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Probability Theory and Related Fields

A Representation of Local Time for Lipschitz Surfaces

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Summary. Suppose that $D \subset \mathbb{R}^n$, $n \geqq 2$, is a Lipschitz domain and let $N_t(r)$ be the number of excursions of Brownian motion inside D with diameter greater than r which started before time t. Then $rN_t(r)$ converges as $r \to 0$ to a constant multiple of local time on ∂D, a.s. and in L^p for all $p < \infty$. The limit need not exist or may be trivial (0 or ∞) in Hölder domains, non-tangentially accessible domains and domains whose boundaries have finite surface area.

1. Introduction

Consider an open domain D in $\mathbb{R}^n$, where $n \geqq 2$, and an n-dimensional Brownian motion X. Let $\{e_t\}_{t \in V}$ be the collection of all excursions of X in D, i.e., $V = \{t > 0: X_t \in \partial D\}$ and each e_t is a piece of X contained in D with endpoints $X(t)$ and $X(t')$ in ∂D (although some excursions e_t may be null). The collection $\{e_t\}$ of excursions may be described in terms of the exit system of Maisonneuve (1975). If D is a half-space this description is completely satisfactory and many explicit formulae have been derived, see, e.g., Burdzy (1987) or Burdzy, Toby and Williams (1989).

Intuition suggests that when ∂D is sufficiently smooth, then the properties of excursions in D are similar to the properties of excursions in a half-space. One may consider two basic types of properties: local and global.

Burdzy and Williams (1986) considered local properties of excursions such as the local law of the iterated logarithm and proved that in every $C^{1,\alpha}$-domain, all excursions have the same local properties as excursions in a half-space. They also constructed a C^1-domain such that w.p.1, all excursions in this domain lack a certain property which characterizes excursions in a half-space.

This paper is devoted to global properties of excursions. An obvious candidate for a global property of an excursion is its size. In the rest of the introduc-

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tion, "size" of an excursion will mean either its diameter or the square root of its lifetime; we have applied the square root function to the lifetime in order to simplify the statement of the results. Of course, we cannot infer anything about the smoothness of ∂D from the size of a single excursion. However, we may count the number of excursions of different sizes. Let $N_t(r)$ be the number of excursions of size greater than r which started before t. If D is a half-space then for each $t>0$,

$$r N_t(r) \to L_t \quad \text{as } r \to 0 \tag{1.1}$$

where L_t is a certain continuous additive functional ("local time", up to a constant multiple). This result follows easily from the one-dimensional version, which is well known (see Itô and McKean (1974)). Burdzy, Toby and Williams (1989) extended (1.1) to $C^{1,\alpha}$-domains, $\alpha>0$. If the boundary of D is not smooth, for example if it has many "crevices", then one would expect to see a very large number of small excursions generated in small crevices and, consequently, $r N_t(r)$ would diverge. Thus, the convergence of $r N_t(r)$ may serve as an indicator of the smoothness of the domain. The main result of our paper is the following (see Sects. 6 and 7 for a more precise statement).

Theorem 1.1. (i) *The limit in* (1.1) *exists in bounded Lipschitz domains. The convergence holds a.s. and in* L^p *for all* $p<\infty$. *The Revuz measure of the limiting continuous additive functional* L_t *is equal (up to a constant multiple) to the surface area measure on the boundary.*

(ii) *There exist Hölder domains, non-tangentially accessible domains and domains whose boundaries have finite surface area such that* $r N_t(r)$ *does not converge to a limit in* $(0, \infty)$.

In Sect. 7, we prove a result slightly stronger than Theorem 1.1 (ii). Lipschitz domains probably form the widest natural class of domains for which (1.1) is true.

The above discussion can be summed up by asking

Question. Which domains have smooth boundaries from the point of view of Brownian excursions?

The answer is contained in the following table.

	Local properties of excursions	Global properties of excursions
Smooth	$C^{1,\alpha}$, $\alpha>0$	Lipschitz
Non-smooth	C^1	Hölder

To prove our main theorem, we derive an inequality for the Green function which may have some interest of its own (see Sect. 3). We also present some results on convergence of continuous additive functionals which hold in situations more general than the ones considered in our paper.

The paper is organized as follows. The next section is devoted to notation and a review of known results. Section 3 presents an inequality for the Green function. Convergence of continuous additive functionals is discussed in Sect. 4. Section 5 is devoted to estimates of excursions laws. Sections 6 and 7 present a rigorous version of Theorem 1.1.

Our main theorems are true as stated for $n \geqq 2$ but we will give the proofs only for $n \geqq 3$ to avoid the usual problems with the recurrence of 2-dimensional Brownian motion. The modifications needed for $n=2$ (i.e. killing) are obvious.

We would like to point out some related results. Bass (1984) studied convergence of continuous additive functionals of Brownian motion, while excursions of reflecting Brownian motion in smooth domains were considered in Hsu (1986).

2. Preliminaries

In this section, we will establish notation and review some known results. In order to save space, we will not go into details, e.g., measurability questions. The reader is referred to Sharpe (1988) for a meticulous exposition of Markov processes and exit systems. Fabes et al. (1986) is an excellent reference for boundary problems in parabolic potential theory. The notation and results on semimartingales used in Sect. 4 may be found in Dellacherie and Meyer (1980) or Durrett (1984).

We start with some general notation. We will consider domains in $\mathbb{R}^n$ where $n \geqq 2$. We will tacitly assume that $n \geqq 3$ in all our proofs. Most of the time, n will be suppressed in the notation. The diameter of a set $A \subset \mathbb{R}^n$ will be denoted $\operatorname{diam}(A)$ and $\operatorname{dist}(x, A)$ will refer to the usual distance between a point x and a set A. The symbol $\operatorname{Dist}(A, B)$ will stand for the Hausdorff distance between sets A and B i.e.,

$$\operatorname{Dist}(A, B) = \max\left(\sup_{x \in A} \operatorname{dist}(x, B), \sup_{x \in B} \operatorname{dist}(x, A)\right).$$

A set $D \subset \mathbb{R}^n$ will be called a Lipschitz domain with character λ if for every $x \in \partial D$ there exist a neighborhood V of x, an orthonormal coordinate system $CS(x)$ and a Lipschitz function f with constant λ, mapping $\mathbb{R}^{n-1}$ into $\mathbb{R}$, such that

$$D \cap V = \{y \in V : y_n > f(y_1, y_2, \ldots, y_{n-1})\},$$

where $y = (y_1, y_2, \ldots, y_n)$ in $CS(x)$. If D is a bounded Lipschitz domain then, by compactness, we may choose a finite number of coordinate systems $CS_1, CS_2, \ldots, CS_m$ so that for each $x \in \partial D$, the coordinate system $CS(x)$ is one of CS_k's.

For $x \in \partial D$, the symbol N_x will stand for the inward normal unit vector at x, provided it exists. Notice that

$$\varepsilon_x \overset{\mathrm{df}}{=} \inf\{\varepsilon > 0 : \operatorname{dist}(x + \varepsilon N_x, \partial D) < \varepsilon/2\} > 0.$$

By abuse of notation, $x + \varepsilon N_x$ will have the usual meaning only for $\varepsilon \leqq \varepsilon_x$ and it will denote $x + \varepsilon_x N_x + (\varepsilon - \varepsilon_x)\,(0, 0, \ldots, 0, 1)$ in $CS(x)$ if $\varepsilon > \varepsilon_x$. If D is a bounded

Lipschitz domain we may find an $\varepsilon_0 > 0$ and choose a finite family of local coordinate systems so that $x + \varepsilon N_x \in D$ for all $\varepsilon \in (0, \varepsilon_0)$ and all x such that N_x is well defined.

For a Greenian domain D, the Green function will be denoted $G_D(\cdot,\cdot)$. See Doob (1984) for the definitions of harmonic and parabolic functions and a detailed review of the corresponding potential theory.

The Harnack inequality easily implies the following inequality. (Recall our convention concerning $x + rN_x$.) Suppose that D is a bounded Lipschitz domain and let $x_0 \in D$. Then there exists $r_0 > 0$ and $c < \infty$ such that for all $x \in \partial D$ with N_x well-defined and all $r < r_0$, we have

$$G_D(x_0, x + (r/4)N_x)/G_D(x_0, x + rN_x) \in (c^{-1}, c)$$

where c depends only on D and x_0 but not on x or r.

The boundary Harnack principle was first proved by Dahlberg (1977). We present a version adapted from Burdzy (1987).

Lemma 2.1. (Boundary Harnack principle.) *Suppose that for some $\lambda > 0$, domain D, $x \in \partial D$ and $r > 0$ we have*

$$D_1 \overset{\mathrm{df}}{=} \{y \in D : |y - x| < r\} = \{y \in D : y_n > f(y_1, y_2, \ldots, y_{n-1})\}$$

where $f: \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function with constant λ. If functions g and h are harmonic in D_1 and vanish on $\partial D_1 \cap \partial D$ then

$$\frac{h(y)}{g(y)} \geq c \frac{h(z)}{g(z)}$$

for all $y, z \in D_1$ such that $|y - x| < r/2$ and $|z - x| < r/2$, where $c > 0$ is a constant which depends only on λ. $\square$

We will work with the same probability setup as in Burdzy et al. (1989). Specifically, let Ω be the space of paths mapping $[0, \infty)$ to $\mathbb{R}^n \cup \{\delta\}$ which are continuous on $[0, R)$ for some $R \leqq \infty$ and equal to δ for $t \geqq R$. Thus, R denotes the lifetime of a path, which may be infinite. Let X be the canonical process. We will use the symbol ω to denote harmonic measure. Denote $\mathscr{F} = \sigma\{X_t, t \geqq 0\}$, $\mathscr{F}_t = \sigma\{X_s, s \leqq t\}$. For a stopping time T let $\mathscr{F}_T$ denote the usual σ-field of pre-T-events and let θ_t, $t \geqq 0$, be the shift operators on Ω. For a set $A \subset \mathbb{R}^n$ let

$$T_A = T(A) = \inf\{t > 0 : X_t \in A\}.$$

Let P^x denote a measure on $(\Omega, \mathscr{F})$ which makes X the standard n-dimensional Brownian motion starting from x. Analogously, P_D^x will denote the distribution of Brownian motion in D, i.e., Brownian motion killed at $T(D^c)$. The corresponding expectations will be denoted E^x and E_D^x, resp.

An excursion law H^x in $D \subset \mathbb{R}^n$ is a σ-finite measure on $(\Omega, \mathscr{F})$ which has the following properties:

(i) $H^x(X_0 \neq x) = 0$,

(ii) H^x is strong Markov for the P_D^x-transition probabilities, i.e.,

$$H^x(a \cdot b(\theta_T)) = H^x(a \cdot P_D^{X(T)}(b))$$

for all stopping times $T > 0$, all nonnegative and $\mathscr{F}$-measurable b, and all nonnegative and $\mathscr{F}_T$-measurable a.

An excursion law H^x in D is called standard if $H^x(T_B < \infty) \in (0, \infty)$ for some compact nonpolar set $B \subset D$. If $D \subset \mathbb{R}^n$ is a Lipschitz domain and $x \in \partial D$, then there exists a standard excursion law H^x in D.

The following is a version of the exit system theorem. See Maisonneuve (1975) for more details on exit systems and see Revuz (1970) or Sharpe (1988) for the definition and properties of continuous additive functionals (CAF's).

Suppose that $D \subset \mathbb{R}^n$ is a Lipschitz domain and let μ denote the surface area measure on ∂D. Let L be the CAF of the Brownian motion X (with associated probability measures $\{P^x, x \in \mathbb{R}^n\}$), whose Revuz measure (relative to Lebesgue measure as invariant measure) is given by μ, i.e.,

$$\mu(A) = \lim_{t \downarrow 0} \frac{1}{t} E^v\left[\int_0^t 1_A(X_s)\, dL_s\right],$$

for all Borel sets $A \subset \mathbb{R}^n$, where v denotes Lebesgue measure on $\mathbb{R}^n$. This continuous additive functional will be called the local time on ∂D. Fix some nonpolar compact set $B \subset D$. For μ-almost all points $x \in \partial D$, the unit inward normal vector N_x is well defined and $\lim_{\varepsilon \to 0} \varepsilon^{-1} P_D^{x+\varepsilon N_x}(T_B < \infty)$ exists. For such x let H^x be the excursion law in D with the property that $H^x(T_B < \infty)$ is equal to the above limit. For all other x, let $H^x \equiv 0$. Then the pair (dL, H) is an exit system in D in the following sense.

For u such that $X_u \in \partial D$ let $e_u = \{e_u(t), t \geqq 0\} \in \Omega$ be the excursion of X in D i.e.,

$$e_u(t) = \begin{cases} X(u+t) & \text{if } \inf\{s > u : X_s \in D^c\} > u + t, \\ \delta & \text{otherwise.} \end{cases}$$

For u such that $X_u \notin \partial D$, define $e_u \equiv \delta$. Then (Burdzy (1987), Theorem 3.2),

$$E^\cdot\Big(\sum_{0 < u < \infty} Z_u \cdot (f \circ e_u)\Big) = E^\cdot\left(\int_0^\infty Z_s H^{X(s)}(f)\, dL_s\right) \tag{2.1}$$

for all universally measurable functions f on Ω which vanish on excursions $e_u \equiv \delta$ and all nonnegative $\mathscr{F}_t$-predictable processes Z.

Now we will present a generalization of Theorem 4.1 and Proposition 4.1 of Burdzy (1987).

Theorem 2.1. *Suppose that D is a Lipschitz domain, $x \in \partial D$ and H^x is a standard excursion law in D. Fix some $s>0$ and let $B=\{z \in \mathbb{R}^n : |z-x| \geqq s\}$. Assume that A_1 and A_2 are events in $\sigma\{X_{T+t}, t \geqq 0\}$, where $T=\min(s, T_B)$. Then*

$$H^x(A_1)/H^x(A_2)=\lim_{\substack{z \to x \\ z \in D \\ t \to 0}} P_D^z(\mathbf{1}_{A_1} \circ \theta_t)/P_D^z(\mathbf{1}_{A_2} \circ \theta_t).$$

Proof. The proof is completely analogous to that of Theorem 4.1 of Burdzy (1987) except that it uses the parabolic version of the boundary Harnack principle proved in Theorem 6.1 of Burdzy et al. (1989). □

The potential of Brownian motion will be denoted in the usual way as $U(\cdot,\cdot)$, i.e., for Borel sets $B \subset \mathbb{R}^n$ we have

$$\int_B U(x, y)\, dy = E^x \int_0^\infty \mathbf{1}_B(X_s)\, ds.$$

Note that U agrees, up to a constant, with the Green function $G_{\mathbb{R}^n}$ for $n \geqq 3$.

Now we will review some facts about A_p-weights. See Garnett (1981) for more details. We will present the results in the form slightly different from the usual one in order to make them readily applicable in the next section.

Suppose that D is a Lipschitz domain and μ is surface area measure on ∂D. If $g \in L^1(\mu)$ is a positive function and $1<p<\infty$, then g belongs to the Muckenhoupt $A_p(\mu)$ class if

$$\sup_{\Delta \subset \partial D} \left(\frac{1}{\mu(\Delta)} \int_\Delta g(x)\, \mu(dx)\right)\left(\frac{1}{\mu(\Delta)} \int_\Delta (g(x))^{-1/(p-1)}\, \mu(dx)\right)^{p-1} < \infty, \tag{2.2}$$

where the sup is taken over all surface balls Δ in ∂D. For $x \in \partial D$ a surface ball is defined by $\Delta(x, r)=B(x, r) \cap \partial D$ where $B(x, r)$ is the ball in $\mathbb{R}^n$ centered at x with radius r. When $p=\infty$ we say that $g \in A_\infty(\mu)$ if there are positive constants c_1, c_2, and α such that for any surface ball $\Delta \subset \partial D$ and any Borel subset $V \subset \Delta$,

$$c_1\left(\frac{\mu(V)}{\mu(\Delta)}\right)^{1/\alpha} \leqq \left(\frac{g(V)}{g(\Delta)}\right) \leqq c_2\left(\frac{\mu(V)}{\mu(\Delta)}\right)^\alpha,$$

where $g(V)=\int_V g(x)\, \mu(dx)$, and similarly for $g(\Delta)$.

Next recall that if $f \in L^1(\mu)$, the Hardy-Littlewood maximal function Mf of f is defined by

$$Mf(x)=\sup\left\{\frac{1}{\mu(\Delta)} \int_\Delta |f(y)|\, \mu(dy)\right\},$$

where the sup is taken over all surface balls Δ which contain x. The following are two well known results on A_p-weights. The proofs may be found in Garnett (1981), Muckenhoupt (1974) and Coifman and Fefferman (1974).

(2.3) If $g \in A_\infty(\mu)$, then there exists a $p_0 \in (1, \infty)$, depending only on c_1, c_2, and α such that $g \in A_p(\mu)$ for all $p > p_0$.

(2.4) If $g \in A_p(\mu)$ for some $p \in (1, \infty)$, then there exists a constant c, independent of f, such that

$$\int_{\partial D} (Mf(x))^p g(x)\, \mu(dx) \leqq c \int_{\partial D} |f(x)|^p g(x)\, \mu(dx).$$

3. An Inequality for the Green Function

We start this section with an elementary estimate of the harmonic measure.

Let D be a bounded Lipschitz domain and let $x_0 \in D$. Let ω denote harmonic measure on ∂D relative to x_0 and let μ be surface area measure on ∂D. Recall that Δ denotes a surface ball i.e.,

$$\Delta = \Delta(x, r) = \{y \in \partial D : |y - x| < r\}.$$

By Dahlberg (1977), there exist $c = c(D, x_0) < \infty$ and $\alpha = \alpha(D) > 1/2$ such that

$$\omega(A) \leqq c(\mu(A))^\alpha$$

for all sets $A \subset \partial D$. Moreover, the bound 1/2 on α cannot be improved. However, we will show that we have a better bound if we limit ourselves to surface balls Δ in place of arbitrary sets A.

Lemma 3.1. *Suppose that D is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geqq 3$. Then there exist $c = c(D, x_0) < \infty$ and $\beta = \beta(D) > (n-2)/(n-1)$ such that*

$$\omega(\Delta) \leqq c(\mu(\Delta))^\beta \tag{3.1}$$

and

$$\omega(\Delta) \leqq c r^{(n-1)\beta} \tag{3.2}$$

for all surface balls $\Delta = \Delta(x, r)$.

Remark 3.1. The bound $(n-2)/(n-1)$ on the exponent β is strictly better than 1/2 only in dimensions $n \geqq 4$. The bound 1/2 is the best possible for $n = 2, 3$.

Proof. In a Lipschitz domain, we have

$$c_1^{-1} r^{n-1} \leqq \mu(\Delta(x, r)) \leqq c_1 r^{n-1} \tag{3.3}$$

for some $c_1 < \infty$. Thus it is sufficient to prove only (3.2).

We start with a special case. Let D be defined in some coordinate system by

$$D = \{y \in \mathbb{R}^n : y_n > -\lambda(y_1^2 + y_2^2 + \ldots + y_{n-1}^2)^{1/2}\},$$

where $\lambda > 0$. Then there exist $\alpha > 0$ and a strictly positive harmonic function h with a pole at infinity such that

$$h(x) = |x|^\alpha h(x/|x|) \qquad \text{for } x \in D,$$

and such that h vanishes continuously on ∂D (see Burkholder (1977)). For $x \in \partial D$, $x \neq 0$, we have

$$\lim_{r \to 0} r^{-1} h(x + rN_x) = c_2 |x|^{-1+\alpha}. \tag{3.4}$$

Let $b > 0$ and let

$$D_1 = D_1(b) = D \cup \{x \in \mathbb{R}^n : |x| > b\}. \tag{3.5}$$

Suppose that x_1 is such that $|x_1| > 2b$. The functions $h(\cdot)$ and $G_{D_1}(x_1, \cdot)$ are harmonic in $\{x \in D_1 : |x| < b/2\}$ and vanish continuously on $\{x \in \partial D_1 : |x| < b/2\}$. The boundary Harnack principle and (3.4) imply

$$\lim_{r \to 0} r^{-1} G_{D_1}(x_1, x + rN_x) \leqq c_3 |x|^{-1+\alpha} \tag{3.6}$$

for $|x| < b/4$, where c_3 does not depend on x_1. The limit in (3.6) is equal to a constant multiple of the density of harmonic measure $\omega = \omega_{x_1}^{D_1}$ with respect to the surface area measure on ∂D_1. Simple integration shows that there exists $c_4 = c_4(\lambda, b) < \infty$ such that for all x_1 and r with $|x_1| > 2b$, $r < b/4$,

$$\omega_{x_1}^{D_1}(\Delta(0, r)) \leqq c_4 r^{n-2+\alpha} \overset{\text{df}}{=} c_4 r^{(n-1)\beta}.$$

We have, of course, $\beta > (n-2)/(n-1)$, since $\alpha > 0$.

Now we turn to the general case. Let D be a bounded Lipschitz domain with character λ_1 and fix an $x_0 \in D$. Let $d_0 = \operatorname{dist}(x_0, \partial D)$ and choose some $b < d_0/4$.

Consider an arbitrary $x \in \partial D$ and a coordinate system CS such that $x = 0$ in CS and the boundary of D may be represented in the ball $B(x, \rho)$ as the graph of a Lipschitz function with constant λ_1. The radius ρ may be chosen independently of x.

Let $\lambda = 2\lambda_1$ and b be as above. For $d > 0$, let $z(d) = (0, \ldots, 0, d)$ in CS and let D_1 be defined in CS by (3.5). Let

$$D_2 = D_2(d) = D_1 + z(d).$$

It is elementary to check that for some $d_1 = d_1(D, x_0) > 0$, $c_5 = c_5(\lambda)$, $c_6 = c_6(\lambda)$ and all $d \in (0, d_1)$,

$$\{y \in \partial D_2(d) : |y - z(d)| > c_5 d\} \subset D^c$$

and

$$\Delta_D(x, c_6 d) \subset D_2^c.$$

Let $D_3 = D \cap D_2$. By the first part of the proof, we obtain

$$\begin{aligned}\omega_{x_0}^{D}(\Delta_D(x, c_6 d)) &\leqq \omega_{x_0}^{D_3}(\Delta_{D_2}(z(d), c_5 d))\\ &\leqq \omega_{x_0}^{D_2}(\Delta_{D_2}(z(d), c_5 d))\\ &\leqq c_4 (c_5 d)^{(n-1)\beta}.\end{aligned}$$

This completes the proof of (3.2). □

By a result of Dahlberg (1977), ω is absolutely continuous with respect to μ and

$$K(x) \overset{\mathrm{df}}{=} (d\omega/d\mu)(x) \in (0, \infty) \quad \mu\text{-a.e.}$$

Let $W(x) = W_{r_0}(x)$ be defined by

$$W(x) = \sup_{r \in (0, r_0)} \left(\frac{r}{G_D(x_0, x + r N_x)} \right) K(x)$$

if N_x is well defined and $W(x) = 0$ otherwise.

Theorem 3.1. *For every bounded Lipschitz domain D there exist $r_0 > 0$, $c < \infty$ and $\alpha > n-2$ such that for all $x \in \partial D$ and $r < r_0$,*

$$\int_{\Delta(x,r)} W_{r_0}(y)\, \mu(dy) \leqq c r^{\alpha}.$$

Proof. We will use the results on A_p-weights which have been reviewed in the previous section.

By Dahlberg (1977, Corollary on p. 276), the function K belongs to the class $A_\infty(\mu)$. Thus, by (2.3), $K \in A_p(\mu)$ for all p greater than some p_0. Since $\mu(\partial D)$ is finite, (2.2) implies that

$$\int_{\partial D} K(x)^{-1/(p-1)} \mu(dx) < \infty \tag{3.7}$$

for all $p > p_0$.

If $\beta > 0$ and p is sufficiently large so that $(p-1)\beta > p_0 - 1$, we have by Hölder's inequality, for all surface balls Δ,

$$\frac{1}{\mu(\Delta)} \int_{\Delta} (K(x))^{-1/(p-1)} \mu(dx) \leqq \left(\frac{1}{\mu(\Delta)} \int_{\Delta} (K(x))^{-1/(p-1)\beta} \mu(dx) \right)^{\beta}$$

and

$$\begin{aligned}\int_{\Delta} (K(x))^{-1/(p-1)} \mu(dx) &\leqq \left(\int_{\Delta} (K(x))^{-1/(p-1)\beta} \mu(dx) \right)^{\beta} (\mu(\Delta))^{1-\beta}\\ &\leqq \left(\int_{\partial D} (K(x))^{-1/(p-1)\beta} \mu(dx) \right)^{\beta} (\mu(\Delta))^{1-\beta}.\end{aligned}$$

By (3.7), we have for sufficiently large p, $c_1 = c_1(D, \beta, p) < \infty$ and all surface balls Δ,

$$\int_\Delta (K(x))^{-1/(p-1)} \mu(dx) \leqq c_1 (\mu(\Delta))^{1-\beta}. \tag{3.8}$$

By Dahlberg (1977), there exist $r_0 > 0$ and $c_2 = c_2(D) < \infty$ such that for all $x \in \partial D$ and $r < r_0$,

$$c_2^{-1} \leqq \frac{\omega(\Delta(x,r))}{r^{n-2} G_D(x_0, x + r N_x)} \leqq c_2 \tag{3.9}$$

and by Lemma 3.1, for some $\gamma = \gamma(D) > (n-2)/(n-1)$,

$$\omega(\Delta(x,r)) \leqq c_2 (\mu(\Delta(x,r)))^\gamma. \tag{3.10}$$

Since D is Lipschitz,

$$c_3^{-1} r^{n-1} \leqq \mu(\Delta(x,r)) \leqq c_3 r^{n-1} \tag{3.11}$$

for some $c_3 = c_3(D) < \infty$ and all x, r. Let $r < r_1 < r_0$. Then, by (3.9)–(3.11),

$$\begin{aligned}
&\left(\frac{r_1}{G_D(x_0, x + r_1 N_x)}\right)\left(\frac{r}{G_D(x_0, x + r N_x)}\right)^{-1} \\
&\leqq \frac{r_1}{G_D(x_0, x + r_1 N_x)} \frac{r^{-1} c_2 \omega(\Delta(x,r))}{r^{n-2}} \\
&\leqq \frac{r_1}{G_D(x_0, x + r_1 N_x)} c_2 r^{-n+1} c_2 (\mu(\Delta(x,r)))^\gamma \\
&\leqq \frac{r_1}{G_D(x_0, x + r_1 N_x)} c_4 r^{(n-1)(\gamma-1)}.
\end{aligned}$$

By compactness, $\sup\limits_{x \in \partial D} \dfrac{1}{G_D(x_0, x + r_1 N_x)} \leqq c_5(D, r_1) < \infty$ and, therefore, for some $c_6 = c_6(D, r_1) < \infty$,

$$\sup_{x \in \partial D} \left(\frac{r_1}{G_D(x_0, x + r_1 N_x)}\right)\left(\frac{r}{G_D(x_0, x + r N_x)}\right)^{-1} \leqq c_6{}^{(n-1)(\gamma-1)}.$$

This implies that

$$W_{r_1}(x) \leqq c_6 r^{(n-1)(\gamma-1)} W_r(x) \tag{3.12}$$

for $r < r_1 < r_0$ and $x \in \partial D$. By Hölder's inequality, we have for surface balls Δ,

$$\left(\frac{1}{\mu(\Delta)} \int_\Delta K(x) \mu(dx)\right)^{-1} \leqq \left(\frac{1}{\mu(\Delta)} \int_\Delta (K(x))^{-1/(p-1)} \mu(dx)\right)^{p-1}$$

for $p > p_0$. Thus, (3.9) and (3.11) yield for $r < r_0$,

$$\begin{aligned}\frac{r}{G_D(x_0, x + rN_x)} &\leqq c_7 \frac{\mu(\Delta(x,r))}{\omega(\Delta(x,r))}\\ &= c_7 \left(\frac{1}{\mu(\Delta(x,r))} \int_{\Delta(x,r)} K(y)\,\mu(dy) \right)^{-1}\\ &\leqq c_7 \left(\frac{1}{\mu(\Delta(x,r))} \int_{\Delta(x,r)} (K(y))^{-1/(p-1)}\,\mu(dx) \right)^{p-1}.\end{aligned}$$

Recall the definition of the maximal function from Sect. 2. The last inequality shows that for $r_2 < r_0$, $y \in \partial D$ and $x \in \Delta(y, r_2)$,

$$\tilde{W}_{r_2}(x) \overset{\mathrm{df}}{=} \sup_{r \in (0, r_2)} \frac{r}{G_D(x_0, x + rN_x)} \leqq c_7 (Mf(x))^{p-1}$$

where $f(x) = (K(x))^{-1/(p-1)} \mathbf{1}_{\Delta(y, 2r_2)}(x)$. In view of the fact that $K \in A_p$ for $p > p_0$, (2.4) and (3.8) show that

$$\begin{aligned}\int_{\Delta(y,r_2)} (\tilde{W}_{r_2}(x))^{1+\frac{1}{p-1}} K(x)\,\mu(dx) &\leqq \int_{\partial D} (c_7 (Mf(x))^{p-1})^{1+\frac{1}{p-1}} K(x)\,\mu(dx)\\ &\leqq c_8 \int_{\partial D} (Mf(x))^p K(x)\,\mu(dx)\\ &\leqq c_9 \int_{\partial D} (f(x))^p K(x)\,\mu(dx)\\ &\leqq c_9 \int_{\Delta(y,2r_2)} (K(x))^{-p/(p-1)} K(x)\,\mu(dx)\\ &\leqq c_9 \int_{\Delta(y,2r_2)} (K(x))^{-1/(p-1)}\,\mu(dx)\\ &\leqq c_{10}\, \mu(\Delta(y, 2r_2))^{1-\beta}\\ &\leqq c_{11}\, r_2^{(n-1)(1-\beta)}. \end{aligned} \tag{3.13}$$

By Hölder's inequality

$$\begin{aligned}&\frac{1}{\int_\Delta K(x)\,\mu(dx)} \int_\Delta \tilde{W}_{r_2}(x) K(x)\,\mu(dx)\\ &\quad\leqq \left(\frac{1}{\int_\Delta K(x)\,\mu(dx)} \int_\Delta (\tilde{W}_{r_2}(x))^{1+\frac{1}{p-1}} K(x)\,\mu(dx) \right)^{(p-1)/p}.\end{aligned}$$

This, (3.10), (3.11) and (3.13) imply

$$\begin{aligned}\int_{\Delta(y,r_2)} \tilde{W}_{r_2}(x)\,K(x)\,\mu(dx)&\\ \leqq\Big(\int_{\Delta(y,r_2)} (\tilde{W}_{r_2}(x))^{1+\frac{1}{p-1}}\,K(x)\,\mu(dx)\Big)^{(p-1)/p}&(\omega(\Delta(y,r_2)))^{1/p}\\ \leqq c_{12}\, r_2^{(n-1)(1-\beta)(p-1)/p}(\mu(\Delta(y,r_2)))^{\gamma/p}&\\ \leqq c_{13}\, r_2^{(n-1)(1-\beta)(p-1)/p+(n-1)\gamma/p}.&\end{aligned}$$

Finally, by (3.12), we have for $r<r_1<r_0$, and $y\in\partial D$,

$$\begin{aligned}\int_{\Delta(y,r)} W_{r_1}(x)\,\mu(dx)\leqq c_6\, r^{(n-1)(\gamma-1)}&\int_{\Delta(y,r)} W_r(x)\,\mu(dx)\\ =c_6\, r^{(n-1)(\gamma-1)}&\int_{\Delta(y,r)} \tilde{W}_r(x)\,K(x)\,\mu(dx)\\ \leqq c_{14}\, r^{(n-1)(\gamma-1)+(n-1)(1-\beta)(p-1)/p+(n-1)\gamma/p}&.\end{aligned}$$

The constant γ is greater than $(n-2)/(n-1)$, β may be chosen arbitrarily close to 0 and p may be arbitrarily large, so the exponent in the last expression may be made greater than $n-2$ which completes the proof. □

Remark 3.2. It follows from the above proof that for some $r_0>0$ and $p>1$,

$$\int_{\partial D} (\tilde{W}_{r_0}(x))^p\,K(x)\,\mu(dx)<\infty$$

and

$$\int_{\partial D} W_{r_0}(x)\,\mu(dx)<\infty. \tag{3.14}$$

A small change in the proof gives

$$\int_{\partial D} (\tilde{W}_{r_0}(x))^\alpha\,\mu(dx)<\infty$$

for some $r_0>0$ and $\alpha>0$.

4. Convergence of Continuous Additive Functionals

Recall that $U(x,y)=c_1|x-y|^{2-n}$ where $c_1=\Gamma((n/2)-1)(2\pi)^{-n/2}$. We will use $\|\cdot\|$ to denote the supremum norm in $\mathbb{R}^n$.

Lemma 4.1. *Suppose that* $\{\nu_a\}_{a\in(0,1]}$ *is a family of positive measures such that*

$$\sup_{a\in(0,1]} \nu_a(\mathbb{R}^n)<\infty \tag{4.1}$$

and suppose there exist constants $c, \alpha > 0$ such that for all $x \in \mathbb{R}^n$, $r > 0$ and $a \in (0, 1]$,

$$\nu_a(B(x, r)) \leqq c r^{n-2} |\log r|^{-1-\alpha}. \tag{4.2}$$

Then

$$\sup_{a \in (0, 1]} \|U \nu_a\| < \infty. \tag{4.3}$$

If in addition $\nu_a \to \nu_0$ weakly as $a \to 0$, then

$$U \nu_a \to U \nu_0 \quad \text{uniformly as } a \to 0. \tag{4.4}$$

Proof. Changing to polar coordinates and integrating by parts,

$$\begin{aligned} U \nu_a(x) &\leqq \int_{|y-x|>1} U(x, y)\, \nu_a(dy) + c_1 \int_{|y-x| \leqq 1} |y-x|^{2-n} \nu_a(dy) \\ &\leqq c_1 \nu_a(\mathbb{R}^n) + c_2(n) \int_0^1 r^{1-n} \nu_a(B(x, r))\, dr + c_2(n)\, \nu_a(B(x, 1)). \end{aligned} \tag{4.5}$$

Using (4.1) and (4.2) gives (4.3).

For each $M > 0$, $U_M(x, y) \overset{\mathrm{df}}{=} \min(U(x, y), M)$ is a bounded Lipschitz function of x. So using (4.1), $\{\int_{\mathbb{R}^n} U_M(x, y)\, \nu_a(dy)\}_{a \in (0, 1]}$ is an equicontinuous family of functions of x. Since $\nu_a \to \nu_0$ weakly,

$$\int_{\mathbb{R}^n} U_M(x, y)\, \nu_a(dy) \to \int_{\mathbb{R}^n} U_M(x, y)\, \nu_0(dy)$$

for each x. By virtue of the equicontinuity, we see that the convergence is uniform.

On the other hand, setting $r(M) = (c_1/M)^{1/(2-n)}$ and bounding the right hand side as in (4.5),

$$\int_{\mathbb{R}^n} (U(x, y) - U_M(x, y))\, \nu_a(dy) \leqq c_1 \int_{|x-y| \leqq r(M)} |x-y|^{2-n} \nu_a(dy)$$

can be made uniformly small by taking M large enough. Hence $U \nu_a$ tends to $U \nu_0$ uniformly. □

Remark 4.1. Of course, (4.2) could be replaced by

$$\int_0^1 r^{1-n} (\sup_{a,x} \nu_a(B(x, r)))\, dr < \infty.$$

Theorem 4.1. *Suppose that for each $a \in [0, 1]$, L_t^a is a continuous additive functional with Revuz measure ν_a and assume that the potentials of the measures ν_a satisfy (4.3) and (4.4). Suppose that for each $a \in (0, 1]$, N_t^a is a pure jump process that* *is identically* 0 *at time* 0, *all the jumps of* N_t^a *are of size* 1, *and the compensator of* aN_t^a *is* L_t^a. *Then for every* $t>0$, $p<\infty$, *and* $x\in\mathbb{R}^n$,

$$\sup_{s\in[0,t]} |L_s^a - L_s^0| \xrightarrow[a\to 0]{} 0 \quad \text{in } L^p(P^x) \tag{4.6}$$

and

$$\sup_{s\in[0,t]} |aN_s^a - L_s^a| \xrightarrow[a\to 0]{} 0 \quad \text{in } L^p(P^x). \tag{4.7}$$

Moreover, if N_t^a *is non-increasing in a for each t, and*

$$\sup_{s\in[0,t]} |L_s^a - L_s^0| \xrightarrow[a\to 0]{} 0 \quad P^x\text{-a.s.},$$

then

$$\sup_{s\in[0,t]} |aN_s^a - L_s^0| \xrightarrow[a\to 0]{} 0 \quad P^x\text{-a.s.} \tag{4.8}$$

Proof. Note that for (4.6) and (4.7), by Jensen's inequality it suffices to consider only the case $p\geqq 2$. Let

$$Y_t^a = U v_a(X_t) - U v_a(X_0) + L_t^a. \tag{4.9}$$

We know by Brosamler (1970) or Wang (1977) that for each a, Y_t^a is a continuous local martingale. Let

$$N = \sup_{b\in(0,1]} \|U v_b\|.$$

Fix a for the moment and let $\varepsilon = \|U v_a - U v_0\|$. By Itô's lemma,

$$\begin{aligned} &E^x(U v_a + U v_0)^2(X_t) - E^x(U v_a - U v_0)^2(X_0) \\ &= E^x\langle Y^a - Y^0\rangle_t - E^x \int_0^t (U v_a - U v_0)(X_s)\, d(L_s^a - L_s^0). \end{aligned}$$

Since for all x and t,

$$E^x \int_0^t (U v_a - U v_0)(X_s)\, d(L_s^a - L_s^0) \leqq \varepsilon E^x(L_t^a + L_t^0) \leqq 2\varepsilon N,$$

then

$$E^x\langle Y^a - Y^0\rangle_t \leqq 2\varepsilon^2 + 2\varepsilon N.$$

But then by the Markov property,

$$\begin{aligned} E^x(\langle Y^a - Y_0\rangle_\infty - \langle Y^a - Y^0\rangle_s | \mathscr{F}_s) &= E^{X(s)}\langle Y^a - Y^0\rangle_\infty \\ &\leqq 2\varepsilon^2 + 2\varepsilon N, \end{aligned}$$

and hence, by Dellacherie and Meyer (1980, p. 188),

$$E^x\langle Y^a - Y^0\rangle_\infty^{p/2} \leqq c(p)(2\varepsilon^2 + 2\varepsilon N)^p, \qquad p\geqq 2.$$

By the Burkholder-Gundy inequalities (Dellacherie and Meyer (1980, p. 304)),

$$E^x \sup_{t<\infty} |Y_t^a - Y_t^0|^p \leqq c(p)\, E^x \langle Y^a - Y^0 \rangle_\infty^{p/2} \xrightarrow[a\to 0]{} 0.$$

Since $U v_a \xrightarrow[a\to 0]{} U v_0$ uniformly, using (4.9) gives us (4.6).

The proof of (4.7) follows similar lines. Let

$$Z_t^a = a N_t^a - L_t^a.$$

Since $E^x L_t^a \leqq U v_a(x) < \infty$ and the jumps of $a N_t^a$ are bounded, we see that Z_t^a is a local martingale. Since N_t^a is a pure jump process,

$$\begin{aligned} E^x([Z^a, Z^a]_\infty - [Z^a, Z^a]_{s-} \,|\, \mathscr{F}_s) &= E^x\Big(\sum_{t\geqq s} (\Delta Z_t^a)^2 \,|\, \mathscr{F}_s\Big) \\ &\leqq a^2 + a^2 E^x(N_\infty^a - N_s^a \,|\, \mathscr{F}_s) \\ &= a^2 + a E^x(L_\infty^a - L_s^a \,|\, \mathscr{F}_s) \\ &= a^2 + a E^{X(s)} L_\infty^a \\ &= a^2 + a\, U v_a(X_s) \\ &\leqq a^2 + aN. \end{aligned}$$

By Dellacherie and Meyer (1980) again,

$$E^x [Z^a, Z^a]_\infty^p \leqq c(p)(a^2 + aN)^p, \qquad p \geqq 1, \tag{4.10}$$

and by Burkholder-Gundy,

$$E^x \sup_{t<\infty} |Z_t^a|^p \leqq c(p)\, E^x [Z^a, Z^a]_\infty^{p/2} \xrightarrow[a\to 0]{} 0, \tag{4.11}$$

which is (4.7).

Now we assume that N_t^a is non-increasing in a for each t and

$$\sup_{s\in[0,t]} |L_s^a - L_s^0| \xrightarrow[a\to 0]{} 0 \qquad P^x-\text{a.s.}$$

Suppose $q \in (0, 1)$ and let $a_n = q^n$. By (4.10) and (4.11) applied with $p=2$ and Chebyshev inequality, for each $\lambda > 0$,

$$P^x(\sup_t |Z_t^{a_n}| > \lambda) \leqq c\, a_n / \lambda^2.$$

Then, by the Borel-Cantelli lemma,

$$P^x(\sup_t |Z_t^{a_n}| > \lambda \text{ i.o.}) = 0,$$

or

$$\sup_t |a_n N_t^{a_n} - L_t^0| \xrightarrow[n\to\infty]{} 0 \qquad P^x\text{-a.s.}$$

Now, if $a_{n+1} < a < a_n$, then

$$\sup_t (a N_t^a - q^{-1} L_t^0) \leqq \sup_t (a_n N_t^{a_{n+1}} - q^{-1} L_t^0)$$
$$\leqq q^{-1} \sup_t (a_{n+1} N_t^{a_{n+1}} - L_t^0).$$

Hence

$$\limsup_{a \to 0} \sup_t (a N_t^a - L_t^0) \leqq (q^{-1} - 1) \sup_t L_t^0 \qquad P^x\text{-a.s.}$$

Similarly, since $a N_t^a \geqq a_{n+1} N_t^{a_n}$ for $a_{n+1} < a < a_n$,

$$\liminf_{a \to 0} \sup_t (a N_t^a - L_t^0) \geqq (q-1) \sup_t L_t^0 \qquad P^x\text{-a.s.}$$

Since q can be taken arbitrarily close to 1 and $\sup_t L_t^0 < \infty$, this proves (4.8).□

5. Estimates for Excursion Laws

First we will present two lemmas relating Brownian excursion laws to the Green function. Consider the following events.

$A_1(x, r) = \{\sup\{|x - X(t)|: t \in (0, R)\} > r\}$,
$A_2(x, r) = \{|x - X(R-)| > r\}$,
$A_3(x, r) = A_3(r) = \{\operatorname{diam}(X(0, R)) > r\}$,
$A_4(x, r) = A_4(r) = \{R > r^2\}$.

Recall our convention concerning $x + r N_x$.

Lemma 5.1. *Let D be a Lipschitz domain with character λ. Fix some $x_0 \in D$ and let*

$$B = \{y \in D: G_D(x_0, y) \geqq 1\}.$$

For $x \in \partial D$, let H^x denote the standard excursion law in D. Then there exist $r_0 = r_0(D, x_0) > 0$ and $c = c(\lambda) < \infty$ such that for $r \in (0, r_0)$, $k = 1, 2, 3, 4$, and $x \in \partial D$,

$$H^x(A_k(x, r)) \leqq c H^x(T_B < \infty)/G_D(x_0, x + r N_x), \tag{5.1}$$

provided N_x is well defined.

Proof. Fix an $x \in \partial D$ and let

$$r_0 = \operatorname{dist}(x_0, \partial D)/2,$$
$$B_1(r) = B_1 = \{y \in \mathbb{R}^n: |y - x| \geqq r\},$$
$$h(y) = P_D^y(T_{B_1} < \infty).$$

Note that

$$G_D(x_0, y) = P_D^y(T_B < \infty)$$

for $y \in D \backslash B$. By Theorem 2.1,

$$\begin{aligned} H^x(A_1(x,r))/H^x(T_B<\infty) &= \lim_{\substack{y\to x\\ y\in D}} P_D^y(T_{B_1}<\infty)/P_D^y(T_B<\infty) \\ &= \lim_{\substack{y\to x\\ y\in D}} h(y)/G_D(x_0,y). \end{aligned} \tag{5.2}$$

The functions $h(\cdot)$ and $G_D(x_0,\cdot)$ are harmonic in $D\backslash B_1$ and vanish on $\partial D\backslash B_1$. By the boundary Harnack principle (Lemma 2.1) there exists $c_1<\infty$ such that

$$h(y)/G_D(x_0,y) \leqq c_1\, h(y_1)/G_D(x_0,y_1) \tag{5.3}$$

for $y_1 = x+(r/4)N_x$ and all $y\in D$ with $|y-x|<r/2$. By the Harnack principle

$$G_D(x_0, x+(r/4)N_x) \geqq c_2\, G_D(x_0, x+rN_x). \tag{5.4}$$

We obviously have $h(y_1)=P_D^{y_1}(T_{B_1}<\infty)\leqq 1$. This and (5.2)–(5.4) yield

$$\begin{aligned} H^x(A_1(x,r))/H^x(T_B<\infty) &\leqq c_1\, h(y_1)/G_D(x_0,y_1) \\ &\leqq (c_1/c_2)/G_D(x_0,x+rN_x). \end{aligned}$$

This proves the lemma for $k=1$.

Since $A_2(x,r)\subset A_1(x,r)$, the case $k=2$ follows immediately from (5.1) with $k=1$.

We have $A_3(x,r)\subset A_1(x,r/2)$ H^x-a.s. By the Harnack principle,

$$G_D(x_0, x+(r/2)N_x) \geqq c_3\, G_D(x_0, x+rN_x).$$

This and (5.1) applied with $k=1$ imply

$$\begin{aligned} H^x(A_3(x,r)) &\leqq H^x(A_1(x,r/2)) \\ &\leqq c\,H^x(T_B<\infty)/G_D(x_0,x+(r/2)N_x) \\ &\leqq (c/c_3)\,H^x(T_B<\infty)/G_D(x_0,x+rN_x). \end{aligned}$$

This completes the proof for $k=3$.

To prove (5.1) with $k=4$, it suffices to show that

$$H^x(A_4(x,r)) \leqq c_4\, H^x(A_1(x,r)). \tag{5.5}$$

By Theorem 2.1,

$$H^x(A_4(r))/H^x(A_1(x,r)) = \lim_{\substack{y\to x\\ y\in D}} P_D^y(A_4(r))/P_D^y(T_{B_1(r)}<\infty). \tag{5.6}$$

Now we apply the parabolic boundary Harnack principle (we use the version proved in Theorem 6.1 of Burdzy et al. (1989)) to obtain

$$P_D^y(A_4(r))/P_D^y(T_{B_1(r)}<\infty) \leqq c_5\, P_D^{y_0}(A_4(r))/P_D^{y_0}(T_{B_1(r)}<\infty), \tag{5.7}$$

where $y_0 = x + (r/32)N_x$ and $y \in D$ with $|y-x| < r/16$. It is easy to see that

$$P_D^{y_0}(T_{B_1(r)} < \infty) \geqq c_6 = c_6(\lambda).$$

By the scaling of Brownian motion, the constant c_6 can be chosen independent of r. The inequality (5.7) implies now

$$P_D^y(A_4(r))/P_D^y(T_{B_1(r)} < \infty) \leqq c_5/c_6.$$

This and (5.6) give (5.5), which completes the proof. □

Lemma 5.2. *Suppose that D is a Lipschitz domain, $x_0 \in D$, $x \in \partial D$ and N_x is well defined. Let*

$$B = \{y \in D\colon G_D(x_0, y) \geqq 1\}$$

and let H^x be the standard excursion law in D. Then

$$\lim_{r \to 0} H^x(A_k(x, r)) \frac{G_D(x_0, x + rN_x)}{H^x(T_B < \infty)} = d_k \tag{5.8}$$

for $k = 1, 2, 3, 4$. We have

$$d_1 = 2\pi^{-1/2}\left[\Gamma\left(\frac{n+2}{2}\right)\Big/\Gamma\left(\frac{n+1}{2}\right)\right],$$

$$d_2 = 2\pi^{-1/2}\left[\Gamma\left(\frac{n}{2}\right)\Big/\Gamma\left(\frac{n-1}{2}\right)\right],$$

$$d_4 = (2/\pi)^{1/2}.$$

The constant d_3 satisfies

$$d_1 \leqq d_3 \leqq 2d_1.$$

Proof. The lemma follows immediately from Theorem 4.1 of Burdzy et al. (1989) for $k = 1, 2, 4$.

We turn to the case $k = 3$. Let

$$T_\varepsilon = \inf\{t > 0\colon |x - X(t)| \geqq \varepsilon\},$$

$$A_5^\varepsilon(x, r) = A_5^\varepsilon(r) = \{\operatorname{diam}(X(T_{\varepsilon r}, R)) > r\},$$

and

$$A_6^\varepsilon(x, r) = A_6^\varepsilon(r) = \{\operatorname{diam}(X(T_{\varepsilon r}, R)) > r - 2\varepsilon r\},$$

for $\varepsilon \in (0, 1/2)$. The functions

$$f_5^\varepsilon(y, t) \overset{\mathrm{df}}{=} P_D^y(A_5^\varepsilon(r))$$

and

$$f_6^\varepsilon(y, t) \overset{\mathrm{df}}{=} P_D^y(A_6^\varepsilon(r))$$

are parabolic in $\{y \in D: |x-y|<\varepsilon r\} \times (0, \infty)$. The arguments of Sect. 4 of Burdzy et al. (1989) apply to the functions f_5^ε and f_6^ε and events A_5^ε and A_6^ε, and therefore Theorem 4.1 of that paper holds for A_5^ε and A_6^ε. Consequently, we have

$$\lim_{r \to 0} H^x(A_k^\varepsilon(x,r)) \frac{G_D(x_0, x+rN_x)}{H^x(T_B<\infty)} = d_k^\varepsilon$$

for $k=5, 6$. Recall from the proof of Theorem 4.1 in Burdzy et al. (1989) that

$$d_k^\varepsilon = r H_*^x(A_k^\varepsilon(x,r)),$$

where H_*^x is an excursion law in $\{y \in \mathbb{R}^n: (y-x)\cdot N_x>0\}$ normalized so that the H_*^x-chance of hitting $\{y \in \mathbb{R}^n: (y-x)\cdot N_x=1\}$ is equal to 1. It is now easy to see that the limits $\lim_{\varepsilon \to 0} d_k^\varepsilon$ exist for $k=5, 6$ and are equal to a constant which we will call d_3. This proves (5.8) for $k=3$ since

$$A_5^\varepsilon(r) \subset A_3(r) \subset A_6^\varepsilon(r).$$

In order to see that $d_1 \leqq d_3 \leqq 2d_1$, note that

$$A_1(x,r) \subset A_3(x,r) \subset A_1(x, r/2). \quad \square$$

Lemma 5.3. *Fix some* $k=1, 2, 3$ *or* 4 *and let*

$$v_r(dx) = r H^x(A_k(x,r))\, \mu(dx).$$

Then $\{v_r\}_{r \in (0,1]}$ *satisfy* (4.1)–(4.2) *and* $v_r \to d_k \mu$ *weakly as* $r \to 0$.

Proof. Fix a point $x_0 \in D$ and let ω be the harmonic measure on ∂D relative to x_0. Let B denote $\{y \in D: G_D(x_0, y) \geqq 1\}$.

We have normalized the excursion laws H^x so that

$$\begin{aligned} H^x(T_B<\infty) &= \lim_{r \to 0} r^{-1} P_D^{x+rN_x}(T_B<\infty) \\ &= \lim_{r \to 0} r^{-1} G_D(x_0, x+rN_x). \end{aligned} \tag{5.9}$$

By Dahlberg (1977), the last limit is equal to $c(d\omega/d\mu)(x)$ where c is an absolute constant. Thus

$$H^x(T_B<\infty) = c(d\omega/d\mu)(x) \quad \mu\text{-a.e.}$$

and, by Lemma 5.1,

$$\begin{aligned} r H^x(A_k(x,r)) &\leqq r c_1 H^x(T_B<\infty)/G_D(x_0, x+rN_x) \\ &\leqq c_1 c(r/G_D(x_0, x+rN_x)) \left(\frac{d\omega}{d\mu}\right)(x) \\ &\leqq c_2 W_{r_0}(x), \end{aligned} \tag{5.10}$$

where $W_{r_0}(x)$ has been defined in Sect. 3. The inequality (5.10) holds for all r less than some $r_0>0$. By Theorem 3.1 and (3.14), for some $\alpha>n-2$, and all surface balls Δ,

$$\begin{aligned} v_r(\Delta(y,r)) &= \int_{\Delta(y,r)} r H^x(A_k(x,r))\,\mu(dx) \\ &\leqq \int_{\Delta(y,r)} c_2 W_{r_0}(x)\,\mu(dx) \\ &\leqq c_3 r^\alpha \end{aligned}$$

and

$$\int_{\partial D} \Big(\sup_{r\in(0,r_0)} r H^x(A_k(x,r))\Big)\,\mu(dx) \leqq \int_{\partial D} c_2 W_{r_0}(x)\,\mu(dx) < \infty. \tag{5.11}$$

Hence, the v_r satisfy (4.1) and (4.2).

Lemma 5.2 and (5.9) imply that

$$\lim_{r\to 0} r H^x(A_k(x,r)) = \lim_{r\to 0} r d_k \frac{H^x(T_B<\infty)}{G_D(x_0, x+rN_x)} = d_k. \tag{5.12}$$

For every set $B_1 \subset \partial D$ we have

$$v_r(B_1) = \int_{B_1} r H^x(A_k(x,r))\,\mu(dx) \xrightarrow[r\to 0]{} d_k\,\mu(B_1)$$

because the integrands converge pointwise and the dominated convergence theorem may be applied by (5.11). We have thus shown that v_r converge weakly to $d_k \mu$, which completes the proof. □

Lemma 5.4. *Suppose that D is a bounded Lipschitz domain. Then there exists $r_0>0$ such that for all $t>0$ and $x\in\mathbb{R}^n$,*

$$E^x \int_0^t W_{r_0}(X_s)\,dL_s < \infty.$$

Proof. Let $W^a(x) = \min(W(x), a)$ and

$$L_t^a = \int_0^t W^a(X_s)\,dL_s$$

for $a \leqq \infty$. The Revuz measure of L_t^a is equal to $W^a(x)\,\mu(dx)$.

Let $U(x,y)$ be the usual potential operator for the Brownian motion (see Sect. 2). By a result of Revuz (1970, Sect. V.1),

$$E^x \int_0^\infty dL_s^a = \int_{\partial D} U(x,y)\,W^a(y)\,\mu(dy). \tag{5.13}$$

The right hand side of (5.13) increases to a finite limit when $a \to \infty$ by Theorem 3.1 and Lemma 4.1 for a suitable $r_0 > 0$. It follows that

$$E^x \int_0^\infty dL_s^\infty < \infty$$

and the proof is complete. □

6. Representations of Local Time

Our main theorem comes next. The excursions e_s of the process X in D and the exist system (dL, H) have been defined in Sect. 2. Events A_k have been defined at the beginning of the previous section and constants d_k in Lemma 5.2. Observe that A_k is a family of paths in Ω so that the expression $e_s \in A_k$ makes sense.

Theorem 6.1. *Suppose that D is a bounded Lipschitz domain. Fix some $k = 1, 2, 3$ or 4. Let N_t^r be the number of excursions e_s of Brownian motion X in D such that $s < t$ and $e_s \in A_k(e_s(0), r)$. Then, for every $t > 0$, $p < \infty$, and $x \in \mathbb{R}^n$,*

$$\sup_{s \in [0,t]} |r N_s^r / d_k - L_s| \to 0 \quad \text{as } r \to 0 \tag{6.1}$$

*in $L^p(P^x)$ and P^x-*a.s.

Proof. Let k be fixed. For $r > 0$ let

$$L_t^r = \int_0^t r H^{X(s)}(A_k(X_s, r))\, dL_s$$

and

$$v_r(dx) = r H^x(A_k(x, r))\, \mu(dx).$$

Since by Lemmas 4.1 and 5.3, $\|U v_r\| < \infty$, it follows from a result of Revuz (1970, Sect. V.1) that L_t^r is a well defined continuous additive functional with Revuz measure v_r, for $r < r_0$.

By the exit system formula (2.1), $r N_t^r$ is a point process with compensator L_t^r.

In view of Lemmas 4.1 and 5.3, the assumptions of Theorem 4.1 are satisfied and this shows that (6.1) holds in the sense of $L^p(P^x)$-convergence.

Now we will show that the convergence in (6.1) also holds P^x-a.s. In view of Theorem 4.1, all we have to show is that

$$\sup_{s \in [0,t]} |L_s^r - d_k L_s| \xrightarrow[r \to 0]{} 0 \qquad P^x\text{-a.s.} \tag{6.2}$$

We have

$$\begin{aligned} \sup_{s \in [0,t]} |L_s^r - d_k L_s| &= \sup_{s \in [0,t]} \left| \int_0^s r H^{X(u)}(A_k(X_u, r))\, dL_u - \int_0^s d_k\, dL_u \right| \\ &\leq \sup_{s \in [0,t]} \int_0^s |r H^{X(u)}(A_k(X_u, r)) - d_k|\, dL_u \\ &\leq \int_0^t |r H^{X(u)}(A_k(X_u, r)) - d_k|\, dL_u. \end{aligned} \tag{6.3}$$

By (5.12), the last integrand converges to 0 pointwise. We have shown in Lemma 5.4 that

$$E^x \int_0^t W_{r_0}(X_u)\,dL_u < \infty$$

for some r_0. Hence

$$\int_0^t W_{r_0}(X_u)\,dL_u < \infty \qquad P^x\text{-a.s.}$$

By (5.10), the last integrand in (6.3) is dominated by $c_2 W_{r_0}(X_u) + d_k$, for $r < r_0$. By the dominated convergence theorem, the right hand side of (6.3) converges to 0 P^x-a.s. as $r \to 0$. This completes the proof of (6.2) and of the theorem. □

Remark 6.1. Theorem 6.1 holds also for unbounded Lipschitz domains, provided that the limit in (6.1) is taken in P^x-probability. This may be proved by stopping X at the hitting times of $\{y \in \mathbb{R}^n : |y| = k\}$ and then letting $k \to \infty$.

For the sake of comparison and completeness we give the following result. The a.s. convergence was proved in Bass (1984, Corollary 3.11).

Proposition 6.1. *Let D, X and L be as in Theorem 6.1. For $r > 0$ let $B(r) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial D) \leq r\}$ and*

$$L_t^r = \frac{1}{2r} \int_0^t \mathbf{1}_{B(r)}(X_s)\,ds.$$

Then for every $t > 0$, $p < \infty$ and $x \in \mathbb{R}^n$,

$$\sup_{s \in [0,t]} |L_s^r - L_s| \to 0 \qquad \text{as } r \to 0$$

in $L^p(P^x)$ and P^x-a.s.

Proof. The Revuz measure ν_r of L_s^r is uniform on $B(r)$. It is elementary to check that the measures ν_r satisfy the assumptions of Theorem 4.1, from which the L^p convergence follows. □

7. Counterexamples in Non-Lipschitz Domains

Suppose that $\lambda: [0, \infty) \to \mathbb{R}$. We will say that an open set $D \subset \mathbb{R}^n$ is a λ-domain if for each $x \in \partial D$ there exist a neighborhood V and x and a coordinate system $CS(x)$ such that $\partial D \cap V$ may be represented in $CS(x)$ as the graph of a function $f: \mathbb{R}^{n-1} \to \mathbb{R}$ with the property

$$|f(s) - f(u)| \leq \lambda(|s-u|) \qquad \text{for all } s, u \in \mathbb{R}^{n-1}.$$

If $\lambda(t) = ct^\alpha$ with $\alpha < 1$, then λ-domain is a Hölder domain.

Proposition 7.1. *Suppose that* $\lambda\colon [0,\infty)\to\mathbb{R}$, $\lambda(0)=0$, λ *is nondecreasing and* $\lambda(r)/r\to\infty$ *as* $r\to 0$. *Then there exists a* λ-*domain* D *with the following property.*

Let $k=1,2,3$, *or* 4. *Let* $N_t^k(r)$ *be the number of excursions* e_s *of Brownian motion* X *in* D *such that* $s<t$ *and* $e_s\in A_k(e_s(0),r)$. *There exists a sequence* $\{r_j\}_{j\geqq 1}$ *of positive numbers converging to* 0 *such that for all* $x\in\mathbb{R}^n$

$$\lim_{j\to\infty} r_j N_t^k(r_j)=\begin{cases}0 & \text{on } t\leqq T(\partial D)\\ \infty & \text{on } t>T(\partial D)\end{cases}\qquad P^x\text{-a.s.}$$

Proof. We will sketch an example in $\mathbb{R}^2$. Examples in higher dimensions may be concocted in a similar way.

Suppose that λ satisfies the assumptions of the proposition. Then there exists another function $\tilde\lambda$ such that $\tilde\lambda\leqq\lambda$, $\tilde\lambda$ satisfies the same assumptions and $\tilde\lambda$ is piecewise linear. Hence, we may assume without loss of generality that λ itself is piecewise linear; more precisely, we will assume that λ is continuous and there exist sequences $\{a_j\}_{j\geqq 1}$ and $\{b_j\}_{j\geqq 1}$ such that

$$\begin{aligned} a_j>a_{j+1}&>0 && \text{for } j\geqq 1,\\ \lambda'(r)&=b_j && \text{for } r\in(a_{j+1},a_j),\quad j\geqq 1,\\ b_j&<b_{j+1} && \text{for } j\geqq 1,\\ \lim_{j\to\infty} b_j&=\infty. \end{aligned}$$

We will construct inductively a sequence of domains $\{D_j\}_{j\geqq 1}$ converging to a domain D. We will do it by describing first curves F_j passing through $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$. The four pieces of F_j will be (by definition) similar to each other so we will define explicitly only the one joining $(0,0)$ and $(1,0)$. The domain D_j is defined as the interior of F_j.

The curve F_1 is the straight line segment joining $(0,0)$ and $(1,0)$. Set $\alpha_1=1$.

Suppose that $D_1,\ldots,D_j$, $f_1,\ldots,f_j$, and $\alpha_1,\ldots,\alpha_j$ have been defined.

In order to construct D_{j+1}, we will need a large integer m which will be specified later. Let

$$c_m=(a_m-a_{m+1})/2.$$

For an integer $s\geqq 0$, let g_s be a continuous function such that

$$\begin{aligned} &g_s(s\cdot c_m)=f_j(s\cdot c_m),\\ &g_s'(x)=b_m \qquad \text{for } x<s\cdot c_m,\\ &g_s'(x)=-b_m \qquad \text{for } x>s\cdot c_m. \end{aligned}$$

Moreover, let g_∞ be continuous and

$$\begin{aligned} &g_\infty(1)=0,\\ &g_\infty'(x)=b_m \qquad \text{for } x<1. \end{aligned}$$

Let

$$f_{j+1}(x)=\max(g_\infty(x), \sup_{\substack{0\leqq s<\infty \\ s\in\mathbb{Z} \\ s\cdot c_m<1}} g_s(x)) \qquad \text{for } 0\leqq x\leqq 1,$$

and let F_{j+1} be the graph of f_{j+1} between 0 and 1.

Now we will impose several conditions on m. First, let m be so large that

$$\operatorname{Dist}(\partial D_{j+1}, \partial D_s)<\alpha_s/2 \qquad \text{for } s\leqq j$$

(see Sect. 2 for the definition of Dist).

Let μ be arc length measure on ∂D_j and let $\nu=\nu_m$ be arc length measure on ∂D_{j+1}. It is elementary to check that $\beta_m \nu_m \to \mu$ weakly as $m\to\infty$ where β_m are suitable constants which tend to 0 as $m\to\infty$. It is also easy to check that $\beta_m \nu_m$ satisfy the assumptions of Lemma 4.1. Let L^j and L^{j+1} be the local times on ∂D_j and ∂D_{j+1}. By choosing m sufficiently large, we have, in view of Theorem 4.1,

$$P^x(\beta_m L^{j+1}_{T+1/j} > L^j_{T+1/j}/2)\geqq 1-2^{-j},$$

where $T=T(\partial D_{j+1})$. Thus, since $\beta_m\to 0$, we take m sufficiently large so that

$$P^x(L^{j+1}_{T+1/j}<4j)<2^{-j}.$$

Fix some k between 1 and 4 and let $\tilde{N}^{j+1}_t$ be defined relative to D_{j+1} as N^k_t was defined relative to D. Since D_{j+1} is a Lipschitz domain, Theorem 6.1 implies that

$$r\tilde{N}^{j+1}_{T+1/j}(r)/d_k - L^{j+1}_{T+1/j}\to 0 \qquad \text{as } r\to 0$$

and, therefore, for some $r_{j+1}>0$,

$$P^x(r_{j+1}\tilde{N}^{j+1}_{T+1/j}(r_{j+1})/d_k<2j)<2^{-j+1}.$$

Now choose α_{j+1} so small that

$$P^x(r_{j+1}\hat{N}_{T+1/j}(r_{j+1})/d_k<j)<2^{-j+2} \tag{7.1}$$

for every domain $\hat{D}$ with $\operatorname{Dist}(\partial\hat{D}, \partial D_{j+1})<\alpha_{j+1}$; here $\hat{N}$ is defined relative to $\hat{D}$.

Finally, define D as $\lim_{j\to\infty} D_j$. It is elementary to check that D is a λ-domain.

By (7.1), for every $\varepsilon>0$,

$$\lim_{j\to\infty} r_{j+1} N^k_{T+\varepsilon}(r_{j+1})/d_k=\infty \qquad P^x\text{-a.s.} \quad \square$$

Remarks 7.1. (i) The last example may be modified to show that there exists a λ-domain such that for each function $a(r)$, the limit

$$\lim_{r\to 0} a(r) N^k_t(r)$$

is either 0, infinite, or does not exist P^x-a.s., for all x and t. In some domains, there is no "right" renormalization of N_t^k.

(ii) Proposition 7.1 comes as no surprise since the potential theoretic properties of Hölder domains are known to differ from those of Lipschitz domains. One may wonder whether Theorem 6.1 extends to non-tangentially accessible (NTA) domains which are similar in some respects to Lipschitz domains. For a definition of an NTA domain, see Jerison and Kenig (1982).

Proposition 7.2. *There exists a non-tangentially accessible domain which satisfies the conclusion of Proposition 7.1.*

Proof. The proof is completely analogous to that of Proposition 7.1 and, therefore, it is omitted. It will suffice to say that one should construct a "snowflake" domain, also called a Koch domain in the manner of Mandelbrot (1982). Then one can use the fact that the boundary of such a domain has infinite length if it is suitably constructed. □

Propositions 7.1 and 7.2 may suggest that the finiteness of the surface area measure is the indicator of applicability of Theorem 6.1. This is false, as our next example shows.

Proposition 7.3. *There exists a domain $D\subset\mathbb{R}^2$ such that ∂D has finite length and such that*

$$\liminf_{r\to 0} r N^4_{T(\partial D)+1}(r)=0 \qquad P^0\text{-a.s.}$$

Proof. We will construct inductively a sequence of domains D_j converging to D. The boundary of each domain D_j will consist of a finite union of disjoint circles.

For a set K and $a>0$, let $B(K,a)=\{x\in\mathbb{R}^2\colon \operatorname{dist}(x,K)<a\}$.

Let $D_1=\{x\in\mathbb{R}^2\colon |x|<1\}$ and $\alpha_1<1/4$. Suppose that D_j and $\alpha_j>0$ have been defined. Let

$$C_j=D_j\cap\bigcap_{m\leqq j} B(\partial D_m,\alpha_m(1-2^{-j})).$$

Let $\{B^j_m\}_{m\geqq 1}$ be the (finite) family of all circles with centers $(m_1 2^{-v_j}, m_2 2^{-v_j})$, $m_1, m_2\in\mathbb{Z}$, radii $b_j>0$, and such that $B^j_m\subset C_j$ for all m. Here v_j is a large integer which will be specified later. The radius $b_j=b_j(v_j)$ is chosen so that the total length of all circles B^j_m, $m\geqq 1$, is as close as possible to 2^{-j}. We will define D_{j+1} by removing from D_j all circles B^j_m, $m\geqq 1$, and their interiors.

Let $T=T(\partial D_j)+1$. The domain D_j is Lipschitz, so by Theorem 6.1

$$|rN^4_T(r)-L_T|\xrightarrow[r\to 0]{}0, \qquad P^0\text{-a.s.}$$

Let $a>1$ be such that

$$P^0(L_T<2a)\geqq 1-2^{-j}$$

and choose $\tilde{r}_j$ so that for $r<\tilde{r}_j$

$$P^0(rN_T^4(r)<a)\geqq 1-2^{-j+1}. \tag{7.2}$$

Let $\tilde{N}_T^4(r)$ be the number of excursions e_s in D_j with $e_s\in A_4(e_s(0), r)$, $s<T$, and such that e_s lies totally inside C_j. Choose $r_j<\tilde{r}_j$ so that

$$P^0\left(\tilde{N}_T^4(r_j)>\left(1-\frac{1}{a2^j}\right)N_T^4(r_j)\right)\geqq 1-2^{-j}. \tag{7.3}$$

Now let $v_j\to\infty$ and vary b_j accordingly. Let $\eta(v_j)$ be length measure on $\bigcup_{m\geqq 1} B_m^j$. As $v_j\to\infty$, the measures $\eta(v_j)$ converge weakly to the uniform measure on C_j with total mass 2^{-j}. It is elementary to check that the assumptions of Theorem 4.1 are satisfied and, consequently, the continuous additive functionals L^{v_j} associated with $\eta(v_j)$ converge uniformly on $[0, T]$ to $L_t \overset{\mathrm{df}}{=} \int_0^t \mathbf{1}_{C_j}(X_s)\,ds$ P^0-a.s.

This implies that the P^0-probability that there exists an interval of length greater or equal to r_j^2 during which L^{v_j} is constant and X_s stays in C_j, goes to 0 as $v_j\to\infty$. Let us translate the last statement into the language of excursions. Define D_{j+1} by removing from D_j all circles B_m^j, $m\geqq 1$, and their interiors. Then the number of excursions in D_{j+1} with lifetime greater than r_j^2 and lying totally in C_j can be made arbitrarily small by taking v_j large. Now choose v_j sufficiently large so that this last fact combined with (7.3) gives $P^0(r_jN_T^4(r_j, D_{j+1})>2^{-j})\leqq 2^{-j+1}$, where we have indicated that now we are counting excursions in D_{j+1}.

By continuity of paths, it is possible to find $\alpha_{j+1}>0$ such that if $\partial D\subset B(\partial D_{j+1}, \alpha_{j+1})$ and $D\subset D_{j+1}$, then

$$P^0(r_jN_T^4(r_j, D)>2^{-j+1})\leqq 2^{-j+2}. \tag{7.4}$$

Finally, let $D=\lim_{j\to\infty} D_j$. By (7.4) and the Borel-Cantelli lemma,

$$\liminf_{r\to 0} rN_T^4(r)=0, \qquad P^0\text{-a.s.}$$

The boundary of D is equal to $\bigcup_{j,m} B_m^j$ and it has a finite length, since the total length of $\bigcup_m B_m^j$ is less than $2\cdot 2^{-j}$. □

Remark 7.2. There exists a domain $\tilde{D}\subset\mathbb{R}^3$ such that $\partial\tilde{D}$ is locally the graph of a continuous function, $\partial\tilde{D}$ has a finite area and Proposition 7.3 holds for $\tilde{D}$. In order to construct such a domain $\tilde{D}$, consider D of the last proposition, let $\tilde{D}=D\times[0, 1]$ and then make some minor adjustments. We leave the details to the reader.

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