d\ell = r^{2d} \int_{I^d} f^2(r \ell) d\ell = r^d \int_{rI^d} f^2(\ell) d\ell \leq 1,$$

and so  $F_r \in \mathcal{K}$ .

Now fix  $\omega$ , and then pick  $k$  large so that there exists  $F \in \mathcal{K}$  with

$$\|Z_{n_{k+1}}(\cdot)/\varphi_{n_{k+1}} - F\|_{\mathcal{A}} \leq 16\varepsilon$$

and  $(n_{k+1}/n_k)^{d/2} \varphi_{n_{k+1}}/\varphi_{n_k} \leq 1 + 3\varepsilon$ . Suppose  $n_k \leq n \leq n_{k+1}$ . Let  $r = n/n_{k+1}$ . Then

$$n^{d/2} Z_n(A) = \sum_j |nA \cap R_j| X_j = \sum_j |n_{k+1}(rA) \cap R_j| X_j = n_{k+1}^{d/2} Z_{n_{k+1}}(rA).$$

Since  $\mathcal{A}$  is contraction closed (see (2.8)), if  $A \in \mathcal{A}$ ,

$$\begin{aligned} |Z_n(A)/\varphi_n - F_r(A)| &\leq |Z_{n_{k+1}}(rA)/\varphi_{n_{k+1}} - F(rA)| \\ &\quad + \left( \frac{\varphi_{n_{k+1}}}{\varphi_n} \left( \frac{n_{k+1}}{n} \right)^{d/2} - 1 \right) |Z_{n_{k+1}}(rA)/\varphi_{n_{k+1}}| \\ &\leq 16\varepsilon + 3\varepsilon(1 + 16\varepsilon) \\ &\leq 22\varepsilon, \end{aligned}$$

using (2.7). So, for all  $k$  sufficiently large,  $n_k \leq n \leq n_{k+1}$ ,  $Z_n(\cdot)(\omega)/\varphi_n \in \mathcal{K}^{22\varepsilon}$ .  $\square$

### 5. Limit Points

We now determine the limit points of  $Z_n/\varphi_n$ .

**Theorem 5.1.** *For each  $F \in \mathcal{K}$  and each  $1 > \varepsilon > 0$ ,*

$$P(\|Z_n(\cdot)/\varphi_n - F\|_{\mathcal{A}} < 6\varepsilon \text{ i.o.}) = 1.$$

*Proof.* We first suppose that for some  $\tau > 0$ ,  $F(A) = \int_A f(\ell) d\ell$  for all  $A \in \mathcal{A}$ , and

$$(5.1) \quad \int_{I^d} f^2(\ell) d\ell \leq 1 - \tau.$$

Since  $Z_n/\varphi_n \in \mathcal{K}^\varepsilon$  for  $n$  large by Theorem 4.1 and  $F \in \mathcal{K}$ , it suffices by (2.7) to show

$$(5.2) \quad P(\max_{B \in \mathcal{A}(\delta)} |Z_n(B)/\varphi_n - F(B)| < 4\varepsilon \text{ i.o.}) = 1$$

for  $\delta = \varepsilon^2/8$ .

Given a set  $B$ , there exists a closed set  $C$  with  $|B \Delta C| < \delta/2$ . Since  $|Q_{1,m}| = m^{-d}$  and  $O_m(C) \downarrow C$  as  $m \rightarrow \infty$ , where

$$O_m(C) = \bigcup_{\substack{Q_{\ell m} \cap C \neq \emptyset \\ \ell \in \{1, \dots, m\}^d}} Q_{\ell m},$$

the outer rectilinear fit of  $C$  by the cubes  $Q_{\ell m}$  defined in (4.6), then if  $m$  is sufficiently large, there is a set  $A = O_m(C) - Q_{1m}$  that is made up of the union of  $Q_{\ell m}$ 's,  $\ell \neq 1$ , with  $|A \Delta B| < \delta$ .

We construct such a set  $A$  for each  $B \in \mathcal{A}(\delta)$ : label the sets of  $\mathcal{A}(\delta)$  by  $B_1, \dots, B_q$ ; then we can find an integer  $m$  and finitely many sets  $A_1, \dots, A_q$  such that each  $A_i$  is made up of the union of  $Q_{\ell m}$ 's,  $\ell \neq 1$ , and  $|A_i \Delta B_i| \leq \delta$ ,  $i = 1, \dots, q$ .

Let  $n_i = m^l$ . By Bernstein, for each  $i = 1, \dots, q$

$$\begin{aligned} \sum_i P(|Z_{n_i}(A_i, 0, a) - Z_{n_i}(B_i, 0, a)| > \varepsilon \varphi_{n_i}) \\ \leq \sum_i 2 \exp(-\varepsilon^2 \varphi_{n_i}^2 / (2\delta + 2(2a/\varphi_{n_i})(\varepsilon \varphi_{n_i})/3)) \\ \leq \sum_i \exp(-2\varphi_{n_i}^2) < \infty, \end{aligned}$$

provided  $a < \delta$ . Using Proposition 4.1 and Borel-Cantelli,

$$P(|Z_{n_i}(A_i) - Z_{n_i}(B_i)| > 2\varepsilon \varphi_{n_i} \text{ i.o.}) = 0,$$

and so to prove (5.2), it suffices to show

$$(5.3) \quad P(\max_{1 \leq i \leq q} |Z_{n_i}(A_i)/\varphi_{n_i} - F(A_i)| < \varepsilon \text{ i.o.}) = 1.$$

Now  $Z_{n_i}$  and  $F$  are finitely additive, and each  $A_i$ ,  $i = 1, \dots, q$ , is the union of  $Q_{\ell m}$ 's,  $\ell \neq 1$ . So to prove (5.3), it suffices to prove

$$(5.4) \quad P(\max_{\ell \in \{1, \dots, m\}^d - \{1\}} |Z_{n_i}(Q_{\ell m})/\varphi_{n_i} - F(Q_{\ell m})| < \varepsilon/m^d \text{ i.o.}) = 1.$$

We now prove (5.4). Let  $D = I^d - Q_{1m}$ . Let

$$A_{\ell i} = (|Z_{n_i}(Q_{\ell m})/\varphi_{n_i} - F(Q_{\ell m})| < \varepsilon/m^d), \quad \ell \in \{1, \dots, m\}^d - \{1\},$$

and let

$$I_i = \bigcap_{\ell \in \{1, \dots, m\}^d - \{1\}} A_{\ell i}.$$

Each  $Z_{n_i}(Q_{\ell m})$  is the sum of  $s_i = n_i^d |Q_{\ell m}|$  iid mean 0, variance 1 random variables, and so by Lemma 2.4 of [4] (his  $\alpha_k$  is our  $s_i^{\frac{1}{2}} \varphi_{s_i}$ , his  $m_k$  our  $s_i$ ), for each  $\eta > 0$ ,

$$P(A_{\ell i}) \geq \exp\left(-\left(\frac{(F(Q_{\ell m}))^2}{2} + \eta\right) \varphi_{n_i}^2\right)$$

for  $l$  sufficiently large. By the proof of (2.6),  $|F(Q_{\ell m})| \leq (1 - \tau)^{\frac{1}{2}} |Q_{\ell m}|^{\frac{1}{2}}$ , and so

$$\sum_{\ell \in \{1, \dots, m\}^d} (F(Q_{\ell m}))^2 \leq 1 - \tau.$$

Since the  $Z_{n_i}(Q_{\ell m}), \ell \in \{1, \dots, m\}^d$  are independent,

$$\begin{aligned}
 P(\Gamma_l) &= \prod_{\ell \in \{1, \dots, m\}^d - \{1\}} P(A_{\ell l}) \geq \exp(-((1-\tau)/2 + \eta m^d) \varphi_{n_i}^2) \\
 &= c(\log n_i)^{-\rho} = c l^{-\rho}
 \end{aligned}$$

for some  $\rho < 1$ , provided  $\eta$  is sufficiently small and  $l$  sufficiently large.

Now  $\Gamma_l$  depends only on  $X_j$ 's with  $j \in m^l D$ , and since  $D, mD, m^2D, \dots$  are disjoint, the  $\Gamma_l$  are independent. Then by Borel-Cantelli,

$$P(\Gamma_l \text{ i.o.}) = 1,$$

which is precisely (5.4).

Finally, if  $F$  is any element of  $\mathcal{X}$ ,  $\|(1-\tau)F - F\|_{\mathcal{A}} \rightarrow 0$  as  $\tau \rightarrow 0$ , while

$$(1-\tau)F(A) = \int_A (1-\tau)f(\ell) d\ell.$$

Hence  $(1-\tau)F$  is an element of  $\mathcal{X}$  that satisfies (5.1). Since  $\tau$  is arbitrary, the theorem follows.  $\square$

*Proof of Theorem 2.1.* This is immediate from Theorems 4.1 and 5.1.  $\square$

### 6. Central Limit Theorem

Let  $\mathcal{C}(\mathcal{A})$  be the uniformly continuous functions on  $\mathcal{A}$  with norm  $\|\cdot\|_{\mathcal{A}}$ . Let  $W(\cdot)$  be Brownian motion indexed by  $\mathcal{A}$ : a mean 0 Gaussian process with sample paths in  $\mathcal{C}(\mathcal{A})$  and  $\text{Cov}(W(A), W(B)) = |A \Delta B|$ . In this section we show how to prove that  $Z_n$  converges weakly to  $W$  in  $\mathcal{C}(\mathcal{A})$ .

Since enlarging  $\mathcal{A}$  just makes a stronger theorem, we may assume without loss of generality that  $\mathcal{A}$  is closed with respect to  $d(A, B) = |A \Delta B|$  and that  $\mathcal{A}$  still satisfies (2.2), (2.3), and (2.8). One can show  $\mathcal{A}$  is complete ( $A \rightarrow 1_A$  maps  $(\mathcal{A}, d(\cdot, \cdot))$  isometrically into a closed subset of  $L_1(d\ell)$ ). With the total boundedness of  $\mathcal{A}$ , we conclude  $\mathcal{A}$  is compact.

One can show that the finite dimensional distributions of  $Z_n$  converges to those of  $W$ ; for an elegant proof of this, see [1]. Since  $\mathcal{C}(\mathcal{A})$  is a complete and separable metric space,  $Z_n$  will converge weakly to  $W$  provided the  $Z_n$ 's are tight, by Prohorov's theorem. Recalling that  $\phi \in \mathcal{A}$  by (2.8) and that  $Z_n(\phi) = 0$  for each  $n$  by (2.1), to show tightness it is easy to see that it suffices to show: given  $\eta > 0$ , for each  $k$ , there exists  $\delta(k)$  such that

$$\sup_n P\left(\sup_{B, C \in \mathcal{A}, |B \Delta C| \leq \delta(k)} |Z_n(B) - Z_n(C)| > 2^{-k}\right) \leq \eta 2^{-k}.$$

We accomplish this by proving

**Theorem 6.1.** *Given  $1 > \lambda, \varepsilon > 0$ , there exists  $\tau$  such that*

$$\sup_n P\left(\sup_{B, C \in \mathcal{A}, |B \Delta C| \leq \tau} |Z_n(B) - Z_n(C)| > 3\lambda\right) \leq 4\varepsilon.$$

*Proof.* We will use the notation of Sect. 3, except that for this proof we take  $\varphi_n \equiv 1$ . With this change, we follow the proof of Theorem 3.1 up through statement (3.12) word for word, but we now proceed to choose the  $\delta_i$ 's,  $\lambda_i$ 's, etc. somewhat differently.

Let  $G(x)$  be a continuous, strictly decreasing function such that  $G(x) \geq x^{-\frac{1}{2}}$ ,  $H(x/2) = o(G(x))$  as  $x \rightarrow 0$ , and  $\gamma = \int_0^1 (G(x)/x)^{\frac{1}{2}} dx < \infty$ . Let  $\bar{\lambda} = \lambda/2$ ,  $\beta = \lambda/40\gamma$ . Let  $M$  be an integer  $\geq 1$  chosen large enough so that  $\exp(-M) \leq \varepsilon/60(\gamma + 1)$ . Choose  $\delta = \delta_0 < 1$  sufficiently small so that  $\delta^{-\frac{1}{2}}\beta^2/60 > M$ ,  $G(x)\beta^2 \geq 120H(x/2)$  for all  $x \leq \delta$ ,  $\lambda^2/56\delta > M$ , and  $\delta H(\delta) \leq \lambda^2/112$ . Define  $\delta_i$  by  $G(\delta_i)/\delta_i = (G(\delta_0)/\delta_0)2^i$ ,  $\lambda_i = (\delta_i G(\delta_i))^{\frac{1}{2}}\beta$ ,  $a_i = 3\delta_i/\lambda_i$ , and  $a = \alpha = \min(a_0/2, \delta(1 + \lambda)^{-1})$ .

As before,  $a < a_0$  and (3.7) and (3.11) are satisfied. From (3.9),

$$(6.1) \quad p_0 \leq 2 \exp\left(\frac{-\lambda^2 + 56\delta H(\delta)}{28\delta}\right) \leq 2 \exp(-\lambda^2/56\delta) < \varepsilon.$$

By Lemma 2.3(a),  $\delta_{i+1} \geq \delta_i/2$ , and so  $H(\delta_{i+1}) \leq H(\delta_i/2)$ . From (3.10),

$$\begin{aligned} r_i &\leq 4 \exp(-\lambda_i^2/8\delta_i - 2H(\delta_i/2)) \\ &\leq 4 \exp(-G(\delta_i)\beta^2/16) \\ &\leq 4 \exp(-M\delta_i^{-\frac{1}{2}}), \end{aligned}$$

and by Lemma 2.3(d) with  $\zeta = \exp(-M)$ ,

$$(6.2) \quad \sum_{i=0}^{\infty} r_i \leq \varepsilon.$$

Similarly, from (3.12),

$$s_i \leq 2 \exp(-M\delta_i^{-\frac{1}{2}}),$$

and

$$(6.3) \quad \sum_{i=1}^{\infty} s_i \leq \varepsilon.$$

Substituting (6.1), (6.2), and (6.3) in (3.8), for the above choices of  $\delta$  and  $a$ ,

$$P\left(\sup_{B, C \in \mathcal{A}, |\bar{B} \Delta C| \leq \delta} |Z_n(B, 0, a) - Z_n(C, 0, a)| \geq \lambda\right) \leq 3\varepsilon.$$

Taking  $\varphi_n \equiv 1$ , the proof of Proposition 4.1 shows that

$$P(\|Z_n(\cdot, a, \infty)\|_{\mathcal{A}} > \lambda) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and so for  $n \geq n_1$  for some  $n_1$  sufficiently large,

$$P\left(\sup_{B, C \in \mathcal{A}, |\bar{B} \Delta C| \leq \delta} |Z_n(B) - Z_n(C)| > 3\lambda\right) \leq 4\varepsilon.$$

Since  $Z_n$  is uniformly continuous for  $n \leq n_1$  and  $\sup_{B, C \in \mathcal{A}, |\bar{B} \Delta C| \leq \tau} |Z_n(B) - Z_n(C)|$  decreases as  $\tau$  decreases, taking  $\tau$  sufficiently small and  $\leq \delta$  proves the theorem.  $\square$

*Acknowledgment.* I would like to thank K. Alexander and R. Pyke for helpful discussions concerning their paper [1].

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Received July 10, 1984